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REAL ANALYTIC REGULARITY OF THE SZEGÖ PROJECTION ON CIRCULAR DOMAINS

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In this note we show that $D \subseteq \mathbb{C}^n$, $n \geq 2$, is a smooth bounded pseudoconvex domain with real analytic defining function $r(z)$ such that $\sum_{k=1}^n z_k(\partial r/\partial z_k) \neq 0$ holds near some $x_0 \in bD$, then if $g \in C^\omega(bD)$, we have that the Szegö projection of g, Sg, is real analytic near x_0. In particular if D is a smooth bounded complete Reinhardt (or Reinhardt) pseudoconvex domain with real analytic boundary, then the Szegö projection S preserves real analyticity globally.

I. Introduction. Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded pseudoconvex domain. Denote by $L^2(bD)$ the space of square-integrable functions on the boundary and by $H^2(bD)$ the closed subspace of $L^2(bD)$ whose Poisson integrals are holomorphic in D. Then we define the Szegö projection S to be the orthogonal projection from $L^2(bD)$ onto $H^2(bD)$. It is represented by integration against the Szegö kernel function $S(\omega, z)$, i.e., for $f \in L^2(bD)$, we have

$$Sf(\omega) = \int_{bD} S(\omega, z) f(z)\, d\sigma_z.$$

The smooth regularity of the Szegö projection for a large class of domains has been established, for instance, see [1] [2], [3], [4], [8], [9], [10], [12]. Hence in this paper we are going to study the real analytic regularity of the Szegö projection on circular domains. The problems can be formulated as follows.

1. (Global version): Suppose that the boundary of D is real analytic, then does the Szegö projection S map analytic functions to analytic functions, i.e., $S\colon C^\omega(bD) \to C^\omega(bD)$?

2. (Local version): Suppose that the boundary of D is real analytic near some point $x_0 \in bD$, then does the Szegö projection S preserve real analyticity near x_0?

These problems are quite open. The only results we know so far are due to D. Tartakoff [13]. He showed the following.

THEOREM (*Tartakoff*). *On a real, real analytic CR-manifold of dimension $2n-1$ whose Levi form is non-degenerate and which satisfies $Y(q)$, then $\Box_b$ is locally real analytic hypoelliptic on (p, q)-forms.*

For global result in this case see also Tartakoff **[14]**.

The condition $Y(q)$ means that the Levi form has $\max(q+1, n-q)$ eigenvalues of the same sign or $\min(q+1, n-q)$ pairs of eigenvalues of opposite sign at each point. Therefore if one has a smooth bounded strictly pseudoconvex domain D in $\mathbb{C}^n$, $n \geq 3$, with real analytic boundary, then one can apply the following formula

$$S = \mathrm{Id} - \overline{\partial}_b^* N_b \overline{\partial}_b, \tag{1.1}$$

where N_b is the boundary Neumann operator, to show that the Szegö projection S preserves real analyticity locally (hence globally too). However, we must point out here that this theorem does not apply to the domains in $\mathbb{C}^2$, because condition $Y(1)$ is violated on such domains.

Very recently M. Derridj and D. Tartakoff [7] showed that if the defining function near $0 \in bD$ can be expressed as

$$\mathrm{Im}\, \omega = h(|z|^2) \tag{1.2}$$

with h real analytic and $h(0) = 0$, then again a local theorem holds near $0 \in bD$.

In this paper we prove the following main results.

