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THE CLASSIFICATION OF FLAT COMPACT COMPLETE SPACE-FORMS WITH METRIC OF SIGNATURE (2, 2)

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Those flat compact complete space-forms with metric of signature (2,2) are classified up to finite covers. The simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ are classified up to conjugation.

1. Introduction.

(1.1) If $\Gamma \subseteq R^4 \rtimes \mathrm{SO}(2, 2)$ and Γ acts on R^{p+q} freely and properly discontinuously with compact quotient, then $X = R^{p+q}/\Gamma$ is a flat compact complete space-form with metric of signature (p, q). Recently D. Fried [3] has classified those flat compact complete space-forms with metric of signature (1,3) upto finite covers. Ravi S. Kulkarni pointed out that Fried's method can be applied to the case $(p, q) = (2, 2)$. The basic idea of Fried's method is in the following theorem:

(1.2) THEOREM. *Suppose X is a flat compact complete space-form with fundamental group $\Gamma \subseteq R^4 \rtimes \mathrm{SO}(2, 2)$. Then there is a uniquely determined subgroup H of $R^4 \rtimes \mathrm{SO}(2, 2)$ that acts simply transitively on R^4 and $H \cap \Gamma = \pi$ has finite index in Γ.*

(1.3) In §2 we classify those subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ that act on R^4 simply transitively, up to the conjugacy of $R^4 \rtimes \mathrm{O}(2, 2)$. Every such subgroup, as a Lie group, is isomorphic to one of the following:

$$R^4, \quad R \times \mathrm{Nil}^3, \quad \mathrm{Nil}^4, \quad R \times \left\{R^2 \rtimes \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} ; t \in R\right\},$$
$$R \times \{R^2 \rtimes \mathrm{SO}(2)\}.$$

All of them, except the last one, correspond to Γ's. Their uniform lattices are known, cf. [3] and [7].

(1.4) To prove Theorem (1.2), we first prove in §3 that Γ is virtually solvable. This result confirms a conjecture by Milnor in a special case. In [6], it is conjectured that the fundamental group of a complete affinely flat manifold is virtually polycyclic. Our result, combined with

Fried's result, shows that this conjecture is true for compact pseudo-Riemannian 4-manifolds.

(1.5) In §4 we complete the proof of Theorem 1.2, using the theory of crystallographic hull developed by Fried and Goldman, cf. [**4**]. In §5, we give our classification. By comparing our list with Fried's, we obtain an interesting fact: as differential manifolds, they are the same coset spaces of the form H/Γ, where H is a Lie group isomorphic to R^4, $R\times \mathrm{Nil}^3$, Nil^4 or $R\times\{R^2\rtimes\left(\begin{smallmatrix} e^t & 0\\ 0 & e^{-t}\end{smallmatrix}\right); t\in R\}$ and Γ is a uniform lattice of H. These Lie groups have simply transitive representations as affine motions and when the signature is (2,2) (resp. (3,1)), the images of the representations are $R^4\rtimes\mathrm{SO}(2,2)$ (resp. $R^4\rtimes\mathrm{SO}(3,1)$).

(1.6) *Notations and some properties of* $\mathrm{SO}(2,2)$ *and* $\mathrm{so}(2,2)$. Throughout this paper we will call $\{e_i\}$, $1\le i\le 4$, a standard basis s.t. the metric Q, w.r.t. this basis, has the form

$$Q(v, v) = v_1v_3 + v_2v_4,$$

where $v=\sum_{i=1}^4 v_ie_i$. The full group of orientation-preserving isometries is $R^4\rtimes\mathrm{SO}(2,2)$ and

$$\mathrm{SO}(2,2)=\left\{g\in \mathrm{SL}_4(R);\ {}^tg\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}g=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right\},$$

where $I=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. The infinitesimal isometries are $R^4\rtimes\mathrm{so}(2,2)$ and

$$\text{(1.6.1)}\qquad \mathrm{so}(2,2)=\left\{X\in \mathrm{gl}_4(R);\ {}^tX\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}+\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}X=0\right\}$$
$$=\left\{\begin{pmatrix}a_{11} & a_{12} & 0 & d\\ a_{21} & a_{22} & -d & 0\\ 0 & c & -a_{11} & -a_{21}\\ -c & 0 & -a_{12} & -a_{22}\end{pmatrix};\ a_{ij}, d, c\in R\right\}.$$

(1.6.2) $\mathrm{so}(2,2)=L_1\oplus L_2$, where $L_i\simeq \mathrm{sl}_2(R)$, $i=1,2$; and

$$L_1=\left\{\begin{pmatrix}a & b & & \\ c & -a & & \\ & & -a & -c\\ & & -b & a\end{pmatrix};\ a, b, c\in R\right\},$$

$$L_2=\left\{\begin{pmatrix}a' & 0 & 0 & d'\\ 0 & a' & -d' & 0\\ 0 & c' & -a' & 0\\ -c' & 0 & 0 & -a'\end{pmatrix};\ a', d', c'\in R\right\}.$$

L_1, L_2 are permuted by an element of $\mathrm{O}(2,2)$.

