SETS OF UNIQUENESS AND SYSTEMS
OF INEQUALITIES HAVING A UNIQUE SOLUTION

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Suppose a number of X-ray pictures is taken of the same object, but from different directions. One typically likes to know to what degree the pictures determine the object and exactly when an object is uniquely determined. Replacing picture taking by projections, that is, images relative to specified mappings, these same problems are easily formulated for higher dimensions and even for abstract spaces. The objects on hand might be data structures.

With this general framework, starting from an arbitrary but fixed collection of mappings, we study a new and very useful class of objects (sets) each of which is uniquely determined by its projections. In the process, we disprove a previously conjectured characterization of uniqueness relative to the one-dimensional projections in R^n. For all situations where the underlying space is finite, a complete and rather simple characterization of uniqueness is obtained.

1. Introduction. Suppose an X-ray picture is made of an object S in R^3 of uniform unit density. This corresponds to the creation of an image of S on a photographic plate Y by means of a central or parallel projection $\pi: R^3 \to Y$. The darkness of the image at $y \in Y$ is directly related to the length $|L \cap S|$ of the part of L inside S, with L as the straight line $L = \pi^{-1}\{y\} = \{x \in R^3 : \pi x = y\}$. Hence, having such a π-photograph of S is equivalent to knowing the precise value $\lambda(S \cap A)$, for each set of the form $A = \pi^{-1}B = \{x \in R^3 : \pi x \in B\}$. Here, B can be any subset of Y while λ is Lebesgue measure on R^3.

Given any finite class of such photographic maps $\pi_j: R^3 \to Y_j$, we would like to know exactly what subsets S of R^3 are such that S is uniquely determined by its set of images.

The subset S of R^3 can be identified with the measure μ_S on R^3 defined by $\mu_S(A) = \lambda(S \cap A)$. Thus μ_S has its density relative to Lebesgue measure λ equal to the function $1_S(x)$ on X (1 on S and 0 on its complement S^c). If $\pi: R^3 \to Y$ is *any* map then the π-projection of μ_S onto Y (also nonchalantly called the π-projection of S) is the mass distribution (measure) $\pi\mu_S$ on Y, whose mass inside any subset of B of Y equals $\mu_S(A) = \lambda(S \cap A)$ with $A = \pi^{-1}B = \{x \in R^3 : \pi x \in B\}$. Knowing the π-photograph of S is the same as knowing the projection $\pi\mu_S$ of μ_S.

More generally, let X be any space supplied with a reference measure λ. Let further $\{\pi_j, j \in J\}$ be a fixed finite or infinite collection of maps $\pi_j\colon X \to Y_j$. A subset S of X will be identified with the associated measure $d\mu_S = 1_S\, d\lambda$ on X, (carried by S and having a uniform density there). In particular, subsets S and T equal a.e. $[\lambda]$ are identified. The π_j-projection of S is defined as the measure $\pi_j\mu_S$ on Y_j whose mass $(\pi_j\mu_S)(B)$ inside any subset B of Y_j equals $\mu_S(A) = \lambda(S \cap A)$ with $A = (\pi_j)^{-1}B = \{x \in X\colon \pi_j x \in B\}$.

In the literature, much attention is given to the following situation, which we will refer to as the *classical case*. Here, $X = R^n$ supplied with n-dimensional Lebesgue measure $\lambda = m_n$ and projections $\pi_1, \ldots, \pi_n$, with π_j as the orthogonal projection of R^n onto the jth coordinate axis. Relative to this classical case, knowing the π_j-projections of a subset S of R^n $(j = 1, \ldots, n)$ is the same as knowing the $(n-1)$-dimensional volume of the cross section of S with each hyperplane $x_j = c$ perpendicular to one of the coordinate axes.

Going back to the general case, we will be interested in the class of subsets S of X that are uniquely determined by the associated collection $\{\pi_j\mu_S\colon j \in J\}$ of projections of $d\mu_S = 1_S\, d\lambda$. It is always assumed that $\lambda(S) < \infty$. We say that S is a *weak set of uniqueness* (relative to X, λ and the π_j) if there is no other set T such that $\pi_j\mu_T = \pi_j\mu_S$, for all $j \in J$. Here, we identify sets which are equal a.e. $[\lambda]$. And we call S a (strong) *set of uniqueness* if besides μ_S there does not exist any measure μ of the form $d\mu = \phi d\lambda$, with $\phi\colon X \to [0, 1]$, such that $\pi_j\mu = \pi_j\mu_S$, for all $j \in J$.

These notions are distinct if X is discrete, see [2]. On the other hand, the two notions of uniqueness coincide in the important case that $X = R^n$ with λ of the form $\lambda(dx) = q(x)\, dx$ while $\{\pi_j\colon j \in J\}$ is an arbitrary but finite set of linear or central projections $\pi_j\colon R^n \to Y_j$, see [1], [4], [5]; $(\dim Y_j \leq n-1)$. This includes the above classical case.

An important class of sets of uniqueness are the so-called *additive sets*, first introduced for the classical case by Fishburn et al. [1], [2].

Let $F(S)$ denote the class of all $f\colon X \to R$ of the special form

$$(1.1) \qquad\qquad f(x) = \sum_{j \in J} f_j(\pi_j x),$$

with the f_j as functions $f_j\colon Y_j \to R \; (j \in J)$ satisfying the integrability

condition (2.10). The importance of this class is due to the fact that

$$(1.2) \qquad \int_X f \, d\mu = \int_X f \, d\mu_s = \int_S f \, d\lambda, \quad \text{for any } f \in F(S),$$

and any measure μ on X having the same marginals as μ_s (relative to the π_j; $j \in J$). A subset S of X with $\lambda(S) < \infty$, is said to be *additive* if it admits a representation of the form

$$(1.3) \qquad S = \left\{ x \in X : f(x) = \sum_{j \in J} f_j(\pi_j x) \geq 0 \right\},$$

with f as a suitable function $f \in F(S)$ as in (1.1). One easily shows that each additive set S in X is a (strong) set of uniqueness, see Lemma 2.14.

An easy illustration, relative to the classical case, would be any product set $S = A_1 \times \cdots \times A_n$ in R^n with $m_1(A_j) < \infty$. For, then S is of the form (1.3) with $f_j(\xi) = +1$ or $-n$, depending on whether $\xi \in A_j$ or $\xi \notin A_j$, respectively, $(\xi \in R)$. Or choose S as an ellipsoid in R^n having its axes parallel to the coordinate axes. On the other hand, if S is an ellipsoid $S = \{x \in R^n : \sum_{i,j} a_{ij} x_i x_j \leq 1\}$ with $a_{1k} \neq 0$ for some $k \geq 2$ then S is not a set of uniqueness, in fact, it has the same projections as the ellipsoid T obtained by replacing all a_{1j} by $-a_{1j}$ $(j \geq 2)$.

Naturally, one wonders whether conversely all sets of uniqueness might be additive. This was shown to be true for the classical case with $n = 2$ independently by Fishburn, Lagarias, Reeds and Shepp [1] and by Kuba and Volcic [7], [9], each making an essential use of the results due to Lorentz [8]. And for a while it was conjectured, for the classical case, that the converse is always true.

One goal of the present paper is to show that this converse is false. In §3, for the classical case with $n \geq 3$, we construct a large class of sets S of uniqueness which are definitely not additive. One example with $n = 3$ (see 3.22) has S as a subset of R^3 equal to the union of a tetrahedron, a prism, and two rectangular blocks parallel to the axes.

Our construction is based on a new generalized notion of additivity. Ordinary additive sets are said to be additive of degree 1. A (stepwise) additive set S of degree ≤ 2 has the form $S = \{x \in X :$ either $f(x) > 0$ or $f(x) = 0$; $g(x) \geq 0\}$ with f, g as suitably chosen functions in the class $F(S)$. In an analogous way, one defines the notion of an additive set S of degree $\leq m$, see Definition 3.1. It is shown that each such generalized additive set S is a (strong) set of

uniqueness, see Theorem 3.1. Whenever the latter set S can be shown to be non-additive, one has an example of a set of uniqueness which is not an additive set. In §3, this program is worked out in detail for the classical case with $n \geq 3$, starting with a suitably chosen additive set of degree ≤ 2. There are still many open problems here.

Let us now restrict ourselves to the case where the set X is *finite*, though we allow for the possibility of infinitely many projections $\pi_j: X \to Y_j$ $(j \in J)$. In this situation, generalized additivity is the same as ordinary additivity, see Lemma 3.8. Moreover, a subset S of X is a (strong) set of uniqueness if and only if it is additive, see Theorem 4.50. For the special case that $X = [1, 2, \ldots, N]^n$ with coordinate projections $x \to x_j$ $(j = 1, \ldots, n)$, the latter result is due to Fishburn, Lagarias, Reeds and Shepp [2]. The analogous result for weak sets of uniqueness is false, see [2] for a counterexample.

The proof of Theorem 4.50 is largely based on a new set of necessary and sufficient conditions in order that a solution x^0 of an arbitrary finite system of linear inequalities of type $\sum_j a_{ij} x_j \geq b_i$ $(i = 1, \ldots, m)$; $\alpha_j \leq x_j \leq \beta_j$ $(j = 1, \ldots, n)$ be the only such solution. One necessary condition is that x^0 be additive in a certain sense. See Theorem 4.15 for further details.