THEOREM 1. *Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded pseudoconvex circular domain with real analytic defining function $r(z)$. Suppose that $\sum_{k=1}^n z_k(\partial r/\partial z_k) \neq 0$ holds near some $x_0 \in bD$ and that f is globally real analytic, i.e., $f \in C^\omega(bD)$. Then Sf is real analytic near x_0.*

We remark here that (i) Theorem 1 is also true in dimension two, (ii) it is not quite a local theorem, because we need f to be globally real analytic. It follows from Theorem 1 that we have

COROLLARY 2. *Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded pseudoconvex circular domain with real analytic defining function $r(z)$. If $\sum_{k=1}^n z_k(\partial r/\partial z_k) \neq 0$ holds for all $z \in bD$, then the Szegö projection preserves real analyticity globally.*

Since the transversal condition always holds on a complete Reinhardt domain, see Chen [**6**], we have

COROLLARY 3. *Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded complete Reinhardt pseudoconvex domain with real analytic boundary. Then the Szegö projection preserves real analyticity globally.*

Next if we drop the transversal condition, we can show

THEOREM 4. *Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded Reinhardt pseudoconvex domain with real analytic boundary. Then the Szegö projection preserves real analyticity globally.*

The author would like to thank Mei-chi Shaw for bringing this problem to his attention.

II. Proof of the main results. A domain D in $\mathbb{C}^n$ is called circular if $(z_1, \dots, z_n) \in D$ implies $(e^{i\theta} z_1, \dots, e^{i\theta} z_n) \in D$ for all $\theta \in \mathbb{R}$, and D is called Reinhardt (or multi-circular) if $(z_1, \dots, z_n) \in D$ implies $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in D$ for $\theta_1, \dots, \theta_n \in \mathbb{R}$. D is called complete Reinhardt if $(z_1, \dots, z_n) \in D$ implies $(\omega_1, \dots, \omega_n) \in D$ with $|\omega_j| \leq |z_j|$ for all $j = 1, \dots, n$.

First we recall that the Szegö kernel function $S(\omega, z)$ can be represented as follows. Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis for $H^2(bD)$. Here we have identified each element $f \in H^2(bD)$ with its Poisson integral. Then we have

$$S(\omega, z) = \sum_{j=1}^{\infty} \varphi_j(\omega) \overline{\varphi_j(z)}, \quad \forall \omega, z \in D, \tag{2.1}$$

and the expression is independent of the choice of the basis.

Next we prove some basic facts on circular domains. Define an S^1-action on D as follows,

$$\begin{aligned} \pi &: S^1 \times \overline{D} \to \overline{D}, \\ &(e^{i\theta}, z) \mapsto \omega = e^{i\theta} \cdot z = (e^{i\theta} z_1, \dots, e^{i\theta} z_n). \end{aligned}$$

Then for each fixed θ, $\pi_b = \pi|_{bD}$ is a CR-diffeomorphism of bD.

LEMMA 2.2. *Let $D \subseteq \mathbb{C}^n$, $n \geq 2$, be a smooth bounded circular domain. Then for each fixed θ we have $\pi_b^* d\sigma_\omega = d\sigma_z$, where $d\sigma$ denotes the surface element on bD.*

Proof. Let $r(\omega)$ be the defining function for D_ω. Here we use subscripts to emphasize the domain we consider. Then $r \circ \pi(z)$ will be a defining function for D_z. By using the $*$-operator (cf. **[11]**), we have

$$\pi_b^* d\sigma_\omega = \pi_b^* \left(\frac{2}{\|dr\|} i_\omega^*(*\partial r) \right),$$

where $i \colon D \hookrightarrow \mathbb{C}^n$ is the inclusion map. Then by direct computation, we get

$$\begin{aligned}
\pi_b^* d\sigma_\omega &= \frac{2}{\|d(r \circ \pi)\|} \pi_b^* i_\omega^* \left(\sum_{k=1}^n \frac{1}{2^{n-1} \cdot i^n} \frac{\partial r}{\partial \omega_k}(\omega) d\omega_k \bigwedge_{j \neq k} (d\overline{\omega}_j \wedge d\omega_j) \right) \\
&= \frac{2}{\|d(r \circ \pi)\|} i_z^* \cdot \pi^* \left(\sum_{k=1}^n \frac{1}{2^{n-1} \cdot i^n} \frac{\partial r}{\partial \omega_k}(\omega) d\omega_k \bigwedge_{j \neq k} (d\overline{\omega}_j \wedge d\omega_j) \right) \\
&= \frac{2}{\|d(r \circ \pi)\|} i_z^*(*\partial(r \circ \pi)) \\
&= d\sigma_z.
\end{aligned}$$

This completes the proof of the lemma.