(1.6.3) It is easy to show that any Cartan subalgebra of so$(2, 2)$ is conjugate under O$(2, 2)$ to one of the following:

(1) $$\left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -b \end{pmatrix} ; a, b, \in R \right\},$$

(2) $$\left\{ \begin{pmatrix} 0 & a & 0 & b \\ -a & 0 & -b & 0 \\ 0 & b & 0 & a \\ -b & 0 & -a & 0 \end{pmatrix} ; a, b, \in R \right\},$$

(3) $$\left\{ \begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & b \\ & & -b & -a \end{pmatrix} ; a, b, \in R \right\}.$$

An immediate corollary is

(1.6.4) *If* X *is in a Cartan subalgebra of* so$(2, 2)$ *and* $\det X = 0$, *then* X *must conjugate under* O$(2, 2)$ *to*

(4) $$\left\{ \begin{pmatrix} a & & & \\ & 0 & & \\ & & -a & \\ & & & 0 \end{pmatrix} \right\},$$

or

(5) $$\left\{ \begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix} \right\}.$$

(1.7) We identify Aff(n), resp. aff(n), with
$$\left\{ \begin{pmatrix} A & v \\ 0 & a \end{pmatrix} ; A \in \mathrm{GL}_4(R), v \in R^4 \right\},$$
resp.
$$\left\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in \mathrm{gl}_4(R), v \in R^4 \right\},$$
w.r.t. a given basis. Let P_l be the natural homomorphism taking an affine transformation (or an infinitesimal affine transformation) to its linear part. Let $L(G)$ be the Lie algebra of a Lie group G and $A(G)$ be the algebraic hull of G. We will need the following well-known lemma.

(1.7.1) LEMMA. *If* $G \subseteq \mathrm{Aff}(n)$ *s.t.* G *acts freely on* R^n, *then every* $A \in P_l(G)$ *has* 1 *as an eigenvalue.*

(1.7.2) LEMMA (*Kostant and Sullivan, cf.* [5]). *If* G *is as in* (1.7.1), *then every* $A \in P_l(A(G))$ *has* 1 *as an eigenvalue.*

(1.7.3) COROLLARY. *If* G *is an in* (1.7.1), *then every* $X \in P_l(L(A(G))$ *or* $X \in L(A(P_l(G))$ *has* 0 *as an eigenvalue.*

2. Simply transitive subgroups. We will classify subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ that act simply transitively on R^4. Our classification is up to the conjugation under $R^4 \rtimes \mathrm{O}(2, 2)$. It is well known that a simply transitive group of affine motions must be solvable, connected, simply connected and of dimension 4, cf. [1]. We will start from a special case when the groups are unipotent. The following lemma from Auslander and Scheuneman plays the key role in this section.

(2.1) LEMMA. *Let* U *be a nilpotent Lie group which has a faithful representation* $\rho\colon U \to \mathrm{Aff}(n)$, *let* ρ_* *be the induced monomorphism of Lie algebras*

$$\rho_* L(U) \to \left\{ \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} ; X \in \mathrm{gl}_n(R), v \in R^n \right\} = \mathrm{aff}(n),$$

and let P_l *be as in* (1.7), *let* P_t *be the projection from an element in* $\mathrm{aff}(n)$ *to its translation part. Then* $\rho(U)$ *acts on* R^n *simply transitively if and only if*

(1) $P_l \circ \rho_*(L(U))$ *is nilpotent, and*

(2) $P_t \circ \rho_*(L(U))$ *is a linear isomorphism of* $L(U)$ *onto* R^n.

For a proof, cf. [1]. So unipotent simply transitive subgroups are exactly the following U's s.t.

$$L(U) = \left\{ \begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} ; v \in R^n \right\}, \tag{2.2}$$

where $X(v)$ is a linear function of v and $P_l(L(U)) = \{X(v) ; v \in R^n\}$ is nilpotent.

(2.3) LEMMA. *There is a vector* $v_0 \in R^4$ *such that*

(i) $P_l(L(U))(v_0) = 0$,

(ii) $Q(v_0, v_0) = 0$.

Proof. If $V = \{v \in R^4 ; P_l(L(U))v = 0\}$, then $V^\perp$ is invariant. By Engel's Theorem on $V^\perp$, $V^\perp$ meets V. □

Let $\{e_i\}$ be our standard basis. Then we choose $v_0 = e_1$ since $O(2, 2)$ is transitive on $\{v ; Q(v, v) = 0\}/v \sim tv$, where $t \in R - \{0\}$.

(2.4) COROLLARY. *W.r.t. the above standard basis, $X \in P_l(L(U))$ has the form*

$$X(v) = \begin{pmatrix} 0 & a & 0 & b \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix},$$

where $a = a(v)$ and $b = b(v)$ are linear functions of v.

To find $a(v)$ and $b(v)$, we compute the commutator of $L(U)$.

$$\left[\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} X(v'') & v'' \\ 0 & 0 \end{pmatrix}, \tag{2.5}$$

where $v'' = X(v)v' - X(v')v$, $X(v'') = X(v)X(v') - X(v')X(v) = 0$. So

$$a(v'') = b(v'') \equiv 0.$$

Write

$$a(v) = \sum_{i=1}^{4} a_i v_i, \qquad b(v) = \sum_{i=1}^{4} b_i v_i. \tag{2.6}$$

Then we have

$$0 = \sum_{i=1}^{4} a_i v_i'', \qquad 0 = \sum_{i=1}^{4} b_i v_i'', \tag{2.7}$$

where v_i'''s are linear functions of a_i, b_i and $v_i v_j'$, $1 \le i, j \le 4$, and all coefficients of $v_i v_j'$ must be zero. We obtain

(2.8) LEMMA.

(i) $a_1 = b_1 = 0$,
(ii) $a_2 b_4 + a_4^2 = 0$,
(iii) $a_2 b_2 + a_4 a_2 = 0$,
(iv) $b_2 b_4 + b_4 a_4 = 0$,
(v) $b_2^2 + b_4 a_2 = 0$.