2. Sets of uniqueness. Let X be a measurable space and J an arbitrary index set. For each $j \in J$, let Y_j be a measurable space and $\pi_j: X \to Y_j$ a measurable map. The collection $\pi = \{\pi_j; j \in J\}$ of projections is kept fixed from now on. If μ is any measure on X then its so-called π_j-marginal (or π_j-projection) is the measure $\pi_j \mu$ on Y_j defined by $(\pi_j \mu)(B) = \mu(\pi_j^{-1} B)$. It follows that, for each measurable function $g_j: Y_j \to R$,

$$\int_X g_j(\pi_j x) \mu(dx) = \int_{Y_j} g_j(y)(\pi_j \mu)(dy),$$

provided g_j is either nonnegative or $(\pi_j \mu)$-integrable. In particular, this integral is fully determined by the marginal $\pi_j \mu$ of μ. Thus, if another measure ν on X has the same marginals as μ, that is, $\pi_j \nu = \pi_j \mu$ $(j \in J)$, then

$$(2.1) \quad \int_X g(x) \nu(dx) = \int_X g(x) \mu(dx)$$

$$\text{if } g: X \to R \text{ is of the form } g(x) = \sum_{j \in J} g_j(\pi_j x).$$

Here, the $g_j: Y_j \to R$ are assumed to be measurable such that $\sum_{j \in J} \int_s |g_j(\pi_j x)| \mu(dx) < \infty$.

2.2. In the sequel, λ is a fixed finite or σ-finite measure on X. We shall *identify* functions on X (or subsets of X) which are equal a.e. $[\lambda]$. Define M_0 as the collection of measurable functions

$$M_0 = \left\{ \phi \colon X \to R \, ; \, 0 \le \phi(x) \le 1 \,\, (x \in X) \, ; \, \int_X \phi \, d\lambda < \infty \right\} ;$$

(the latter condition is automatic when λ is finite). Further M will denote the associated class of (finite) measures $d\mu_\phi = \phi \, d\lambda$ on X having density $\phi \in M_0$. If $\phi = 1_S$ is the indicator function of a subset S of X then μ_ϕ is also denoted as μ_S; thus, $\mu_S(dx) = 1_S(x)\lambda(dx)$. In the sequel, S always denotes a measurable subset of X satisfying $\lambda(S) < \infty$. Hence, $1_S \in M_0$, equivalently, $\mu_S \in M$.

For each $\mu \in M$, let

(2.3) $\qquad M(\mu) = \{\nu \in M \colon \pi_j \mu = \pi_j \nu \text{ for all } j \in J\},$

be the collection of all $\nu \in M$ having exactly the same marginals as μ. Similarly for $\phi \in M_0$, let $M_0(\phi)$ be the set of all $\psi \in M_0$ such that μ_ψ has the same marginals as μ_ϕ; that is, $\mu_\psi \in M(\mu_\phi)$. In particular, $\mu \in M(\mu)$ and $\phi \in M_0(\phi)$.

2.4. DEFINITION. We will call a function $\phi \colon X \to R$ a *uniqueness function* if $\phi \in M_0$ and, moreover, $M_0(\phi) = \{\phi\}$. Equivalently, $M(\mu_\phi) = \{\mu_\phi\}$.

We will say that a set S is a *weak set of uniqueness* if there exists no measurable subset T of X distinct from S, such that $d\mu_T = 1_T \, d\lambda$ has the same marginals as $d\mu_S = 1_S \, d\lambda$. Equivalently, $M_0(1_S)$ contains no *indicator* function 1_T different from 1_S. But we do not rule out the possibility that $\psi \in M_0(1_S)$ for some *non-indicator* function $0 \le \psi \le 1$ distinct from 1_S. If also that possibility is ruled out then we will say that S is a *set of uniqueness*. In other words, a subset S of X is a (strong) *set of uniqueness* if its indicator function 1_S is a uniqueness function, equivalently, if $M_0(1_S) = \{1_S\}$.

All these notions are relative to the given choice of the measure λ on X and maps $\pi_j \colon X \to Y_j$ $(j \in J)$ and the standing assumption that $\lambda(S) < \infty$.

2.5. Since we identify sets S, T or functions ϕ, ψ differing only on a λ-null set, one could identify a measurable set S or a function $\phi \in M_0$ with the corresponding measure $d\mu_S = 1_S \, d\lambda$ or $\phi \, d\lambda$, respectively. For instance, one could speak of the marginals of S or

ϕ. These marginals of S are analogous to the X-ray pictures from different directions of an object in R^3.

The set S is a set of uniqueness set precisely when there is no $\psi \in M_0$ different from 1_s, such that $d\mu_\psi = \psi d\lambda$ has exactly the same marginals as $d\mu_s = 1_s d\lambda$. It implies that S is also a weak set of uniqueness, meaning that there exists no other subset T of X can have the same marginals as S.

In many applications, the two notions are equivalent, in other words, any weak set S of uniqueness is also a (strong) set of uniqueness. This equivalence holds, for instance, when X, λ and π_j ($j \in J$) are as in the following Example 2.76 and moreover, λ is finite, see [1] and [4]. However, it fails in Example 2.7, see [2].

2.6. EXAMPLE. The following system $\{X, \lambda, \pi_j \ (j \in J)\}$ will be referred to as the *classical case*. Here, $m_n(dx) = dx$ denotes n-dimensional Lebesgue measure on R^n, while 'measurable' is the same as m_n-measurable. Namely, choose X as a measurable subset of R^n and λ as an (absolutely continuous) measure $\lambda(dx) = q(x)\,dx$. Often one will choose λ as the restriction of m_n to X; it would be finite when $m_n(X) < \infty$. In addition, choose $J = \{1, \dots, n\}$ and $\pi_j \colon X \to R$ as the projection which maps $x = (x_1, \dots, x_n)$ to its jth coordinate x_j ($j = 1, \cdots, n$).

We are interested in finite measures μ on X of the form $d\mu = \phi\,d\lambda = \phi(x)q(x)dx$; $0 \le \phi \le 1$, (often $\phi = 1_s$). In the present example, the $\pi_j\mu$ ($j = 1, \cdots, n$) are precisely the one-dimensional marginals of μ. Clearly, $\pi_j\mu$ is an absolutely continuous measure on R with density function

$$h_j(\xi) = \int_X \phi(x)q(x)dx_1 \cdots dx_{j-1}dx_{j+1} \cdots dx_n,$$
$$(x_j = \xi \in R; j = 1, \dots, n).$$

Moreover, ϕ is a uniqueness function precisely when no other function $0 \le \psi \le 1$ leads to precisely the same marginal densities h_j ($j = 1, \cdots, n$).

2.7. EXAMPLE. Let $X = \{1, 2, \dots, a\}^n$ and choose J and the π_j as in Example 2.6. Let $\lambda(x) = \lambda(\{x\})$. Presently, the jth marginal $\pi_j\mu$ of a measure $d\mu = \phi dx$ on X is the measure on $\{1, \dots, a\}$

with discrete density function

$$h_j(\xi) = (\pi_j \mu)(\{\xi\})$$
$$= \sum \{\phi(x_1, \ldots, x_n)\lambda(x_1, \ldots, x_n) : x \in X ; x_j = \xi\},$$
$$\{\xi = 1, \ldots, a ; j = 1, \ldots, n\}.$$

2.8. DEFINITION. Given a set S with $\lambda(S) < \infty$, let $F(S)$ denote the linear set consisting of all functions $f : X \to R$ having the special form

$$(2.9) \qquad\qquad f(x) = \sum_{j \in J} f_j(\pi_j x),$$

with the $f_j : Y_j \to R \ (j \in J)$ as measurable functions such that

$$(2.10) \qquad\qquad \sum_{j \in J} \int_S |f_j(\pi_j x)| \lambda(dx) < \infty.$$

Note from (2.1) that each $f \in F(S)$ is integrable relative to each $\mu \in M(\mu_s)$ while, moreover, the corresponding integral $\int f \, d\mu$ is fully determined by the marginals $\pi_j \mu_s \ (j \in J)$ of μ_s.

Let further F be the class of all $f : X \to R$ of the form (2.9) and satisfying

$$\sum_{j \in J} \int_X |f_j(\pi_j x)| \lambda(dx) < \infty.$$

The latter condition requires in particular that f_j is $(\pi_j \lambda)$-integrable, as happens when λ is finite and f_j is bounded. Clearly, $F(S) \supset F$ for any S. Further each $f \in F$ is integrable relative to each $\mu \in M$. Of special interest is the case that all but finitely many $f_j \ (j \in J)$ are identically zero.

2.11. DEFINITION. A subset S of X is said to be *additive* if S is measurable, with $\lambda(S) < \infty$, and, moreover, S is of the form

$$(2.12) \qquad S = S_f = \{x \in X : f(x) \geq 0\}, \qquad \text{for some } f \in F(S).$$

Note that 1_s and S attain the suprema

$$(2.13) \qquad \sup \left\{ \int f \phi \, d\lambda : 0 \leq \phi \leq 1 \right\}$$
$$= \int f_+ \, d\lambda = \sup_T \left\{ \int_T f \, d\lambda \right\} = \int_S f \, d\lambda,$$

(where $f_+ = \max(0, f)$). For instance, relative to Example (2.6), the closed unit ball S in R^n is an additive set, as is seen by choosing

$f(x) = 1 - \sum_j x_j^2 = 1 - ||x||^2$. In fact, S maximizes $\int_T (1 - ||x||^2)\, dx$; therefore, S minimizes $\int_T ||x||^2\, dx$ among subsets T of R^n volume $\int_T dx = \int_S dx$.

The following result is a straightforward generalization of a result due to Fishburn et al. [2].