From Lemma 2.2 and the representation of the Szegö kernel function we obtain immediately the following transformation law for the Szegö kernel on circular domains.

LEMMA 2.3. *Under the hypotheses of Lemma* 2.2, *we have* $S(\omega, z) = S(\pi(\omega), \pi(z))$.

Now we define the crucial vector field T on bD as follows. Let z be a point near the boundary. Define

$$\begin{aligned}
\Lambda_z \colon S^1 &\to \overline{D} \\
e^{i\theta} &\mapsto e^{i\theta} \cdot z = (e^{i\theta} z_1, \dots, e^{i\theta} z_n),
\end{aligned}$$

and denote by $\Lambda_{z,*}$ the differential mapping of Λ_z. Then define the vector field T at z by

$$\begin{aligned}
T(z) = \frac{1}{2} \Lambda_{z,*} \left(\left. \frac{\partial}{\partial \theta} \right|_{\theta=0} \right) &= \frac{i}{2} \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} - \frac{i}{2} \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \\
&= \frac{i}{2} X - \frac{i}{2} \overline{X},
\end{aligned}$$

where $X = \sum_{j=1}^n z_j (\partial / \partial z_j)$.

The vector field T defined in this way has many nice properties. First of all, if $(\partial r/\partial z_n)(x_0) \neq 0$, then

$$L_k = \frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_n} - \frac{\partial r}{\partial z_n}\frac{\partial}{\partial z_k}, \qquad k = 1, \dots, n-1,$$

forms a local basis for $T^{1,0}(bD)$. Hence by assumption $Xr \neq 0$ near x_0, $L_1, \dots, L_{n-1}$, $\overline{L}_1, \dots, \overline{L}_{n-1}$ and T form a basis for complex tangent space on bD near x_0, and it is shown in **[6]** that we have

$$(2.4) \quad [T, L_k] = -iL_k, \quad [T, \overline{L}_k] = i\overline{L}_k, \quad \text{for } k = 1, \dots, n-1.$$

The next lemma shows that T commutes with the Szegö projection S on such domains.

LEMMA 2.5. *Under the hypotheses of Theorem* 1, *we have* $TSu = STu$ *for all* $u \in C^\infty(bD)$.

Proof. Since $Su \in C^\infty(bD)$, one can apply T to Su. Also by the construction of T, we get by writing $z = e^{it} \cdot \eta$,

$$\begin{aligned} 2TSu(\omega) &= \frac{d}{dt}\int_{bD} S(e^{it}\cdot\omega, z)u(z)\, d\sigma_z|_{t=0} \\ &= \frac{d}{dt}\int_{bD} S(e^{it}\cdot\omega, e^{it}\cdot\eta)u(e^{it}\cdot\eta)\, d\sigma_\eta|_{t=0} \\ &= \frac{d}{dt}\int_{bD} S(\omega, \eta)u(e^{it}\cdot\eta)\, d\sigma_\eta|_{t=0} \\ &= 2STu(\omega). \end{aligned}$$

This completes the proof of the lemma.

Now we begin to prove Theorem 1. So we assume that $f \in C^\omega(bD)$. It follows from the smooth regularity of the Szegö projection that we have $Sf \in C^\infty(bD)$. Denote by L (or $\overline{L}$) any of the L_i's (or $\overline{L}_i$'s) with $i \leq n-1$, and by $\overset{(-)}{L}$ any of L_i's or $\overline{L}_i$'s with $i \leq n-1$. Also denote by Z either $\overset{(-)}{L}$ or T. Denote by $a_{(1)}$ any of the finite collection of analytic functions that occur in commutators of L, $\overline{L}$ and T and integration by parts. Let $a_{(j+1)} = a_{(1)} \cdot a_{(j)}$ or $a_{(j+1)} = Za_{(j)}$. Hence there exists a constant $R > 0$ such that for all $k \in \mathbb{N}$, we have

$$(2.6) \qquad |a_{(k)}| \leq RR^k k!.$$

The proof for Theorem 1 will be complete if one can show

$$\|Z_1 \cdots Z_q S f\|_{L^2(U)} \leq M M^q q!, \tag{2.7}$$

for all $q \in \mathbb{N}$ and some open neighborhood U of x_0 and some $M > 0$ uniformly in q. We may assume that $Xr \neq 0$ holds on some neighborhood of $\overline{U}$.