(2.9) COROLLARY.

(i) $b_4(b_2 + a_4) = 0$,
(ii) $a_2(b_2 + a_4) = 0$,
(iii) $(b_2 - a_4)(b_2 + a_4) = 0$.

(2.10) Now we can get some necessary conditions for the nontranslation unipotent simply transitive subgroups. If $b_2 + a_4 \neq 0$, then $b_4 = a_2 = 0$. By (2.8) (ii) and (v), $b_2^2 = a_4^2 = 0$ and we get a contradiction. So $b_2 + a_4 = 0$, and we have three subcases:

(2.10.1) $b_2 = a_4 = b_4 = a_2 = 0$, but $(a_3, b_3) \neq (0, 0)$, i.e.,

$$\begin{cases} a(v) = a_3 v_3 \\ b(v) = b_3 v_3. \end{cases}$$

(2.10.2) $b_2 + a_4 = 0$ but $b_2 \neq 0$, $a_4 \neq 0$. Then by (2.8) $b_4 \neq 0$, $a_2 \neq 0$, i.e.

$$\begin{cases} a(v) = a_2 v_2 + a_3 v_3 + a_4 v_4 \\ b(v) = b_2 v_2 + b_3 v_3 + b_4 v_4. \end{cases}$$

(2.10.3) $b_2 = 0$, $a_4 = 0$, $(a_2, b_4) \neq (0, 0)$. By (2.8), $b_4 a_2 = 0$, so

$$\begin{cases} a(v) = a_2 v_2 + a_3 v_3 \\ b(v) = b_3 v_3, \end{cases}$$

or

$$\begin{cases} a(v) = a_3 v_3 \\ b(v) = b_3 v_3 + b_4 v_4. \end{cases}$$

(2.11) THEOREM. *Up to conjugacy under $R^4 \rtimes \mathrm{O}(2, 2)$, the nontranslation unipotent simply transitive groups U of $R^4 \rtimes \mathrm{SO}(2, 2)$, have the following Lie algebras*:

$$L(U) = \left\{ \begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\},$$

where

$$X(v) = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

$a(v)$ and $b(v)$ are listed in the following table:

Type of $L(U)$	$a(v)$	$b(v)$	isomorphism type as an abstract Lie algebra
I-1	v_3	v_3	$N_3 \oplus R$
I-2	v_3	$-v_3$	$N_3 \oplus R$
I-3	v_3	0	$N_3 \oplus R$
II-1	$v_2 + v_4 + tv_3$, $(t \geq 0)$	$-v_2 - v_4$	N_4
II-2	$-v_2 + v_4 + tv_3$, $(t \geq 0)$	$-v_2 + v_4$	N_4
II-3	v_2	v_3	N_4

The equivalence classes are uniquely determined by the type of $L(U)$ and the parameter t (in Type II).

Proof. The discussion of the conjugacy under $R^4 \rtimes \mathrm{O}(2, 2)$ is long and tedious. We will only write down a brief one for subcase (2.10.2). We give the following lemma without proof.

(2.11.1) LEMMA. *If $a(v) \not\equiv 0$, $b(v) \not\equiv 0$, $a'(v') \not\equiv 0$, $b'(v') \not\equiv 0$, and if there is a matrix $A = (a_{ij}) \in \mathrm{O}(2, 2)$ such that*

$$A^{-1}\begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} A = \begin{pmatrix} 0 & a'(v') & 0 & b'(v') \\ 0 & 0 & -b'(v') & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a'(v') & 0 \end{pmatrix},$$

then either

$$(1) \quad \begin{cases} a'(v') = \dfrac{a_2 a_{22}^2}{a_{11}} v_2' + \left\{ \dfrac{a_{22}a_{23}}{a_{11}} a_2 + \dfrac{a_{22}}{a_{11}^2} a_3 + \dfrac{a_{22}a_{43}}{a_{11}} a_4 \right\} v_3' \\ \qquad + \dfrac{a_4}{a_{11}} v_4' \\ b'(v') = \dfrac{b_2}{a_{11}} v_2' + \left\{ \dfrac{a_{23}}{a_{11}a_{22}} b_2 + \dfrac{1}{a_{11}^2 a_{22}} b_3 + \dfrac{a_{43}}{a_{11}a_{22}} b_4 \right\} v_3' \\ \qquad + \dfrac{b_4}{a_{11}a_{22}^2} v_4' \end{cases}$$

where $a_{11}a_{22} \neq 0$; *or*

$$(2)\quad \begin{cases} a'(v') = \dfrac{b_4 a_{42}^2}{a_{11}} v_2' + \left\{ \dfrac{a_{42}a_{23}}{a_{11}} b_2 + \dfrac{a_{42}}{a_{11}^2} b_3 + \dfrac{a_{42}a_{43}}{a_{11}} b_4 \right\} v_3' \\ \qquad + \dfrac{b_2}{a_{11}} v_4' \\ b'(v') = \dfrac{a_4}{a_{11}} v_2' + \left\{ \dfrac{a_{23}}{a_{11}a_{42}} a_2 + \dfrac{1}{a_{11}^2 a_{42}} a_3 + \dfrac{a_{43}}{a_{11}a_{42}} a_4 \right\} v_3' \\ \qquad + \dfrac{a_4}{a_{11}a_{41}^2} v_4' \end{cases}$$

where $a_{11}a_{42} \neq 0$.