2.14. LEMMA. *Each additive set S is also a (strong) set of uniqueness.*

2.15. *Proof.* Let S be of the form (2.12), with $\lambda(S) < \infty$, and suppose that $d\nu = \psi\, d\lambda$ with $0 \leq \psi \leq 1$ has the same marginals as $d\mu = 1_s\, d\lambda$ (thus, $\nu(X) = \mu(X) = \lambda(S) < \infty$ and $\psi \in M_0$). We must show that $\nu = \mu$, that is, $\psi(x) = 1_s(x)$ a.e. $[\lambda]$. It follows from (2.1) that $\int f\, d\nu = \int f\, d\mu$, that is,

$$\int_X f(x)(1_s(x) - \psi(x))\lambda(dx) = 0.$$

From $0 \leq \psi \leq 1$ and the definition of S, the latter integrand is nonnegative and thus zero a.e. $[\lambda]$. This shows that $\psi(x) = 1_s(x)$ a.e. $[\lambda]$ outside $D = \{x \in X: f(x) = 0\}$. Here, $S \supset D$ thus $\lambda(D) < \infty$. Moreover,

$$\int_D (1 - \psi(x))\lambda(dx) = \int_D (1_S(x) - \psi(x))\lambda(dx)$$

$$= \int_X (1_S(x) - \psi(x))\lambda(dx) = 0,$$

the latter since $\int d\mu = \int d\mu = \lambda(S) < \infty$. Here, $1 - \psi(x) \geq 0$; hence $\psi(x) = 1 = 1_S(x)$ a.e. on D.

2.16. *Property* (A). We will say that the system $\{X, \lambda; \pi_j \ (j \in J)\}$ has *property* (A) when, conversely, each set S of uniqueness in X is necessarily additive.

2.17. In this direction, we can report the following results.

   (i) Property (A) does hold in the classical case (Example 2.6) when $n = 2$ and $X = R^2$ while λ is two-dimensional Lebesgue measure. This result is essentially due to Lorentz [8]; see Fishburn et al. [1]; Kuba and Volcic [7], [9] for related results.

   (ii) Property (A) will be shown to be *false* in the classical case (Example 2.6) when $n \geq 3$ even when $X = [0, 2]^n$ and λ is Lebesgue measure. See §3 for details.

(iii) Property (A) does hold for the discrete case of Example 2.7, as was shown in [2].

(iv) More generally, property (A) holds whenever X is finite (even when J is infinite). This result is obtained in §4 as a corollary to a uniqueness result for systems of linear inequalities.

3. Generalized additivity. Here, we use the notations of §2. We always assume that $\lambda(S) < \infty$. Recall the linear classes $F(S)$ and F of functions $f: X \to R$ defined in 2.8. Further, an additive set S is a subset of X such that $S = \{x \in X: f(x) \geq 0\}$ for some $f \in F(S)$. From Lemma 2.12, each additive set is a set of uniqueness.

3.1. DEFINITION. Let S be a subset of X with $\lambda(S) < \infty$. We will way that S is (stepwise) additive of degree ≤ 2 if it is of the form

$$(3.2) \qquad S = \{x \in X: f(x) > 0\} \cup \{x \in X: f(x) = 0;\ g(x) \geq 0\},$$

for some $f, g \in F(S)$. More generally, S will be said to be (stepwise) *additive of degree* $\leq m$ (with m as a positive integer), if it has the following structure relative to a suitably chosen m-tuple of functions $f^{(r)} \in F(S)$ $(r = 1, \ldots, m)$.

(i) An element $x \in X$ belongs to S if and only if the first non-zero element (if any) in the sequence $f^{(1)}(x), \ldots, f^{(m)}(x)$ is positive.

(ii) In particular, $x \in S$ whenever $f^{(r)}(x) = 0$ for all $r = 1, \ldots, m$.

Equivalently, S is said to be *additive of degree* $\leq m$ if it is of the form

$$(3.3) \qquad\qquad S = U_1 \cup U_2 \cup \cdots \cup U_m \cup Z_m,$$

for some choice of the functions $f^{(r)} \in F(S)$ $(r = 1, \ldots, m)$. Here,

$$(3.4) \qquad U_r = \{x \in X: f^{(k)}(x) = 0 \text{ for } 1 \leq k < r;\ f^{(r)}(x) > 0\};$$
$$Z_r = \{x \in X: f^{(r)}(x) = 0 \text{ for } 1 \leq k \leq r\}.$$

3.5. We will further use the notation

$$(3.6) \qquad Q_r = \{x \in X: f^{(k)}(x) = 0 \text{ for } 1 \leq k < r;\ f^{(r)}(x) \neq 0\}.$$

Note that X is the *disjoint* union of the sets $Q_1, \ldots, Q_m$ and Z_m. If S is defined as in (3.3) and $x \in Q_r$, then $x \in S$ if and only if $f^{(r)}(x) > 0$ $(r = 1, 2, \ldots, m)$.

Additivity of degree $\leq m$ implies additivity of degree $\leq m + 1$. This can be seen by choosing $f^{(m+1)} = f^{(m)}$, in which case Q_{m+1} and thus U_{m+1} is empty. We will say that S is additive of degree m if

it is additive of degree $\leq m$ and, moreover, m is minimal, that is, S fails to be additive of any degree $\leq m'$ for each $1 \leq m' < m$. In particular, additivity of degree ≤ 1, additivity of degree 1 and ordinary additivity (as in Definiton 2.11) are all equivalent.

3.7. Analogously, one defines the notion of a subset S of X to be additive of degree $\leq \infty$. This means that, for some infinite sequence $f^{(r)} \in F(S)$ $(r = 1, 2, \ldots,)$, S happens to the union of all the associated sets U_r $(r \geq 1)$ as in (3.4), together with the set $Z_\infty = \{x \in X : f^{(r)}(x) = 0 \text{ for all } r \geq 1\}$. If the latter set S is not additive of any finite degree, we will say that S is additive of infinite degree.

The following result shows that additivity of higher degree is no more general than ordinary additivity in the special case that X is finite; (one may as well assume that $\lambda(X) < \infty$). Another proof for this discrete case would be as follows: (i) Theorem 3.11 below shows that an additive set of degree $\leq m$ is always a set of uniqueness. (ii) Theorem 4.50 below shows that each set of uniqueness is additive, as long as X is finite.

3.8. LEMMA. *If X is finite then any subset S of X which is additive of degree $\leq m$, for some $1 \leq m \leq \infty$, is already additive (that is, additive of degree 1).*

3.9. *Proof.* Let $1 \leq m \leq \infty$ and suppose S is additive of degree $\leq m$ as in (3.3), (3.4), (or 3.7 when $m = +\infty$), where $f^{(r)} \in F(S)$. Functions $f^{(r)}$, such that Q_r is empty, serve no purpose and may as well be dropped. In other words, one can assume without loss of generality that all sets Q_r are non-empty $(1 \leq r \leq m)$. But X is finite and the Q_r are disjoint, hence, m can be assumed to be finite. Let

$$a = \inf_{x,r}\{|f^{(r)}(x)| : f^{(r)}(x) \neq 0\};$$

$$b = \sup_{x,r}\{|f^{(r)}(x)|\}; \quad \text{thus } 0 < a \leq b < \infty,$$

(where, $x \in X$; $1 \leq r \leq m$). Choose further $0 < \rho < a/(a+b)$ thus $\rho/(1 - \rho) < a/b$, and consider

$$f = f^{(1)} + \rho f^{(2)} + \rho^2 f^{(3)} + \cdots + \rho^{r-1} f^{(r)}, \quad \text{hence, } f \in F(S).$$

It suffices to show that S coincides with the additive set

$$T = \{x \in X : f(x) \geq 0\}.$$

If $x_0 \in Z_m$, that is, $f^{(k)}(x_0) = 0$ for $1 \le k \le m$ then clearly $x_0 \in S \cap T$. Next, let $x_0 \notin Z_m$ and define $r = r(x_0)$ by $1 \le r \le m$; $f^{(k)}(x_0) = 0$ for $1 \le k < r$ and $f^{(r)}(x_0) \ne 0$. From (3.3), one has $x_0 \in S$ or $x_0 \in S^c$ depending on whether $f^{(r)}(x_0) > 0$ or $f^{(r)}(x_0) < 0$, respectively. It suffices to show that in these cases one has $f(x_0) > 0$ or $f(x_0) < 0$, respectively (to the effect that $x_0 \in T$ or $x_0 \notin T$, respectively). And that property follows immediately from

$$\left| \sum_{r < k \le m} \rho^{k-1} f^{(k)}(x_0) \right| \le b\rho^r/(1 - \rho) < a\rho^{r-1} \le |\rho^{r-1} f^{(r)}(x_0)|.$$

3.10. Let us return to the general case as in §2. For the classicial case described in Example 2.6 (with $n \ge 3$), Theorem 3.11 will enable us to construct sets of uniqueness which are not additive.

3.11. THEOREM. *Let S be a subset of X which is additive of degree $\le m$, for some $1 \le m \le \infty$. Then S is a (strong) set of uniqueness.*

3.12. *Proof.* Let S be of the form (3.3) (or as in 3.7 when $m = +\infty$), $\lambda(S) < \infty$. Let $0 \le \psi \le 1$ be such that the measure $d\nu = \psi \, d\lambda$ has the same marginals as $d\mu = 1_S \, d\lambda$. In particular, from (2.1) and $f^{(r)} \in F(S)$, one has $\int f^{(r)} \, d\nu = \int f^{(r)} \, d\mu$, that is,

$$(3.13) \qquad \int f^{(r)}(x)(1_S(x) - \psi(x))\lambda(dx) = 0, \quad \text{for all } 1 \le r \le m.$$

One must show that $\nu = \mu$, that is $\psi(x) = 1_S(x)$ a.e. $[\lambda]$.