Let φ be a cut-off function with $\varphi \equiv 1$ in some neighborhood of x_0 and supp φ contained in U. Denote by φ' any first derivative of φ. Then by Lemma 2.5 one can estimate the pure terms quite easily,

$$\|\varphi T^p S f\| = \|\varphi S T^p f\| \leq \|T^p f\| \leq R_f R_f^p p!, \tag{2.8}$$

for some $R_f > 0$. To estimate the mixed terms we use $\mathrm{Op}(k, q)$ to denote any differential operator of order q formed out of the $\overset{(-)}{L}$ and T in all order with precisely $k\,\overset{(-)}{L}$'s. Thus if $k = 0$, by (2.8), we have

$$\|\varphi\, \mathrm{Op}(0, q) S f\| = \|\varphi T^q S f\| \leq R_f R_f^q q!.$$

Hence what remains in this note is to estimate the term $\mathrm{Op}(s, q)Sf$ with $s \geq 1$. We will use underline to mean at most such terms are being considered and c is a constant depending only on n. First we write

$$\begin{aligned} \mathrm{Op}(s, p) &= T^r \overset{(-)}{L}\, \mathrm{Op}(s-1, p-r-1) \\ &= \overset{(-)}{L}\, T^r\, \mathrm{Op}(s-1, p-r-1) \\ &\quad + \sum_{j=1}^{r} (\pm) i T^{r-j} \overset{(-)}{L}\, T^{j-1}\, \mathrm{Op}(s-1, p-r-1) \\ &= \overset{(-)}{L}\, \mathrm{Op}(s-1, p-1) + \sum_{j=1}^{r} (\pm) i\, \mathrm{Op}(s, p-1). \end{aligned} \tag{2.9}$$

Define

$$\begin{aligned} I_p(\varphi) = \|\varphi\, \mathrm{Op}(s, p) S f\| &+ \|\varphi T\, \mathrm{Op}(s-1, p-1) S f\| \\ &+ \sum_{i=1}^{n-1} (\|\varphi L_i\, \mathrm{Op}(s-1, p-1) S f\| \\ &\qquad + \|\varphi \overline{L}_i\, \mathrm{Op}(s-1, p-1) S f\|). \end{aligned}$$

By (2.9) we see that

$$\begin{aligned} I_p(\varphi) \leq 2 \sum_{i=1}^{n-1} &(\|\varphi L_i\, \mathrm{Op}(s-1, p-1) S f\| + \|\varphi \overline{L}_i\, \mathrm{Op}(s-1, p-1) S f\|) \\ &+ \|\varphi T\, \mathrm{Op}(s-1, p-1) S f\| + (p-1) \|\varphi\, \mathrm{Op}(s, p-1) S f\|. \end{aligned}$$

We estimate the first term as follows:

$$\begin{aligned}
&\|\varphi L\,\mathrm{Op}(s-1,p-1)Sf\|^2\\
&\quad=(\varphi L\,\mathrm{Op}(s-1,p-1)Sf,\ \varphi L\,\mathrm{Op}(s-1,p-1)Sf)\\
&\quad=\|\varphi\overline{L}\,\mathrm{Op}(s-1,p-1)Sf\|^2\\
&\qquad+(\varphi\,\mathrm{Op}(s-1,p-1)Sf,\ -\varphi[\overline{L},L]\,\mathrm{Op}(s-1,p-1)Sf)\\
&\qquad+2(\varphi'\,\mathrm{Op}(s-1,p-1)Sf,\ \varphi L\,\mathrm{Op}(s-1,p-1)Sf)\\
&\qquad+2(\varphi'\,\mathrm{Op}(s-1,p-1)Sf,\ \varphi\overline{L}\,\mathrm{Op}(s-1,p-1)Sf)\\
&\qquad+(a_{(1)}\varphi\,\mathrm{Op}(s-1,p-1)Sf,\ \varphi L\,\mathrm{Op}(s-1,p-1)Sf)\\
&\qquad+(a_{(1)}\varphi\,\mathrm{Op}(s-1,p-1)Sf,\ \varphi\overline{L}\,\mathrm{Op}(s-1,p-1)Sf).
\end{aligned}$$

It shows that for $C_1>0$ we have

$$\begin{aligned}
&\|\varphi L\,\mathrm{Op}(s-1,p-1)Sf\|\\
&\quad\le\|\varphi\overline{L}\,\mathrm{Op}(s-1,p-1)Sf\|\\
&\qquad+C_1(\|\varphi\,\mathrm{Op}(s-1,p-1)Sf\|+4\|\varphi'\,\mathrm{Op}(s-1,p-1)Sf\|\\
&\qquad\qquad+2\|a_{(1)}\varphi\,\mathrm{Op}(s-1,p-1)Sf\|)\\
&\qquad+\frac{1}{C_1}(3\|\varphi L\,\mathrm{Op}(s-1,p-1)Sf\|+3\|\varphi\overline{L}\,\mathrm{Op}(s-1,p-1)Sf\|\\
&\qquad\qquad+\|\varphi[\overline{L},L]\,\mathrm{Op}(s-1,p-1)Sf\|).
\end{aligned}$$

LEMMA 2.11. (i) $[\overline{L},L]=\underline{(2n-1)}a_{(1)}Z$.
(ii)

$$\begin{aligned}
\overline{L}\,\mathrm{Op}(s-1,p-1)&=\mathrm{Op}(s-1,p-1)\overline{L}\\
&\quad+\sum_{j=1}^{p-1}\underline{(2(2n-1))^j\binom{p-1}{j}}a_{(j)}Z_jZ_{j+1}\cdots Z_{p-1},
\end{aligned}$$

where Z_j denotes any Z. Hence one may choose $c=2(2n-1)$.

Proof. (i) is trivial. For (ii) we have

$$\begin{aligned}
[\overline{L},\ \mathrm{Op}(s-1,p-1)]&=[\overline{L},Z_1\cdots Z_{p-1}]\\
&=\sum_{j=1}^{p-1}\underline{\binom{p-1}{j}}[[\cdots[\overline{L},Z_1]\cdots],Z_j]Z_{j+1}\cdots Z_{p-1}.
\end{aligned}$$

Hence the conclusion follows immediately.

Since Sf is annihilated by $\overline{L}$, we obtain

$$
\begin{aligned}
(2.12)\qquad I_p(\varphi) \le 2\sum_{i=1}^{n-1}\{2\|\varphi\overline{L}_i\operatorname{Op}(s-1,p-1)Sf\| \\
&+ C_1(\|\varphi\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + 4\|\varphi'\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + 2R^2\|\varphi\operatorname{Op}(s-1,p-1)Sf\|) \\
&+ \frac{1}{C_1}(3\|\varphi L_i\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + 3\|\varphi\overline{L}_i\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + cR^2\|\varphi Z\operatorname{Op}(s-1,p-1)Sf\|)\} \\
&+ \|\varphi T\operatorname{Op}(s-1,p-1)Sf\| \\
&+ (p-1)\|\varphi\operatorname{Op}(s,p-1)Sf\|.
\end{aligned}
$$