Write $a'(v') = \sum_{i=2}^4 a_i' v_i'$ and $b'(v') = \sum_{i=2}^4 b_i' v_i'$, then from (2.11.1)

$$a_2' b_4' = a_4' b_2' = \frac{a_4 b_2}{a_{11}^2} = -\frac{a_4^2}{a_{11}^2} < 0,$$

since $a_4 = -b_2 \neq 0$. So we can choose a_{11} such that $a_2'b_4' = a_4'b_2' = -1$, i.e. $a_4/a_{11} = \pm 1$. Next we use (1) (resp. (2)) if $a_4/a_{11} = 1$ (resp. -1), and choose a_{22} (resp. a_{42}) to reduce

$$\begin{pmatrix} a_2' & a_4' \\ b_2' & b_4' \end{pmatrix}$$

to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{if } a_2 a_4 > 0,$$

or

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{if } a_2 a_4 < 0.$$

Now a_3', b_3' have the form

$$\begin{cases} a_3' = z_1 + \dfrac{a_{22}}{a_{11}^2} a_3 \\ b_3' = \pm z_1 + \dfrac{1}{a_{11}^2 a_{22}} b_3, \end{cases}, \quad \text{or} \quad \begin{cases} a_3' = z_2 + \dfrac{a_{42}}{a_{11}^2} b_3 \\ b_3' = \pm z_2 + \dfrac{1}{a_{11}^2 a_{42}} a_3, \end{cases},$$

where z_1 (resp. z_2) depends on a_{23}, a_{43} (resp. a_{23}, a_{43}) and z_i, $i = 1, 2$ can assume any real number. We can choose z_i so that $b_3' = 0$ and we can choose the sign of a_{22} (resp. a_{42}) so that $a_3' \geq 0$. So we can find an $A \in \mathrm{O}(2, 2)$ such that

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is of Type II-1 or Type II-2. We can replace $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} A & w \\ 0 & 1 \end{smallmatrix}\right)$ and show that the translation part doesn't contribute to the classification.

We omit the rest of the proof. □

(2.12) To handle the general case, namely when the simply transitive group of affine motion is non-unipotent solvable, we need the following lemma from Auslander, cf. [**1**].

(2.12.1) LEMMA. *Let H be an n-dimensional, connected, simply connected, solvable Lie group acting simply transitively as affine motions on R^n. Let $A(H)$ be the algebraic hull of H and let U be the unipotent radical of $A(H)$. Then U operates simply transitively as affine motions on R^n.*

Now all such nontranslation U's are known from (2.11), and we'll study them first.

(2.12.2) LEMMA. *Let H, U be as in* (2.12.1) *and assume that U is not the translation group T. Then $H = U$.*

Proof. W.r.t. the standard basis $\{e_i\}$, $1 \le i \le 4$, we know

$$L(P_l(U)) = \left\{ \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

Notice that $A(H)$ is contained in the normalizer of U, we have

$$[L(A(H)), L(U)] \subseteq L(U), \qquad [L(P_l(A(H))), L(P_l(U))] \subseteq L(P_l(U)).$$

Since for

$$Y = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix},$$

we have

$$[Y, X] = \begin{pmatrix} -a(v)a_{21} + b(v)c & a(v)(a_{11} - a_{22}) & 0 & b(v)(a_{11} + a_{22}) \\ 0 & a(v)a_{21} + b(v)c & -b(v)(a_{11} + a_{22}) & 0 \\ 0 & 0 & a(v)a_{21} - b(v)c & 0 \\ 0 & 0 & -a(v)(a_{11} - a_{22}) & -a(v)a_{21} - b(v)c \end{pmatrix}.$$

So

$$\begin{cases} -a(v)a_{21} + b(v)c = 0, \\ a(v)a_{21} + b(v)c = 0, \end{cases}$$

i.e.

$$\begin{cases} a(v)a_{21} = 0 \\ b(v)c = 0 \end{cases}$$

for any a_{21}, c, $a(v)$ and $b(v)$, $v \in R^4$.

By (2.11), we can always find a v so that $a(v) \neq 0$, so we must have $a_{21} = 0$. Similarly $c = 0$, unless $b(v) \equiv 0$. So we have two cases.

Case 1. Type of $L(U)$ is I-1, I-2 or II.
$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ 0 & a_{22} & -d & 0 \\ 0 & 0 & -a_{11} & 0 \\ 0 & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, d \in R \right\}.$$

Case 2. Type of $L(U)$ is I-3.
$L(P_l(A(H)))$ is contained in

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & \\ 0 & a_{22} & 0 & 0 \\ 0 & c & -a_{11} & 0 \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix} ; a_{11}, a_{12}, a_{22}, c \in R \right\}.$$

It's easy to show that matrices in Case 1 and Case 2 are conjugate under O(2, 2). We will only write down a proof for Case 1; a proof for Case 2 can be obtained similarly.

Again let $Y \in L(P_l(A(H)))$. Then by (1.7.3) $\det Y = 0$, so $a_{11}a_{22} = 0$, i.e. $a_{11} = 0$ or $a_{22} = 0$.