Let Z_r be as in (3.4) (as in 3.7 when $r = \infty$), $Z_0 = X$, and let Q_r be as in (3.6). In particular, $Z_0 = X$ is the disjoint union of the $m+1$ sets Q_r and Z_m ($1 \le r \le m$; $r < \infty$). We first prove, by an induction on r, that $\psi(x) = 1_S(x)$ a.e. $[\lambda]$ on Q_r ($r = 1, \ldots, m$; $r < \infty$). Let $1 \le r \le m$; r finite, and suppose we already know that $\psi(x) = 1_S(x)$ a.e. $[\lambda]$ or Q_k, for $1 \le k < r$ and, thus, on the full complement $Q_1 \cup \cdots \cup Q_{r-1} = X/Z_{r-1}$ of Z_{r-1}. Hence, (3.13) yields that

$$(3.14) \qquad \int_{Z_{r-1}} f^{(r)}(x)(1_S(x) - \psi(x))\lambda(dx) = 0.$$

Here, $Z_{r-1} = Q_r \cup Z_r$ with Q_r and Z_r disjoint. If $x \in Z_r$ then $f^{(r)}(x) = 0$. On the other hand, if $x \in Q_r$ (thus $f^{(r)}(x) \ne 0$) we have from (3.3) and (3.4) that x belongs to S if and only if $f^{(r)}(x) > 0$. Therefore, the integrand in (3.14) (where $0 \le \psi \le 1$) is *nonnegative* and thus equal to zero a.e. $[\lambda]$ on Z_{r-1} and thus on Q_r. Hence, $\psi(x) = 1_S(x)$ a.e. $[\lambda]$ on Q_r which completes our proof by induction.

We now know that $\psi(x) = 1_S(x)$ a.e. $[\lambda]$ on the full complement $Q_1 \cup \cdots \cup Q_m = X/Z_m$. Further,

$$\nu(x) = \int \psi \, d\lambda = \int 1_S \, d\lambda < \infty, \quad \text{thus} \quad \int_{Z_m} (1_S(x) - \psi(x))\lambda(dx) = 0.$$

But $S \supset Z_m$, thus, the latter integrand is nonnegative, showing that $\psi(x) = 1_S$ a.e. $[\lambda]$ on Z_m. This completes the proof that $\psi(x) = 1_S$ a.e. $[\lambda]$ on X.

3.15. REMARK. Theorem 3.11 and its proof even generalizes to sets S which are additive of degree $\leq \alpha$, with α as any countable ordinal number. Here, an additive set S of degree $\leq \alpha$ would be a subset S with $\lambda(S) < \infty$, which is determined in the following manner by a suitable family $\{f^{(r)}; r \leq \alpha\}$ in $F(S)$, with r running through the ordinal numbers $r \leq \alpha$. If $x \in X$ is such that $f^{(r)}(x) = 0$ for all $r \leq \alpha$ then $x \in S$. Otherwise, let $r = r(x)$ be the smallest ordinal number with $f^{(r)}(x) \neq 0$ and assign x to S or S^c, respectively, depending on whether $f^{(r)}(x) > 0$ or $f^{(r)}(x) < 0$, respectively. If α is minimal (given S) one would say that S is additive of degree α. It would be interesting to know (for instance, in the classical case 2.6) for what ordinal numbers α there does exist a subset S of X, which is additive of degree α. And also what systems $\{X, \lambda, \pi_j \, (j \in J)\}$ have the following Property (B): To every set S of uniqueness in X there corresponds a countable ordinal number α, such that S is additive of degree $\leq \alpha$. This property is of a similar type as Property (A), which was discussed in 2.16 and 2.17.

The proof of Theorem 3.11 also yields that $T = U_1 \cup U_2 \cup \cdots \cup U_m$ is a set of uniqueness. However, this is nothing new, in fact, T is additive of degree $\leq m + 1$, as is seen by introducing $f^{(m+1)} \equiv -1$.

3.16. *A non-additive set of uniqueness.* From now on, in this section, we will restrict ourselves to the classical case as in 2.6. We will take X as a bounded subset of R^n supplied with Lebesgue measure. Further the $\pi_j \mu$ $(j = 1, \ldots, n)$ are the one-dimensional marginals of $d\mu = \phi \, d\lambda$, with $0 \leq \phi \leq 1$. Assuming that $n \geq 3$ (as is necessary by 2.17), we would like to have an example of a subset of S of R^n which is additive of exact degree 2. This means that:

  (i) S is additive of degree ≤ 2, (as in (3.2)). Thus, from Theorem 3.11, S is also a set of uniqueness.
  (ii) S is not additive.

3.17. *Construction.* Let m be a fixed integer, such that $2 \leq m \leq n - 1$ (as is only possible when $n \geq 3$). We will choose X as the block

$$X = [0, 2)^m \times [0, 1)^{n-m}$$
$$= \{x \in R^n : 0 \leq x_j < 2 \text{ for } j \leq m; \ 0 \leq x_j < 1 \text{ for } j > m\}.$$

The reference measure λ is chosen as Lebesgue measure m_n restricted to X. One has $F(S) \supset F$ with F as the class of all $f : X \to R$ of the form $f(x) = \sum_j f_j(x_j)$ with f_j integrable on $[0, 2)$ or $[0, 1)$.

It will be convenient to regard the block X as the disjoint union of the 2^m unit cubes X_δ defined by

$$X_\delta = \{x \in X : \delta_j \leq x_j < \delta_j + 1, \ \text{for } j = 1, \ldots, m\}$$

$(\delta_j \in \{0, 1\})$. Here, δ runs through all the 2^m sequences $\delta = (\delta_1, \ldots, \delta_m)$ with $\delta_j \in \{0, 1\}$. The choice $\delta = 0$ leads to the special unit cube $X_0 = \{x \in R^n : 0 \leq x_j < 1, \ \text{for } j = 1, \ldots, n\}$. Let $S_\delta = S \cap S_\delta$ denote the part of S in S_δ.

The subset S of X will be defined as in (3.2), with $f, g \in F$; hence S is additive of degree ≤ 2. Here, f and g are chosen to be of the special form

$$(3.18) \quad f(x) = \sum_{j=1}^{m} f_j(x_j); \qquad g(x) = \sum_{j=1}^{n} g_j(x_j) = \sum_{j=1}^{n} x_j - n + 1.$$

Thus, $f_j \equiv 0$ for $m < j \leq n$. Further

$$(3.19) \quad f_j(x_j) = 0 \quad \text{if } 0 \leq x_j < 1;$$
$$f_j(x_j) = \frac{1}{m} - (x_j - 1) \quad \text{if } 1 \leq x_j < 2, \ (j = 1, \cdots, m);$$

(much more general functions f and g would work equally well, see Remark 3.32 below). Observe that $f(x) = 0$ for all $x \in X_0$. Hence, from (3.2),

$$(3.20) \qquad S_0 = S \cap X_0 = \{x \in X_0 : g(x) \geq 0\}$$
$$= \{x \in X_0 : x_1 + \cdots + x_n \geq n - 1\}.$$

Clearly, S_0 is a *simplex* (pyramid) in R^n having the points $(0, 1, \ldots, 1); \cdots, (1, \ldots, 1, 0)$ and $(1, \ldots, 1)$ as its $n + 1$ vertices.

Ignoring the set $\{x \in S/X_0; \ f(x) = 0\}$, which has m_n-measure zero, the remaining part of S is of the form

$$S/X_0 = \{x \in X : f(x) > 0\}$$

and thus independent of the function g. Let $\delta \neq 0$ where $\delta = (\delta_1, \ldots, \delta_m)$ $(\delta_j \in \{0, 1\})$. It is immediate, from the definition of X_δ and (3.19), that $S_\delta = S \cap X_\delta$ consists of all $x \in R^n$ satisfying

    (i) $0 \leq x_j < 1$ when $j > m$ or $\delta_j = 0$ $(j = 1, \cdots, n)$;

    (ii) If $\delta_j = 1$ then $1 \leq x_j < 2$ $(j = 1, \cdots, m)$ in such a way that

$$(3.21) \qquad \sum \{x_j - 1: \quad \delta_j = 1; 1 \leq j \leq m\} < d/m.$$

Here, $d = |\delta| = \delta_1 + \cdots + \delta_m$; hence, $1 \leq d \leq m$. Thus, S_δ is a prism $S_\delta = B_\delta \times [0, 1)^{n-d}$ having as its base B_δ a simplex in R^d, with vertices $(y_1, \ldots, y_d)$ for which $y_j = 1$, with at most one exception, in which case $y_j = 1 + d/m$. If $d = 1$ this base is an interval and S_δ is a rectangular block. For instance, if $\eta = (1, 0, \ldots, 0)$ then $S_\eta = [1, 1 + 1/m) \times [0, 1)^{n-1}$. If $d = 2$ then the base B_δ is a triangle.

3.23. Consider the special case $n = 3$ where $m = 2$. Then our set S of uniqueness in R^3 consists of: (i) The pyramid (tetrahedron) with vertices $(1, 1, 1)$; $(0, 1, 1)$; $(1, 0, 1)$; $(1, 1, 0)$. (ii) The rectangular blocks $[1, 3/2) \times [0, 1) \times [0, 1)$ and $[0, 1) \times [1, 3/2) \times [0, 1)$. (iii) The prism $B \times [0, 1)$ with B as the triangle $B = \{(x_1, x_2) : x_1 \geq 1; x_2 \geq 1; x_1 + x_2 < 3\}$.