Choose $C_1 = \max(24, 8ncR^2)$; then we get

$$
\begin{aligned}
(2.13)\quad I_p(\varphi) &\le 8\sum_{i=1}^{n-1}\sum_{j=1}^{p-1}\underline{c^j\binom{p-1}{j}}\|a_{(j)}\varphi Z_jZ_{j+1}\cdots Z_{p-1}Sf\| \\
&\quad + 4(n-1)C_1(\|\varphi\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + 4\|\varphi'\operatorname{Op}(s-1,p-1)Sf\| \\
&\qquad + 2R^2\|\varphi\operatorname{Op}(s-1,p-1)Sf\|) \\
&\quad + 2\|\varphi T\operatorname{Op}(s-1,p-1)Sf\| \\
&\quad + 2(p-1)\|\varphi\operatorname{Op}(s,p-1)Sf\| \\
&\le 8(n-1)\sum_{j=1}^{p-1}\underline{c^j\binom{p-1}{j}}\cdot RR^j j! I_{p-1}(\varphi) \\
&\quad + 16(n-1)C_1 I_{p-1}(\varphi') \\
&\quad + 4(n-1)C_1(1+2R^2)I_{p-1}(\varphi) \\
&\quad + 2\|\varphi\operatorname{Op}(s-1,p)Sf\| + 2(p-1)I_{p-1}(\varphi).
\end{aligned}
$$

Since the vector field $\overset{(-)}{L}$ in general is defined only locally, we need a special cut-off function which is of compact support, but behaves like an analytic function up to certain order.

LEMMA (*Ehrenpreis*). *Let $x_0 \in bD$ and U_1, U_2 be two neighborhoods of x_0 with $U_1 \Subset U_2 \Subset U$; then there exists a constant $M > 0$ such that for any integer k one can find $\varphi_k \in C_0^\infty(U_2)$ with*

$0 \le \varphi_k \le 1$, $\varphi_k \equiv 1$ *on* U_1 *and satisfying*

$$|D^\alpha \varphi_k| \le M(Mk)^{|\alpha|}, \qquad \textit{for } |\alpha| \le k+1. \tag{2.14}$$

Now one can replace φ by $\varphi_p^{(b)}$, some derivative of φ_p of order $\le b$, and p by $p-b$ in (2.13). Then we obtain

$$\begin{aligned}(2.15)\, I_{p-b}(\varphi_p^{(b)}) \le 8(n-1) & \sum_{j=1}^{p-b-1} \underline{c^j \tbinom{p-b-1}{j}} \cdot RR^j j! I_{p-b-j}(\varphi_p^{(b)}) \\ & + 16(n-1)C_1 I_{p-b-1}(\varphi_p^{(b+1)}) \\ & + 4(n-1)C_1(1+2R^2) I_{p-b-1}(\varphi_p^{(b)}) \\ & + 2\|\varphi_p^{(b)} \operatorname{Op}(s-1, p-b) Sf\| \\ & + 2(p-b-1) I_{p-b-1}(\varphi_p^{(b)}).\end{aligned}$$

There are five terms in (2.15). Inductively we will show that for $p \ge b \ge 0$, we have

$$I_{p-b}(\varphi_p^{(b)}) \le R_1 (R_2 p)^b (R_3 p)^{p-b} R_4^s, \tag{2.16}$$

where R_1, R_2, R_3 and R_4 are some constants, i.e., we will show that each term can be bounded by $\frac{1}{5} R_1 (R_2 p)^b (R_3 p)^{p-b} R_4^s$. The initial step $s=0$ or $p=b$ is easy to check. Hence we assume that (2.16) is true for $b > b_0$ or $s < s_0$. Then we prove the case $b = b_0$ and $s = s_0$.