If $a_{22} = 0$, then an element in $L(A(H))$ has the form

$$\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 & d(v) & v_1 \\ 0 & 0 & -d & 0 & v_2 \\ 0 & 0 & -a_{11} & 0 & v_3 \\ 0 & 0 & -a_{12} & 0 & v_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{for some } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} - a(v) & 0 & d - b(v) & 0 \\ 0 & 0 & -d + b(v) & 0 & 0 \\ 0 & 0 & -a_{11} & 0 & 0 \\ 0 & 0 & -a_{12} + a(v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

For any $\begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 070 & \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v_1' + (a_{12} - a(v))v_2' + (d - b(v))v_4' \\ 0 & 0 & -a_{11}b(v') & 0 & -(d - b(v))v_3' \\ 0 & 0 & 0 & 0 & -a_{11}v_3' \\ 0 & 0 & -a_{11}a(v') & 0 & -(a_{12} - a(v))v_3' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

But we know that

$$\begin{pmatrix} 0 & a_{11}a(v') & 0 & a_{11}b(v') & a_{11}v_1' \\ 0 & 0 & -a_{11}b(v') & 0 & a_{11}v_2' \\ 0 & 0 & 0 & 0 & a_{11}v_3' \\ 0 & 0 & -a_{11}a(v') & 0 & a_{11}v_4' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

So we have

(1) $a_{11}v_1' = a_{11}v_1' + (a_{12} - a(v))v_2' + (d - b(v))v_4'$;
(2) $a_{11}v_2' = -(d - b(v))v_3'$;
(3) $a_{11}v_3' = -a_{11}v_3'$;
(4) $a_{11}v_4' = -(a_{12} - a(v))v_3'$.

From (3) we get $a_{11} = 0$. Then (2), resp. (4), implies $d = b(v)$, resp. $a_{12} = a(v)$, i.e. $Y = X(v)$. So $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(U)$.

If $a_{11} = 0$, let $\begin{pmatrix} Y & v \\ 0 & 0 \end{pmatrix} \in L(A(H))$. By subtracting an element $\begin{pmatrix} X(v) & v \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a - 12 - a(v) & 0 & d - b(v) & 0 \\ 0 & a_{22} & -(d - b(v)) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(a_{12} - a(v)) & -a_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(A(H)).$$

Then for any $\begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix} \in L(U)$, we have

$$\left[\begin{pmatrix} Y - X(v) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} X(v') & v' \\ 0 & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & -a_{22}a(v') & 0 & a_{22}b(v') & (a_{12}-a(v))v'_2 + (d-b(v))v'_4 \\ 0 & 0 & -a_{22}b(v') & 0 & a_{22}v'_2 - (d-b(v))v'_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{22}a(v') & 0 & -(a_{12}-a(v))v'_3 - a_{22}v'_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in L(U).$$

Let $a(v) = \sum a_i v_i$, $b(v) = \sum b_i v_i$, $2 \le i \le 4$ as in (2.6) ($a_1 = b_1 = 0$) and let

$$v'' = \begin{pmatrix} (a_{12}-a(v))v'_2 + (d-b(v))v'_4 \\ a_{22}v'_2 - (d-b(v))v'_3 \\ 0 \\ -(a_{12}-a(v))v'_3 - a_{22}v'_4 \end{pmatrix}.$$

Then

$$\begin{cases} -a_{22}a(v') = a(v'') = a_2(a_{22}v'_2 - (d-b(v))v'_3) \\ \qquad\qquad + a_4(-(a_{12}-a(v))v'_3 - a_{22}v'_4), \\ a_{22}b(v') = b(v'') = b_2(a_{22}v'_2 - (d-b(v))v'_3) \\ \qquad\qquad + b_4(-(a_{12}-a(v))v'_3 - a_{22}v'_4), \end{cases}$$

i.e.

$$\begin{cases} -a_{22}(a_2v'_2 + a_3v'_3 + a_4 + v'_4) = a_2(a_{22}v'_2 - (d-b(v))v'_3) \\ \qquad\qquad + a_4(-(a_{12}-a(v))v'_3 - a_{22}v'_4), \\ a_{22}(b_2v'_2 + b_3v'_3 + b_4v'_4) = b_2(a_{22}v'_2 - (d-b(v))v'_3) \\ \qquad\qquad + b_4(-(a_{12}-a(v))v'_3 - a_{22}v'_4), \end{cases}$$

i.e.

$$\begin{cases} 2a_{22}a_2v'_2 + (a_3a_{22} - a_4(a_{12}-a(v)) - a_2(d-b(v)))v'_3 = 0, \\ 2a_{22}b_4v'_4 + (b_3a_{22} + b_4(a_{12}-a(v)) + b_2(d-b(v)))v'_3 = 0. \end{cases}$$

Letting v'_i's vary, we have

(1) $a_{22}a_2 = 0$;
(2) $a_{22}b_4 = 0$;
(3) $a_3a_{22} - a_4(a_{12}-a(v)) - a_2(d-b(v)) = 0$;
(4) $b_3a_{22} + b_4(a_{12}-a(v)) + b_2(d-b(v)) = 0$.

If $a_{22} \neq 0$, we must have $a_2 = b_4 = 0$ by (1) and (2). According to (2.10), this implies $b_2 = a_4 = b_4 = a_2 = 0$. Then (3) and (4) lead to

$$\begin{cases} a_3 a_{22} = 0, \\ b_3 a_{22} = 0, \end{cases}$$

i.e. $a_3 = b_3 = 0$. So $U = T$, and we have a contradiction. So $a_{22} = 0$. We always have $a_{11} = a_{22} = 0$, i.e. $A(H)$ is unipotent; so H is unipotent. But any unipotent connected Lie group is Zariski closed, so $H = A(H)$. U, as the unipotent radical of H must be H itself. □

(2.12.3) Now consider the case when the unipotent radical $A(H)$ is precisely the group T of translations of R^4. Suppose $H \neq T$, i.e. H is not unipotent.