3.23. THEOREM. *The set S of uniqueness as constructed above is not additive.*

3.24. *Proof.* On the contrary, suppose there exist measurable functions $h_1, \ldots, h_m$ on $[0, 2)$; further measurable functions $h_{m+1}, \ldots, h_n$ on $[0, 1)$ and a subset N of X, with $m_n(N) = 0$, such that if $x = (x_1, \ldots, x_n) \in X/N$ then

$$(3.25) \qquad \sum_j h_j(x_j) \geq 0 \text{ if and only if } x \in S.$$

In the sequel, indices j, r and s will run through $\{1, \ldots, n\}$; $\{1, \ldots, m\}$ and $\{m + 1, \ldots, n\}$, respectively. We write $x = (x_1, \ldots, x_n) \in R^n$ as $x = (y, z)$, where $y = (y_1, \ldots, y_m) = (x_1, \ldots, x_m)$ and $z = (z_{m+1}, \ldots, z_n) = (x_{m+1}, \ldots, x_n)$. Let further

$$H_1(y) = \sum_r h_r(y_r); \quad H_2(z) = \sum_s h_s(z_s);$$

$$L_1(y) = \sum_r y_r; \qquad L_2(z) = \sum_s z_s.$$

Thus, (3.25) can be written as if $(y, z) \in X/N$ then

(3.26) $H_1(y) + H_2(z) \geq 0$ if and only if $(y, z) \in S$.

In particular, (3.20) implies that if $(y, z) \in X_0/N$ then

(3.27) $H_1(y) + H_2(z) \geq 0$ if and only if $L_1(y) + L_2(z) \geq n - 1$.

Next, let us apply (3.26) for points $(y, z) \in X_\zeta$ where $\zeta = (1, \ldots, 1)$. Note that $X_\zeta = [1, 2)^m \times (0, 1)^{n-m} = e + X_0$, where $e \in R^n$ has coordinates $e_r = 1$ $(r = 1, \ldots, m)$ and $e_s = 0$ $(s = m + 1, \ldots, n)$. We see from (3.21) with $\delta = \zeta$ (thus $d = |\delta| = m$) that

$$S \cap X_\zeta = \left\{ x \in X_\zeta : \sum_r (x_r - 1) < 1 \right\} = e + \{(y, z) \in X_0 : L_1(y) < 1\}.$$

Hence, it follows from (3.27) that if $(y, z) \in X_0/N_1$ then

(3.28) $K_1(y) + H_2(z) \geq 0$ if and only if $L_1(Y) < 1$.

Here,

$$K(y) = \sum_r k_r(y_r) \quad \text{where } k_r(\xi) = h_r(1 + \xi) \qquad (r = 1, \ldots, m)$$

$(y \in [0, 1)^m; \xi \in [0, 1))$. Further, $N_1 = \{x \in X_0 : e + x \in N\}$ thus $m_n(N_1) = 0$. Note that $\{(y, z) \in X_0 : L_1(y) = 0\}$ has m_n-measure zero and can be ignored.

We will show that (3.27), (3.28) together lead to a contradiction. In the sequel, y ranges through I^m and z through I^{n-m}, where $I = [0, 1)$. Put

(3.29) $\alpha = \operatorname{ess\,inf}\{H_2(z)\}: \qquad \beta = \operatorname{ess\,sup}\{H_2(z)\};$

thus, $\alpha \leq \beta$, (possibly $\alpha = -\infty$ or $\beta = +\infty$). We claim that $\alpha < \beta$.

Suppose not, equivalently, α is finite and $H_2(z) = \alpha$ a.e. on I^{n-m}. Since $m_n(N) = 0$, there exists a subset E of I^{n-m} with $m_{n-m}(E) = 0$, such that, for each $z \in I^{n-m}/E$, one has $(y, z) \notin N$ for almost all $y \in I^m$. Note that the linear function $L_1(y) = \sum_r y_r$ on I^m takes all values in $[0, m)$, while the linear function $L_2(z) = \sum_s z_s$ on I^{n-m} takes all values in $[0, n - m)$. Clearly, there exist $z', z'' \in I^{n-m}/E$ such that $H_2(z') = H_2(z'') = \alpha$; further $L_2(z') < L_2(z'')$ and $n - m - 1 < L_2(z'') < n - m$. Next, there exists $y \in I^m$ such that

$$(y, z') \notin N, \ (y, z'') \notin N \quad \text{and}$$
$$n - 1 - L_2(z'') < L_1(y) < n - 1 - L_2(z');$$

(just choose y such that $L_1(y)$ is slightly larger than the value $p = n - 1 - L_2(z'')$, which satisfies $m - 1 < p < m$). Applying (3.27), we see that $(y, z') \notin N$ and $L_1(y) + L_2(z') < n - 1$ imply $H_1(y) + \alpha < 0$. Similarly, $(y, z'') \notin N$ and $L_1(y) + L_2(z'') > n - 1$ imply that $H_1(y) + \alpha \geq 0$ and we have a contradiction. This proves that $\alpha < \beta$.

Since $m_n(N_1) = 0$ there exists a subset D of I^m of (m-dimensional Lebesgue) measure zero, such that, for each $y \in D$, one has $(y, z) \notin N_1$ for almost all $z \in I^{n-m}$. It follows from (3.28) and (3.29) that

$$(3.30) \quad \text{if } y \in I^m/D \quad \text{and} \quad L_1(y) < 1 \quad \text{then } K_1(Y) + \alpha \geq 0;$$
$$\text{if } y \in I^m/D \quad \text{and} \quad L_1(y) \geq 1 \quad \text{then } K_1(y) + \beta \leq 0.$$

Here, $L_1(y) = \sum y_r$ takes all values in $[0, m)$, thus, α and β are finite. Let $\Delta = \beta - \alpha$; thus $\Delta > 0$. At least intuitively, (3.30) seems impossible. For, it would imply that, roughly speaking, $K_1(y) = \sum_r k_r(y_r)$ makes a downward jump of size $\geq \Delta$ *everywhere* across the hyperplane $L_1(y) = \sum y_r = 1$. Here, $r = 1, \ldots, m$ where $m \geq 2$.

Let us write $y \in I^m$ as $y = (u, v)$, where $u = (u_1, \ldots, u_{m-1}) = (y_1, \ldots, y_{m-1}) \in I^{m-1}$ and $v = y_m \in I = [0, 1)$. Thus $L_1(y) = L_1(u, v) = L_0(y) + v$ where $L_0(u) = u_1 + \cdots + u_{m-1}$. Further $K_1(y) = K_1(u, v) = K_0(u) + k(v)$, where $K_0(u) = k_1(u_1) + \cdots + k_{m-1}(u_{m-1})$ and $k(v) = k_m(v)$.

Let C be a subset of $I = [0, 1)$ of full measure $m_1(C) = 1$ such that $(u, v) \notin D$, for almost all $u \in I^{m-1}$, whenever $v \in C$. Let $v', v'' \in C$ with $v' < v''$ be fixed for the moment. There exists $u \in I^{m-1}$ such that $(u, v') \notin D$; $(u, v'') \notin D$ and $1 - v'' < L_0(u) < 1 - v'$; (after all, $L_0(u)$ is a linear function on I^{m-1} taking all values in $(0, m - 1)$ and thus all values in $(1 - v'', 1 - v')$; it is mainly here that $m \geq 2$ is used). Since $(u, v') \notin D$ and $L_1(u, v') = L_0(u) + v' < 1$, it follows from (3.30) that $K_1(u, v') = K_0(u) + k(v') \geq -\alpha$. Similarly, since $(u, v'') \notin D$ and $L_0(u) + v'' > 1$, one has that $K_0(u) + k(v'') \leq -\beta$. Consequently, $k(v'') - k(v') \leq -\beta + \alpha = -\Delta < 0$ whenever $v', v'' \in C$ with $v' < v''$. Choosing $v_i \in C$ with $v' < v_1 < \cdots < v_{M-1} < v''$, it would follow that $k(v'') - k(v') \leq M\Delta$ for M arbitrarily large and this is clearly impossible. This completes the proof of Theorem 3.23.

3.31. REMARK. Omitting the proof, we assert that (3.25) restricted to X_0 (which is the same as (3.27)) has only trivial solutions $h = (h_1, \ldots, h_n)$, provided $n \geq 3$. More precisely, suppose that $h_j: I = [0, 1) \to R$ ($j = 1, \cdots, n$) are measurable functions such that, for

all $x \in I^n/N$, one has $\sum_j h_j(x_j) \geq 0$ if and only if $\sum_j x_j \geq n - 1$. Here, the subset N of I^n satisfies $m_n(N) = 0$. Then there exist constants $a > 0$ and b_j such that $h_j(x_j) = ax_j - b_j$ for almost all $x_j = I$ $(j = 1, \cdots, n)$. Naturally, $\sum_j b_j = (n - 1)a$. The analogous result with $n = 2$ is false. After all, for each choice of the strictly increasing function $\theta : [0, 1] \rightarrow R$, $x_1 + x_2 \geq 1$ a.e. on $[0, 1)^2$ is equivalent to $\theta(x_1) - \theta(1 - x_2) \geq 0$ a.e. on $[0, 1)^2$.