$$\begin{aligned}\text{Term } 1 & \le 8(n-1) \sum_{j=1}^{p-b_0-1} \frac{(p-b_0-1)!}{(p-b_0-j-1)!j!} \\ & \qquad \cdot R(cR)^j j! R_1 (R_2 p)^{b_0} (R_3 p)^{p-b_0-j} \cdot R_4^{s_0} \\ & = R_1 (R_2 p)^{b_0} (R_3 p)^{p-b_0} \cdot R_4^{s_0} \\ & \qquad \cdot \sum_{j=1}^{p-b_0-1} \frac{(p-b_0-1)!}{(p-b_0-j-1)! p^j} \cdot \frac{8(n-1)R(cR)^j}{R_3^j} \\ & \le \frac{1}{5} R_1 (R_2 p)^{b_0} (R_3 p)^{p-b_0} \cdot R_4^{s_0},\end{aligned}$$

provided R_3 is chosen large enough.

$$\begin{aligned}\text{Term } 2 & \le 16(n-1)C_1 R_1 (R_2 p)^{b_0+1} \cdot (R_3 p)^{p-b_0-1} \cdot R_4^{s_0} \\ & = R_1 (R_2 p)^{b_0} \cdot (R_3 p)^{p-b_0} \cdot R_4^{s_0} \cdot \left(\frac{16(n-1)C_1 R_2}{R_3} \right) \\ & \le \frac{1}{5} R_1 (R_2 p)^{b_0} (R_3 p)^{p-b_0} \cdot R_4^{s_0},\end{aligned}$$

provided $R_3 \ge 80(n-1)C_1 R_2$.

$$\begin{aligned}\text{Term } 3 &\le 4(n-1)C_1(1+2R^2)\cdot R_1(R_2p)^{b_0}(R_3p)^{p-b_0-1}\cdot R_4^{s_0}\\ &= R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0}\cdot\left(\frac{4(n-1)C_1(1+2R^2)}{R_3p}\right)\\ &\le \frac{1}{5}R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0},\end{aligned}$$

provided $R_3 \ge 20(n-1)C_1(1+2R^2)$.

$$\begin{aligned}\text{Term } 4 &\le 2R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0-1}\\ &\le \frac{1}{5}R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0},\end{aligned}$$

provided $R_4 \ge 10$.

$$\begin{aligned}\text{Term } 5 &\le 2(p-b_0-1)R_1(R_2p)^{b_0}(R_3p)^{p-b_0-1}\cdot R_4^{s_0}\\ &= R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0}\left(\frac{2(p-b_0-1)}{R_3p}\right)\\ &\le \frac{1}{5}R_1(R_2p)^{b_0}(R_3p)^{p-b_0}\cdot R_4^{s_0},\end{aligned}$$

provided $R_3 \ge 10$. This completes the proof of (2.16). In particular, we have shown that for $k \le p$,

$$\begin{aligned}\|\varphi_p \operatorname{Op}(k,p)Sf\| &\le I_p(\varphi_p) \le R_1(R_3p)^pR_4^k\\ &\le R_1(R_3R_4p)^p\\ &\le MM^pp!,\end{aligned}\tag{2.17}$$

for some large constant $M > 0$ depending only on R_1, R_3 and R_4. This also completes the proof of Theorem 1.

To prove Theorem 4 one can apply the above techniques almost verbally with slight modification. The key point is that for every point $x_0 \in bD$ we have to choose a vector field like T for Theorem 1 which is transversal to $T^{1,0}(bD)\oplus T^{0,1}(bD)$ locally near x_0 and commutes nicely with $\overset{(-)}{L}$ and the Szegö projection. This can be done easily. By rotational symmetry of the domain one can choose a direction, say n, such that $(z_n(\partial r/\partial z_n))(x_0) \ne 0$ holds in some neighborhood of x_0. Then define S^1-action on $\overline{D}$ as follows.

$$\begin{aligned}&\pi\colon S^1\times\overline{D}\to\overline{D}\\ &(e^{i\theta}, z)\mapsto e^{i\theta}\cdot z = (z_1,\dots,z_{n-1}, e^{i\theta}z_n).\end{aligned}$$

Such an action will generate a vector field $T_n = iz_n(\partial/\partial z_n) - i\overline{z}_n(\partial/\partial\overline{z}_n)$ with the desired properties. So we are done. For more details in this case see [**5**].

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