(2.12.3.1) LEMMA. *$P_l(H)$ is abelian.*

Proof. $P_l(H) \simeq H/\operatorname{Ker}(P_l|_H)) = H/(H \cap T) \subseteq A(H)/T$, but $A(H)/T$ is abelian (cf. [2], $A(H)/U(H)$ is abelian, since $A(H)$ is solvable and algebraic). □

(2.12.3.2) LEMMA. $\dim P_l(H) = 1$; *$P_l(H)$ is diagnolizable in* $\mathbf{C}$.

Proof. $P_l(H)$ is a connected abelian subgroup of $SO_0(2, 2)$, so $\dim P_l(H) \leq 2$. By (1.7.3) $\det X = 0$ for every $X \in L(P_l(H))$, i.e. 0 is an eigenvalue of X. Since $X \in so(2, 2)$, so

$$X = \begin{pmatrix} a_{11} & a_{12} & 0 & d \\ a_{21} & a_{22} & -d & 0 \\ 0 & c & -a_{11} & -a_{21} \\ -c & 0 & -a_{12} & -a_{22} \end{pmatrix},$$

and

$$\begin{aligned} \det(X - \lambda I) &= \lambda^4 + (2dc - 2a_{12}a_{21} - a_{11}^2 - a_{22}^2)\lambda^2 \\ &\quad + (-a_{11}a_{22} + a_{12}a_{21} + dc)^2 \\ &= \lambda^4 + \{-4a_{12}a_{21} - (a_{11} - a_{12})^2\}\lambda^2, \end{aligned}$$

since 0 is an eigenvalue. So the eigenvalues of X are $\{0, 0, 0, 0\}$ or $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. If $\dim P_l(H) = 2$, then by (1.6.2) $so(2, 2) = L_1 \oplus L_2$, $L_i \simeq sl_2(R)$. So $L(P_l(H)) = RX_1 + RX_2$ where $X_i \in L_i$, $i = 1, 2$. But by (1.6.2)

$\det(X_1 - \lambda I) = \lambda^4 + 2(a^2 + bc)\lambda^2 + (a^2 + bc)^2$, and

$$\det(X_2 - \lambda I) = \lambda^4 + 2(b'c' - a'^2)\lambda^2 + (b'c' - a'^2)^2.$$

So zero is an eigenvalue of X_i, $i = 1, 2$, if and only if all the eigenvalues of X_i are zero. This means $P_l(H)$ is unipotent and leads to a contradiction. So $\dim P_l(H) = 1$, $L(P_l(H)) = RX$ and X has eigenvalues $\{0, 0, \lambda, -\lambda\}$, $\lambda \neq 0$, $\lambda \in R$ or $\sqrt{-1}R$. Since X is an infinitesimal isometry, it is diagnolizable. □

(2.12.3.3) Corollary. *$L(P_l(H))$ is contained in a Cartan subalgebra of* so(2, 2) *and is conjugate under* O(2, 2) *to*

(1)
$$\begin{pmatrix} a & & & 0 \\ & 0 & & \\ & & -a & \\ 0 & & & 0 \end{pmatrix}$$

(2)
$$\begin{pmatrix} 0 & a & 0 & a \\ -a & 0 & -a & 0 \\ 0 & a & 0 & a \\ -a & 0 & -a & 0 \end{pmatrix}.$$

Proof. By (1.6.4). □

Since H is simply transitive, the map $P_t \colon L(H) \to R^4$ is a linear isomorphism, so in (2.12.3.3) we have $a = \sum_{i=1}^4 a_i v_i$, where

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

is the corresponding translation part. Since T is the unipotent radical of $A(H)$, we have $[L(H), L(H)] \subseteq L(T) = R^4$. By computing the commutator and using the fact that H is simply transitive, we must have $a(v) = a_2v_2 + a_4v_4$, $(a_2, a_4) \neq (0, 0)$ in Case (1) and $a(v) = a_1(v_1 - v_3) + a_2(v_2 - v_4)$, $(a_1, a_2) \neq (0, 0)$ in Case (2). Finally, by considering the conjugation under $R^4 \rtimes O(2, 2)$, we get

(2.12.4) Theorem. *If $H \subseteq R^4 \rtimes SO(2, 2)$ acts simply transitively on R^4 and H is not unipotent, then H is conjugate under $R^4 \rtimes O(2, 2)$*

to one of the following:

(i) *Type* III-1:

$$\left(\begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \end{pmatrix}\right),$$

where $a(v) = tv_2 + v_4$, $t > 0$ *and* $L(H) \simeq R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$.

(ii) *Type* III-2:

$$\left(\begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} \begin{pmatrix} \\ v \\ \\ \end{pmatrix}\right),$$

where $a(v) = t(v_1 - v_3)$, $t > 0$ *and* $L(H) = R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$. *The type and the parameter* t *determine the equivalence classes uniquely.*

(2.13) Combining (2.11) with (2.12.4) and denoting $H = T_4$ as Type 0, we complete the classification of simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$. We summarize our result in the following table. We denote

$$A(a, b, v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & b(v) \\ 0 & 0 & -b(v) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\},$$

$$B(a, v) = \left\{ \begin{pmatrix} \begin{pmatrix} a(v) & & & 0 \\ & 0 & & \\ & & -a(v) & \\ 0 & & & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} , v \in R^4 \right\},$$

$$C(a, v) = \left\{ \begin{pmatrix} \begin{pmatrix} 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \\ 0 & a(v) & 0 & a(v) \\ -a(v) & 0 & -a(v) & 0 \end{pmatrix} & v \\ 0 & 0 \end{pmatrix} ; v \in R^4 \right\}.$$

Table of equivalence classes of simply transitive subgroups of $R^4 \rtimes \mathrm{SO}(2, 2)$ (given in the form of subalgebras of $\mathrm{aff}(n)$ w.r.t. a standard basis).