3.32. REMARK. The proof of Theorem 3.23 is quite robust and uses very few of the properties of the functions f, $g \in F$ defining S as in (3.2). Let us maintain the structure of X and also the assumptions that $2 \leq m \leq n - 1$ and

$$f(x) = \sum_{j=1}^{m} f_j(x_j), \quad \text{with } f_j(x_j) = 0 \text{ if } 0 \leq x_j < 1;$$

$$g(x) = \sum_{j=1}^{n} g_j(x_j) \quad (x \in X).$$

In particular, $f_j \equiv 0$ if $j > m$. One has $f(x) = 0$ for all $x \in X_0 = [0, 1)^n$, thus, $S_0 = S \cap X_0 = \{x \in X_0 : g(x) \geq 0\}$ is determined by the restrictions $g_j : [0, 1) \rightarrow R$ $(j = 1, \cdots, n)$.

Writing $x \in X_0$ as $x = (y, z)$ with $y \in [0, 1)^m$ and $z \in [0, 1)^{n-m}$, we want that the property $(y, z) \in S_0$ depends in an essential way on z. This in order that, as in the above proof, α and β as in (3.29) will satisfy $\alpha < \beta$. It would be sufficient that

$$\operatorname*{ess\,inf}_{z} \sum_{s} g_s(z_s) < -\sum_{r} g_r(y_r) < \operatorname*{ess\,sup}_{z} \sum_{s} g_s(z_s), \qquad z \in [0, 1)^{n-m}$$

$(r - 1, \dots, m; s = m+1, \dots, n)$ holds for a set of points $y \in [0, 1)^m$ of positive measure. For each such y, the set $\{(y, z) : z \in [0, 1)^{n-m}\}$ meets both S and S^c in a set of positive measure.

The above method of proof easily leads to a contradiction (from the assumption that S is additive), provided that, in addition, the functions $f_1, \dots, f_m$ are *continuous* on $(1,2)$ in such a way that

$$\sum_{r=1}^{m-1} f_r(x_r) < -f(x_m) < -f(y_m) < \sum_{r=1}^{m-1} f_r(y_r),$$

for at least one choice of the numbers $x_r, y_r \in (1, 2)$ $(r = 1, \dots, m)$. This is a rather weak assumption. We omit the details.

4. The finite case. Here, we characterize systems of linear inequalities having a unique solution. Let m and n be fixed positive integers and $I = \{1, \ldots, m\}$; $J = \{1, \ldots, n\}$. Unspecified indices i and j run through I and J, respectively. Further $A = (a_{ij}) \in R^{mn}$ is a given $m \times n$ matrix and $b = (b_1, \ldots, b_m)^T \in R^m$ a given column vector. We will be interested in the system of inequalities $Ax \geq b$, with $x = (x_1, \ldots, x_n)^T$. That is,

$$(4.1) \qquad \sum_{j \in J} a_{ij} x_j \geq b_i, \quad \text{for all } i \in I.$$

In addition, we impose the conditions

$$(4.2) \qquad \alpha_j \leq x_j \leq \beta_j, \quad \text{for all } j \in J.$$

Here, and from now on, the α_j and β_j are given extended real numbers such that

$$(4.3) \qquad -\infty \leq \alpha_j < \beta_j \leq +\infty \qquad (j \in J);$$

(the case $\alpha_j = \beta_j$, where $x_j = \alpha_j$ is prescribed, can be eliminated by lowering n).

Specifically, we shall be interested in characterizing the situations where (4.1), (4.2) have a unique solution x^0. The special case where (4.2) is void (that is $\alpha_j = -\infty$; $\beta_j = +\infty$, for all $j \in J$) leads to the following definition.

4.4. DEFINITION. The matrix $A = (a_{ij})$ is said to be a *matrix of uniqueness* if

$$(4.5) \qquad x \in R^n; \quad Ax \geq 0 \text{ imply that } x = 0.$$

Naturally, this would be equivalent to $Ax \leq 0$ implying that $x = 0$.

Since the matrix A represents a map $R^n \rightarrow R^m$, one is not allowed to identify all empty matrices (having $m = 0$ or $n = 0$). In fact, (4.5) forces the convention that an empty matrix with $m = 0$ and $n \geq 1$ is never a matrix of uniqueness. It will be convenient to regard an empty $m \times n$ matrix with $n = 0$ columns as being a matrix of uniqueness. The following result is known.

4.6. LEMMA. *In order that A be a matrix of uniqueness it is necessary and sufficient that*

(4.7) *for each $z \in R^n$ there exists $y \in R^m$*

such that $y \geq 0$ and $yA = z$.

Here, y and z are row vectors.

4.8. *Proof*. Assume property (4.7) and let $x \in R^n$ satisfy $Ax \geq 0$. Applying (4.7) with $z = -x^T$, there exists $y \in R^m$ with $y \geq 0$ and $yA = -x^T$. Hence,

$$0 \leq y(Ax) = (yA)x = -x^T x = -||x||^2, \quad \text{thus}, \, x = 0.$$

Next, suppose property (4.7) is false. That is, some $z_0 \in R^n$ is not contained in the convex cone $K = \{yA : y \geq 0\}$. It follows that K is contained in some closed half space $H_+ = \{z \in R^n : zc \geq 0\}$ with $c \in R^n$ as a *non-zero* column vector. Thus $y(Ac)c \geq 0$, whenever $y \geq 0$; hence, $Ac \geq 0$ contradicting (4.5).

4.9. REMARK. Geometrically, Lemma 4.6 is very intuitive. For (4.5) requires that the finite collection of closed half spaces $\sum_j a_{ij} x_j \geq 0 \, (i \in I)$ has intersection $\{0\}$. And property (4.7) says that the convex cone spanned by the associated normals (the rows of A) is all of R^n.

4.10. REMARK. The system of *equalities* $Ax = 0$ would be equivalent to $Bx \geq 0$, where B is the $2m \times n$ matrix $B = \left(\begin{smallmatrix} A \\ -A \end{smallmatrix} \right)$. Clearly, B is a matrix of uniqueness if and only if A has rank n. Condition (4.7) applied to B (instead of A) says that the rows of A span all of R^n.

4.11. In the sequel, x^0 is a *fixed solution* of (4.1) and (4.2), while I^0 and J^0 denote the associated index sets

$$(4.12) \qquad I^0 = \left\{ i \in I : \sum_j a_{ij} x_j^0 = b_i \right\};$$

$$(4.13) \qquad J^0 = \{j \in J : \alpha_j < x_j^0 < \beta_j\}.$$

Further A^0 denotes the (possibly empty) associated submatrix of A defined by

$$(4.14) \qquad A^0 = \{a_{ij} ; \, i \in I^0 ; \, j \in J^0\}.$$

4.15. THEOREM. *Let x^0 be a fixed solution of* (4.1), (4.2) *with associated I^0, J^0 and A^0 as above. Then in order that x^0 be the only solution of* (4.1), (4.2), *it is necessary and sufficient that x^0 satisfies the following two conditions.*

(i) *There must exist a vector* $f = (f_1, \ldots, f_m)$ *of the form*

(4.16) $f_j = \sum_{i \in I} y_i a_{ij};$ $y_i \geq 0 \ (i \in I);$ $y_i = 0 \ (i \notin I^0)$

and satisfying

(4.17) $f_j < 0$ *iff* $x_j^0 = \alpha_j;$

$f_j > 0$ *iff* $x_j^0 = \beta_j;$

$f_j = 0$ *iff* $\alpha_j < x_j^0 < \beta_j;$ *that is, iff* $j \in J^0.$

(ii) *The matrix* A^0 *defined by* (4.14) *is a matrix of uniqueness.*

4.18. REMARKS. Note that the above uniqueness criterion depends on $b \in R^m$ only through the index set I^0 defined by (4.12). The vector f in (ii) is often far from unique. Condition (ii) is automatically satisfied when J^0 is empty, that is, when $x_j^0 \in \{\alpha_j, \beta_j\}$ for all $j \in J$.

The *necessity* of condition (ii) is easily seen. For, if (ii) were false then, from Definition 4.4, there would exist a *non-zero* function $z: J \to R$ carried by J^0 (that is, $z_j = 0$ if $x_j^0 = \alpha_j$ or $x_j^0 = \beta_j$) such that $\sum_j a_{ij} z_j \geq 0$ for all $i \in I^0$. Replacing x^0 by $x = x^0 + pz$, with $p > 0$ small, one obtains a different solution of (4.1), (4.2).

4.19. As a simple illustration, let $n = 1$. Here, $x \in R$ is subject to $a_i x \geq b_i \ (i \in I)$ and $\alpha \leq x \leq \beta$ (where $\alpha < \beta$). A solution x^0 is unique if and only if one of the following happens. Here, $I^0 = \{i \in I: a_i x^0 = b_i\}$.

(i) One has $\alpha < x^0 < \beta$, that is, $J^0 = \{1\}$, and, moreover, there exist $r, s \in I^0$ with $a_r < 0$ and $a_s > 0$. Here, the above assertions hold with $y_i = 0$, and thus $f = \sum_i y_i a_i = 0$.

(ii) One has $\alpha = x^0 < \beta$ (thus J^0 is empty) and, in addition, $a_r < 0$ for some $r \in I^0$ (thus $a_r x^0 = b_r$). Then choose $y_r = 1$ and $y_i = 0 \ (i \neq r)$; hence, $f = \sum_i y_i a_i = a_r < 0$.

(iii) One has $\alpha < x^0 = \beta$ (thus J^0 is empty) and $a_r > 0$ for some $r \in I^0$. Then choose $y_r = 1$ and $y_i = (i \neq r)$ so that $f = a_r > 0$.

4.20. Consider the case that I^0 is empty, that is, $A x^0 > b$. Then $Ax \geq b$ holds for all x in a sufficiently small neighborhood of x^0, showing that x^0 cannot possibly be a unique solution of (4.1), (4.2), which must mean that either (i) or (ii) is false. Indeed, A^0 is not a matrix of uniqueness unless possibly J^0 is empty, (see 4.4). But in the latter case, (i) is false. Namely, (i) with I^0 empty would imply that $f_j = 0$ for all $j \in J$, thus, that $J^0 = J \neq \varnothing$.

4.21. *Proof of Theorem* 4.15. *Sufficiency.* Suppose x^0 satisfies not only (4.1), (4.2) but also (i) and (ii). Consider any solution x of (4.1), (4.2). Multiplying (4.1) by $y_i \geq 0$ and summing, and using (4.16), one finds that $f(x) := \sum_j f_j x_j \geq \sum_i b_i y_i = f(x^0)$, and thus, $\sum_j f_j(x_j^0 - x_j) \leq 0$. But, in view of (4.2) and (4.17), each term in the latter sum is nonnegative. Consequently, $x_j = x_j^0$ whenever $f_j \neq 0$, that is, whenever $j \notin J^0$. This already implies the uniqueness of x^0 when J^0 is empty.

Otherwise, it suffices to prove $z = 0$ with z as the restriction of $x - x^0$ to J^0. In fact, $\sum_j a_{ij} x_j \geq b_i = \sum_j a_{ij} x_j^0$, for each $i \in I^0$. We already know that $x_j = x_j^0$, for $j \notin J^0$, and hence $A^0 z \geq 0$. But this implies $z = 0$ since A^0 is a matrix of uniqueness.

4.22. *Necessity.* The necessity of condition (ii) was already established in 4.18. It remains to show that (4.16), (4.17) hold for some vector $f = (f_1, \ldots, f_n)$. The proof will be by induction with respect to $n = |J|$. The case $n = 0$ is trivially true, while the case $n = 1$ was verified in 4.19.

Decreasing x_j, α_j and β_j by x_j^0 ($j \in J$) and decreasing b_i by $\sum_j a_{ij} x_j^0$ ($i \in I$), one may without loss of generality assume that $x^0 = 0$. Thus, $x^0 = 0$ is the unique solution of (4.1), (4.2). Therefore

$$\alpha_j \leq 0 \leq \beta_j \ (j \in J); \quad J^0 = \{j \in J : \alpha_j < 0 < \beta_j\};$$
$$b_i \leq 0 \ (i \in I); \quad I^0 = \{i \in I : b_i = 0\}.$$

We first observe that $x^0 = 0$ is also the unique solution of (4.2) together with the (homogeneous) system

$$(4.23) \qquad\qquad \sum_{j \in J} a_{ij} x_j \geq 0 \quad \text{for all } i \in I^0$$

(itself a subsystem of (4.1)). For, otherwise, there would exist $x \neq 0$ satisfying both (4.2) and (4.23). Since $b_i = 0$ ($i \in I^0$) and $b_i < 0$ ($i \notin I^0$), $x' = \rho x = (1 - \rho) \cdot 0 + \rho x$ would satisfy both (4.2) and the full system (4.1), as soon as $0 < \rho < 1$ is sufficiently small. But then $x^0 = 0$ would not be the unique solution of (4.1), (4.2).

The just established uniqueness property says that $K \cap Q = \{0\}$, where

$$K = \left\{ x \in R^n : \sum_{j \in J} a_{ij} x_j \geq 0, \ \text{for all } i \in I^0 \right\}$$

and

$$Q = \{x \in R^n : \alpha_j \leq x_j \leq \beta_j, \ \text{for all } j \in J\}.$$

Here, K is a closed and convex cone in R^n, possibly of the form $K = \{0\}$. Further Q is a closed and convex subset of R^n having a non-empty interior int(Q) (because $\alpha_j < \beta_j$ for all $j \in J$). One can separate K and Q by a hyperplane as soon as K is disjoint from int(Q).

First, consider the case that K is not disjoint from int(Q). Since $K \cap Q = \{0\}$, this is equivalent to $0 \in \text{int}(Q)$, that is, $\alpha_j < 0 < \beta_j$ for all $j \in J$; equivalently, $J^0 = J$. But then property (i) holds with $y_i = 0$ for all $i \in I$; thus, $f_j = 0$ for all $j \in J$.

It remains to consider the situation that $K \cap \text{int}(Q)$ is empty. Then there exists a hyperplane $H = \{x \in R^n : \sum_j f_j x_j = 0\}$ in R^n with $f = (f_1, \ldots, f_n)$ non-zero such that K is a subset of $H_+ = \{x \in R^n : \sum_j f_j x_j \geq 0\}$ and Q is a subset of $H_- = \{x \in R^n : \sum_j f_j x_j \leq 0\}$. From a classical result, see Gale [3], p. 44, the fact $H_+ \supset K$ implies that there exist numbers y_i ($i \in I$) as in (4.16).

Next, the fact $H_- \supset Q$ means that $\alpha_j \leq x_j \leq \beta_j$ ($j \in J$) implies $\sum_j f_j x_j \leq 0$. Recall that $\alpha_j \leq 0 \leq \beta_j$ ($j \in J$). Choosing all but one x_j equal to zero, we have, for each $j \in J$, that $\alpha_j \leq x_j \leq \beta_j$ implies $f_j x_j \leq 0$. Equivalently,

$$f_j < 0 \text{ implies } \alpha_j = 0; \quad f_j > 0 \text{ implies } \beta_j = 0.$$

Since $J^0 = \{j \in J : \alpha_j < 0 < \beta_j\}$, it follows that

$$J \supset J_f \supset J^0, \quad \text{where } J_f = \{j \in J : f_j = 0\}.$$

Note that J_f is a *proper* subset of J since $f_j \neq 0$ for at least one $j \in J$. The required property (4.17) (with $x_j^0 = 0$) holds as soon as $J_f = J^0$. In particular, one is ready if J_f is empty. However, it is quite possible that J_f is strictly larger than J^0, that is, there might exist indices $j \in J$, such that $f_j = 0$ and either $\alpha_j = 0$ or $\beta_j = 0$.

Since $x_j^0 = 0$ ($j \in J$) is the unique solution of (4.2), (4.23), it follows that $x_j^0 = 0$ ($j \in J$) is certainly the unique solution of (4.2), (4.23) but with J replaced by the proper subset J_f of J. Therefore, our induction assumption implies the existence of numbers $z_i \geq 0$ ($i \in I$) with $z_i = 0$, for $i \notin I^0$, such that the associated numbers $g_j = \sum_i z_i a_{ij}$ ($j \in J$) satisfy

(4.25) $g_j < 0$ if $\alpha_j = 0$; $\quad g_j > 0$ if $\beta_j = 0$; $\quad g_j = 0$ if $\alpha_j < 0 < \beta_j$,

provided $j \in J_f$. Now let $\rho > 0$ be small and consider $\eta_i = y_i + \rho z_i$ ($i \in I$) and the associated numbers $\phi_j = \sum_i \eta_i a_{ij} = f_j + \rho g_j$ ($j \in J$). One has $\eta_i \geq 0$ ($i \in I$) and $\eta_i = 0$ ($i \in I^0$). Choosing

$\rho > 0$ sufficiently small, one has $f_j + \rho g_j < 0$ when $f_j < 0$ and $f_j + \rho g_j > 0$ when $f_j > 0$. Using (4.24) and (4.25), one now easily verifies that (4.16), (4.17) are true for y_i replaced by η_i $(i \in I)$ and f_j replaced by ϕ_j $(j \in J)$. This completes the proof of Theorem 4.15.

4.26. REMARK. Another necessary and sufficient condition for the uniqueness of the solution x^0 of (4.1), (4.2) would be the existence of a vector $f = (f_1, \ldots, f_n)$ as in (4.16) satisfying

$$x_j^0 = \alpha_j \text{ if } f_j < 0; \quad x_j^0 = \beta_j \text{ if } f_j > 0; \text{ and}$$

$$A^f = (a_{ij}; i \in I^0; f_j = 0) \text{ is a matrix of uniqueness.}$$

In particular, J^0 is a subset of $J^f = \{i \in J; f_j = 0\}$. The necessity follows from Theorem 4.15 which shows that one can even attain that $J^f = J^0$. The sufficiency follows by the same reasoning as before, see 4.21.

4.27. THEOREM. *Let $x^0 \in R^n$ satisfy*

(4.28) $$\alpha_j \le x_j \le \beta_j \quad (j \in J).$$

Let further $J^0 = \{j \in J : \alpha_j < x_j^0 < \beta_j\}$. Then in order that x^0 be uniquely determined by the vector Ax^0 together with the bounds (4.28), it is necessary and sufficient that: (i) *Some linear combination*

(4.29) $$f_j = \sum_{i \in I} y_i a_{ij} \quad (y_i \in R; i \in I)$$

of the rows of A has property (4.17).

(ii) *The submatrix $A^0 = (a_{ij}; i \in I; j \in J^0)$ of A has full rank $|J^0|$. Equivalently, if $Az = 0$ and $z_j = 0$ for $j \notin J^0$ then $z = 0$.*

4.30. *Proof.* Apply Theorem 4.15 with A replaced by $B = \begin{pmatrix} A \\ -A \end{pmatrix}$ and b replaced by $\begin{pmatrix} b \\ -b \end{pmatrix}$ with $b = Ax^0$. Presently, $I^0 = I$. The matrix B^0 associated to B is a matrix of uniqueness if and only if A^0 has rank $|J^0|$ (see 4.10).