type of $L(H)$	affine form of $L(H)$	isomorphism type as abstract Lie algebra
0	$\{\left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right) ; v \in R^4\}$;	R^4
I-1	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = v_3 \end{cases}$	$R \oplus N_3$
I-2	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = -v_3 \end{cases}$	$R \oplus N_3$
I-3	$A(a, b, v), \begin{cases} a(v) = v_3, \\ b(v) = 0 \end{cases}$	$R \oplus N_3$
II-1	$A(a, b, v), \begin{cases} a(v) = v_2 + v_4 + tv_3, \\ b(v) = -v_2 - v_4, \ t \geq 0, \end{cases}$	N_4
II-2	$A(a, b, v), \begin{cases} a(v) = -v_2 + v_4 + tv_3, \\ b(v) = -v_2 + v_4, \ t \geq 0 \end{cases}$	N_4
II-3	$A(a, b, v), \begin{cases} a(v) = v_2, \\ b(v) = v_3 \end{cases}$	N_4
III-1	$B(a, v), a(v) = tv_2 + v_4, t \in R$	$R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\}$
III-2	$C(a, v), a(v) = t(v_1 - v_3), t > 0$	$R \oplus \{R^2 \rtimes R\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\}$

The type of $L(H)$ and the parameter t determine the equivalence classes uniquely.

3. Γ is virtually solvable. A group with a solvable subgroup of finite index is called virtually solvable.

(3.1) THEOREM. *If $\Gamma \subset R^4 \rtimes \mathrm{SO}(2, 2)$ and Γ acts freely and properly discontinuously on R^4 with compact quotient, then Γ is virtually solvable.*

Proof. Let $\pi = P_l(\Gamma)$ and $A(\pi)$ be the algebraic hull of Γ. The identity component A_0 is of finite index in $A(\pi)$. We will show A_0 is solvable. The following lemma is due to D. Fried.

(3.2) LEMMA. *If A_0 fixes a vector $v \in R^4$ s.t. $Q(v, v) \neq 0$, then A_0 is solvable.*

For a proof, cf. [**3**].

Assume that A_0 is not solvable. As in (1.7.2), for every $g \in A(\pi)$, $\det(g - I) = 0$. This shows $\det = 0$ on $L(A_0)$ and $\dim A_0 < \dim \mathrm{SO}(2, 2)$. So A_0 contains a semisimpleconnected subgroup S

such that $\dim S = 3$ and $L(S) \simeq \mathrm{sl}_2(R)$. By (1.6.2) $\det \neq 0$ on l_i so $L(S) \neq L_i$, $i = 1, 2$. So $L(S)$ must be a maximal subalgebra of $\mathrm{so}(2, 2)$, so $A_0 = S$. Let $P_i \colon L(S) \to L_i$, $i = 1, 2$ be the projection map, then $P_i(L(S)) = L_i$, $i = 1, 2$.

(3.3) *Claim.* There is a nonzero vector $v \in R^4$ such that

(i) $Q(v, v) \neq 0$;
(ii) $A_0(v) = v$.

To prove the claim, let $0 \neq X \in L(A_0)$ such that RX is a split Cartan subalgebra of $L(A_0)$. Then $h = P_1(RX) \oplus P_2(RX)$ is a split Cartan subalgebra of $\mathrm{so}(2, 2)$. By (1.6.3) h is conjugate under $\mathrm{O}(2, 2)$ to $\{\mathrm{diag}\cdot(a, b, -a, -b); a, b \in R\}$. Since $\det X = 0$ we can rescale and permute coordinates so $X = \mathrm{diag}\cdot(1, 0, -1, 0)$. Let $\{X, Y, Z\}$ be the basis of $L(A)$ such that $[X, Y] = 2Y$, $[X, Z] = -2Z$, $[Y, Z] = X$ and $X = \mathrm{diag}\cdot(1, 0, -1, 0)$. Then $\mathrm{ad}\, X$ has three real eigenvalues on $\mathrm{so}(2, 2)$: $\{2, 0, -2\}$. Let E_λ be the corresponding eigenspaces, then

$$E_2 = \left\{ \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} ; c, e \in R \right\},$$

$$E_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix} ; d, f, \in R \right\}, \quad \text{and}$$

$$[E_2, E_{-2}] = \left\{ \begin{pmatrix} cd - ef & & & 0 \\ & -cd - ef & & \\ & & -cd + ef & \\ 0 & & & cd + ef \end{pmatrix} ; \right.$$
$$\left. c, d, e, f \in R \right\}.$$

So there are $c, e, d, f \in R$ such that

$$Y = \begin{pmatrix} 0 & c & 0 & e \\ 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & f & 0 & -d \\ -f & 0 & 0 & 0 \end{pmatrix}$$

and $[X, Z] = X$ implies

$$\begin{cases} cd - ef = 1, \\ cd + ef = 0, \end{cases}$$

i.e. $cd = -ef = \frac{1}{2}$, $cdef \neq 0$. Let $v = \frac{1}{c}e_2 - \frac{1}{e}e_4$. It's easy to check that $Q(v, v) = \frac{1}{ce} \neq 0$, $A_0(v) = v$.