4.31. COROLLARY. *Let $x^0 \in R^n$ satisfy*

(4.32) $$x_j^0 \in \{\alpha_j, \beta_j\}, \quad \text{for all } j \in J.$$

Then in order that x^0 be unquely determiined by the vector Ax^0 together with the conditions (4.28), it is necessary and sufficient that there exists a linear combination as in (4.29) such that

(4.33) $$f_j < 0 \text{ if } x_j = \alpha_j; \quad f_j > 0 \text{ if } x_j = \beta_j.$$

In particular, $f_j \ne 0$ for all $j \in J$.

4.34. REMARK. Theorem 4.27 and Corollary 4.31 remain valid when I is infinite, provided we add to (4.29) the condition that $y_i = 0$ for all but finitely many $i \in I$. In such a situation, the condition $Ax = Ax^0$ amounts to an infinite system of equalities.

It suffices to show that (i), (ii) are necessary. For each finite subset H of I, let D_H be the closed convex set consisting of all $x \in R^n$ satisfying (4.28) and $\sum_j a_{ij} x_j = \sum_j a_{ij} x_j^0$, for each $i \in H$. Assuming that $\bigcap_H D_H = \{x^0\}$, it suffices to show that H exists with $D_H = \{x^0\}$, that is, $\dim D_H = 0$. Let H^* be such that $\dim D_{H^*} = \inf_H \dim D_H$. One easily verifies that $D_H = D_{H^*}$ for each $H \supset H^*$ implying that $D_{H^*} = \bigcap_H D_H = \{x^0\}$. Consequently, if x^0 is a vector of uniqueness relative to (4.1), (4.2) then also relative to (4.2) together with a suitably chosen finite subsystem of (4.1).

4.35. REMARK. Relative to the prescribed $m \times n$ matrix $A = (a_{ij})$ and extended real numbers α_j, β_j $(\alpha_j < \beta_j ; j \in J)$, let us say that $x^0 \in R^n$ is a vector of uniqueness if it satisfies (4.28) and moreover, there is no other vector x satisfying (4.28) and $Ax = Ax^0$. If, in addition, x^0 satisfies (4.32), then we will say that x^0 is a *special* vector (of uniqueness). Corollary 4.31 amounts to a recipe for finding all such special vectors x. Namely, define $x_j = \alpha_j$ if $f_j < 0$; $x_j = \beta_j$ if $f_j > 0$ with f as any linear combination, as in (4.29), such that $f_j \neq 0$ for all $j \in J$; (α_j finite if $f_j < 0$; β_j finite if $f_j > 0$). In many applications, such a sequence $\{f_j\}$ has only a limited number of changes of sign thus severely limiting the possible special vectors x^0.

4.36. As is often true, suppose there exists $z \in R^m$ such that

$$(4.37) \qquad w_j := \sum_{i \in I} z_i a_{ij} > 0 \quad \text{for all } j \in J.$$

Then an equivalent recipe would be to define

$$(4.38) \qquad x_j = \alpha_j \text{ if } f_j < 0; \quad x_j = \beta_j \text{ if } f_j \geq 0, \quad \text{for all } j \in J.$$

with f as any linear combination, as in (4.29), such that α_j is finite when $f_j < 0$; β_j is finite with $f_j \geq 0$.

We already know that every special vector (of uniqueness) can be realized in this manner even with $f_j \neq 0$ for all $j \in J$. That every vector of the type (4.38) (with f as indicated) is a vector of uniqueness can be seen either by an easy direct proof (analogous to the proof

of Lemma 2.14) or else by introducing $\phi_j = f_j + \rho w_j$ $(j \in J)$ with $\rho > 0$ so small that $f_j < 0$ implies $\phi_j < 0$).

4.39. *Measures determined by their marginals.* Here, we change some of the previous notations. Let X be a fixed *finite* non-empty set and let $\pi_j: X \to Y_j$ $(j \in J)$ be given mappings, (where J can have any cardinality). Let further $-\infty \le \alpha(x) < \beta(x) \le +\infty$, $(x \in X)$, be given extended real numbers and let

(4.40) $M_0 = \{\phi: X \to R$ such that

$$\alpha(x) \le \phi(x) \le \beta(x), \text{ for all } x \in X\}.$$

Each such $\phi \in M_0$ can also be regarded as a finite signed measure μ on X with mass $\mu(\{x\}) = \phi(x)$ at $x \in X$. The associated π_j-marginal (of ϕ or μ) is given by

$$g_j(y) = (\pi_j \mu)(\{y\}) = \sum\{\phi(x): \pi_j x = y\}, \qquad (y \in Y_j; j \in J).$$

The following result characterizes the $\phi \in M_0$ which are uniquely determined by these marginals.

4.42. THEOREM. *Suppose X is finite. Let $\phi \in M_0$ and put*

(4.43) $X^0 = \{x \in X: \alpha(x) < \phi(x) < \beta(x)\}.$

In order that there exists no other $\psi \in M_0$ having the same marginals as ϕ, it is necessary and sufficient that:

(i) *There does not exist any non-zero function $h: X \to R$ which is supported by X^0 (that is, $h(x) = 0$ if $x \notin X^0$) and has all its marginals equal to zero.*

(ii) *There exists a function $f: X \to R$ of the special form*

(4.44) $f(x) = \sum_{j \in J} f_j(\pi_j x)$ *with $f_j: Y_j \to R$ $(j \in J)$,*

with only finitely many non-zero functions f_j, and such that

(4.45) $f(x) < 0$ *if* $\phi(x) = \alpha(x);$ $f(x) > 0$ *if* $\phi(x) = \beta(x);$
$$f(x) = 0 \text{ if } x \in X^0.$$

4.46. *Proof.* Apply Theorem 4.27, *extended* as in Remark 4.34. In the previous discussion, replace $J = \{1, \ldots, n\}$ by X; J^0 by X^0; j by x; x_j by $\phi(x)$; α_j by $\alpha(x)$ and β_j by $\beta(x)$. And finally replace the index set I by the set of pairs (j, y) where $j \in J$ and $y \in Y_j$. In view of (4.41), the condition that ϕ and ψ have the

same marginals is analogous to $Ax^0 = Ax$ with A as a matrix (one row for each choice of $j \in J$ and $y \in Y_j$; if J is infinite then there are infinity many rows). Clearly, the matrix element $a_{(j,y),x}$ which corresponds to the pair of indices $(j, y) \in I$ and $x \in X$, equals 1 if $\pi_j x = y$ and equals 0, otherwise. Therefore, (4.29) now takes the form (4.44), while condition (4.17) of Theorem 4.27 takes the form (4.45). Finally, the present condition (i) is precisely condition (ii) of Theorem 4.27.

4.47. *Additive sets.* For convenience assume that $\alpha(x), \beta(x)$ $(x \in X)$ are finite. Call $\phi: X \to R$ *special* if $\phi(x) \in \{\alpha(x), \beta(x)\}$, for all $x \in X$, and further no other $\psi \in M_0$ has the same marginals as ϕ. From Theorem 4.42, $\phi: X \to R$ is special if and only if

(4.48) $\phi(x) = \alpha(x)$ if $f(x) < 0$; $\phi(x) = \beta(x)$ if $f(x) > 0$,

for some $f: Y \to R$ of the form (4.44) and such that $f(x) \neq 0$ for all $x \in X$.

As an equivalent criterion, $\phi: X \to R$ is special if and only if

(4.49) $\phi(x) = \alpha(x)$ if $f(x) < 0$; $\phi(x) = \beta(x)$ if $f(x) \geq 0$,

for some $f: X \to R$ of the form (4.44). This follows from Remark 4.36. After all, $\sum_y a_{(j,y),x} = 1$, for all j and x so that condition (4.37) is satisfied.

The case $\alpha(x) = 0$; $\beta(x) = q(x)$ can be stated as follows, showing that the converse of Lemma 2.14 holds when X is finite. For the case that X and the π_j are as in Example 2.7, the result is largely due to Fishburn et al. [2].

4.50. THEOREM. *Let S be a subset of X, where X is finite. Then the following are equivalent.*

(i) *S is a set of uniqueness.*

(ii) *S is additive, that is, $S = \{x \in X: f(x) \geq 0\}$, for some $f \in F(S)$, see Definition 2.11.*

(iii) *$S = \{x \in X: f(x) \geq 0\}$ for some $f: X \to R$ of the form $f(x) = \sum_{j \in J} f_j(\pi_j x)$, with all but finitely many $f_j: Y_j \to R$ equal to zero.*

(iv) *Same as (iii), except that we require in addition that $f(x) \neq 0$, for all $x \in X$.*

4.51. REMARK. The reader should keep in mind that, when X is finite, there may exist subsets S of X such that, for no other *set T*

does $1_T\,d\lambda$ have the same projections as $1_S\,d\lambda$, while nevertheless $\phi = 1_S$ is *not* a (special) vector of uniqueness; (this cannot happen in the situation of Example 2.6, see [1], [4]). From Lemma 2.14, such a set S cannot possibly be additive. Natually, the explanation is that in such a situation there exists a function $\psi: X \to R$, not of the form $\psi(x) \in \{0, 1\}$, such that $\psi\,d\lambda$ has the same projections as $1_S\,d\lambda$. An example of such a set S was given in [2]. It uses the structure of Example 2.7 with $n = 3$; $|S| = 66$ and $X = \{1, 2, 3, 4, 5\}^3$ thus $|X| = 125$.

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