Combining (3.3) with Lemma (3.2), we have a contradiction, so A_0 must be solvable. □

4. Proof of Theorem (1.2). The principal tool is the following theorem from [**4**].

(4.1) THEOREM (*Fried and Goldman*). *Let* $\Gamma \subseteq \mathrm{Aff}(n)$ *be virtually polycyclic and suppose that* Γ *acts properly discontinuously on* R^n. *Then there exists at least one subgroup* $H \subseteq \mathrm{Aff}(n)$ *containing* Γ *such that*:

(a) H *has finitely many components and each component meets* Γ;

(b) H/Γ *is compact*;

(c) H *and* Γ *have the same algebraic hull in* $\mathrm{Aff}(n)$;

(d) *if* Γ *has a subgroup* Γ_1 *of finite index such that every element of* $P_l(\Gamma_1)$ *has all real eigenvalues, then* H *is uniquely determined by the above conditions*;

(e) *the identity component* H_0 *of* H *acts simply transitively on* R^n *and* $H_0 \cap \Gamma$ *is a discrete cocompact subgroup of* H_0 *and is of finite index in* Γ.

Such a subgroup H in (4.1) is called a crystallographic hull for Γ. Since a discrete solvable subgroup of Lie with finitely many components is polycyclic and we proved in §3 that Γ in (1.2) is virtually solvable, by (4.1) we need only to check for the uniqueness of H. By (4.1)-(d), we need only to show that $P_l(\Gamma)$ has a subgroup of finite index with real eigenvalues only. Since H_0 must occur in our table of simply transitive motions and all these simply transitive motions, except Type III-2, have linear parts with only real eigenvalues, we need only to check Type III-2. By Bieberbach's theorem (cf. [**8**]), any discrete subgroup of Type III-2 meets T in a subgroup of finite index. □

5. Classification of Γ.

(5.1) LEMMA. *Let Γ be a uniform lattice in a simply transitive group $H \subseteq R^4 \rtimes \mathrm{SO}(2,2)$. Then H is the identity component of the crystallographic hull of Γ if and only if H is not of Type* III-2.

Proof. If H is of Type III-2, then Γ has a subgroup of finite index, say Γ_1, such that $\Gamma_1 \subset T$. So Γ is virtually abelian. By **[4]**, the crystallographic hull of a virtually abelian affine polycyclic group is itself virtually abelian, so H doesn't arise from any Γ.

In the unipotent cases, the algebraic hull of H is H itself. So $A(\Gamma)$, the algebraic hull of Γ, is contained in H. Since H_0', the identity component of the crystallographic hull H' of Γ, acts simply transitively on R^4, the dimension of H_0' must be four, and then by (4.1)-(C) we have

$$H_0' \subseteq H' \subseteq A(H') = A(\Gamma) \subseteq H.$$

So $H = H_0'$; then $H' = H$.

The only remaining case is Type III-1. Since Γ is not unipotent, H_0', the identity component of the crystallographic hull H' of Γ, must be nonunipotent solvable, i.e. H_0' is of Type III-1 and $\Gamma \subseteq H \cap H_0'$. Then it's easy to show that $H_0' = H$. □

(5.2) COROLLARY. *Up to finite covers, every flat compact complete space-form with metric of signature* (2,2) *is of the form H/Γ, where H is a simply transitive subgroup of $R^4 \times \mathrm{SO}(2,2)$ of Type* 0, *Type* I, *Type* II *or Type* III-1 *and Γ is a uniform lattice of H.*

(5.3) *Uniform lattices.* The uniform lattices depend only on the structure of H as a Lie group and do not depend on its embedding in $R^4 \rtimes \mathrm{SO}(2,2)$. Since Type 0 $\simeq R^4$, Type I $\simeq R \times \mathrm{Nil}^3$, Type II $\simeq \mathrm{Nil}^4$ and Type III-1 $\simeq R \times \{R^2 \rtimes \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right); t \in R\}$, as Lie groups, they are exactly the same group as that listed in **[8]**, and D. Fried gave a list of their uniform lattices there. C. T. C. Wall also studied them, cf. **[7]**. Here we only write them down to complete our classification.

(5.3.1) The uniform lattices of H are semidirect products $Z^3 \rtimes Z_A$, where $A \in \mathrm{SL}(Z)$ has a characteristic polynomial

$$\det(t - A) = (t-1)(t^2 - bt + 1),$$

where $b \geq 2$ is an integer, and A and b satisfy:

(i) Type 0: $A = I$, $b = 2$;
(ii) Type I: $(A - I)^2 = 0$, $A \neq I$, $b = 2$;
(iii) Type II: $(A - I)^2 \neq 0$, $(A - I)^3 = 0$, $b = 2$;
(iv) Type III-1: $b \geq 3$.

(Cf. **[3]** and **[7]** for a proof.)

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REFERENCES

[1] L. Auslander, *Simply transitive groups of affine motions*, Amer. J. Math., **99** (1977), 809–821.
[2] A. Borel, *Linear Algebraic Groups*, Benjamin, New York, 1969.
[3] D. Fried, *Flat spacetimes*, J. Differential Geometry, **26** (1987), 385–396.
[4] D. Fried and W. Goldman, *Three-dimensional affine crystallographic groups*, Adv. in Math., **47** (1983), 1–49.
[5] B. Kostant and D. Sullivan, *The Euler characteristic of an affine space form is zero*, Bull. Amer. Math. Soc., **81** (1975), 937–938.
[6] J. Milnor, *On fundamental groups of complete affinely flat manifolds*, Adv. in Math., **525** (1977), 178–187.
[7] C. T. C. Wall, *Geometric structures on compact complete analytic surfaces*, Topology, **25** No. 2, (1986), 119–153.
[8] H. Zassenhaus, *Beweis eines Satzes uber diskrete Grupen*, Abh. Math. Sem. Hamb. Univ., **12** (1938), 289–312.

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