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AN INTRINSIC CHARACTERIZATION OF A CLASS OF MINIMAL SURFACES IN CONSTANT CURVATURE MANIFOLDS

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Let X be an N-manifold of constant sectional curvature. A class of minimal surfaces in X, called exceptional minimal surfaces, will be defined in terms of the structure of their normal bundles. It will be shown that these surfaces can be characterized intrinsically in a way that generalizes the Ricci condition for minimal surfaces in Euclidean 3-space. It will also be shown that these surfaces are rigid when N is even and belong to 1-parameter families of isometric surfaces when N is odd.

0. Introduction. Let $X^N(c)$ denote an N-dimensional manifold of constant sectional curvature c, and suppose that M is a minimal surface in $X^N(c)$ with Riemannian metric ds^2 and Gauss curvature K. The classical theorem of Ricci, as extended by Lawson [3], says that when $N = 3$ minimal surfaces of $X^3(c)$ are characterized by the conditions that $K \leq c$ and at points where $K < c$ the metric $d\hat{s}^2 = \sqrt{c-k}\,ds^2$ is flat. Moreover, for each minimal surface M in $X^3(c)$, there is a 1-parameter family of isometric minimal surfaces M_τ, $0 \leq \tau < 2\pi$, such that M is congruent to one of the members of this family.

This paper will describe a class of minimal surfaces in $X^N(c)$, called exceptional minimal surfaces, and a sequence of functions $A_1^c, A_2^c, \ldots$ on each surface such that when $N = 2n+1$, these surfaces are characterized by the conditions that $A_r^c \geq 0$, $1 \leq r \leq n$, and at points where each $A_r^c > 0$, the metric $d\hat{s}^2 = (A_n^c)^{1/(n+1)}\,ds^2$ is flat. This reduces to the Ricci-Lawson condition when $n = 1$, in that $A_1^c = c - K$. The exceptional minimal surfaces in $X^{2n+1}(c)$ will be seen to belong to 1-parameter families of isometric surfaces, just as happens in $X^3(c)$.

In $X^{2n+2}(c)$, the exceptional minimal surfaces will be characterized by the conditions that $A_r^c \geq 0$, $1 \leq r \leq n$, and $A_{n+1}^c \equiv 0$. Additionally, in $X^{2n+2}(c)$ the exceptional minimal surfaces will be rigid. These results given here for the case where $N = 2n+2$ are actually implicit in [2], although they are stated there in terms of minimal immersions of the 2-sphere S^2 into $X^{2n+2}(c)$.

Sections 1 and 2 summarize the structure equations for minimal surfaces and some results from [2] that are needed here. Section 3 contains the statements of the theorems, which are proved in §§4 and 5. Section 6 contains two corollaries on isometric minimal surfaces.

1. Structure equations of surfaces. Suppose M is a Riemannian surface with Gauss curvature K. Let e_1, e_2 be a local orthonormal frame field on M, and let θ_1, θ_2 be the coframe dual to e_1, e_2. Then the structure equations of M are

(1) $d\theta_1 = \omega_{12} \wedge \theta_2, \quad d\theta_2 = -\omega_{12} \wedge \theta_2, \quad \text{and} \quad d\omega_{12} = -K\theta_1 \wedge \theta_2,$

where ω_{12} $(= -\omega_{21})$ is the connection form on M, and K is the Gauss curvature of M.

If $f\colon M \to \mathbb{R}$ is a smooth function, let f_1, f_2 be given by

$$df = f_1\theta_1 + f_2\theta_2.$$

Taking the exterior derivative of this expression and applying Cartan's Lemma gives f_{11}, f_{12}, f_{21}, and f_{22}, with $f_{12} = f_{21}$, such that

$$\begin{aligned} df_1 - f_2\omega_{12} &= f_{11}\theta_1 + f_{12}\theta_2, \\ df_2 + f_1\omega_{12} &= f_{21}\theta_1 + f_{22}\theta_2. \end{aligned}$$

If Δ is the Laplace-Beltrami operator of M, then $\Delta f = f_{11} + f_{22}$. Let $\overline{\partial} f = (f_1 + if_2)/2$ and $\varphi = \theta_1 + i\theta_2$. Then

$$\Delta f \varphi \wedge \overline{\varphi} = 4(d\overline{\partial} f + i\overline{\partial} f \omega_{12}) \wedge \overline{\varphi}. \tag{2}$$

Note that this is not the usual $\overline{\partial}$-operator. If $z = x + iy$ gives local isothermal coordinates on M, then there is a positive function λ such that $ds^2 = \lambda^2|dz|^2$. The $\overline{\partial}$ defined here is λ times the usual $\overline{\partial}$-operator.

2. Structure equations and normal planes. Suppose M is a minimal surface in $X^N(c)$. When clear from context, this latter manifold will be denoted simply as X. Assume that M *lies fully* in X, i.e., does not lie in a totally geodesic submanifold of X. Let the integer n be given by $N = 2n + 1$ or $2n + 2$, and let indices have the following ranges unless otherwise indicated:

$$1 \le j, k \le 2, \quad 3 \le \alpha, \beta \le N, \quad 1 \le A, B, C \le N, \quad 1 \le p, q, r \le n.$$

(The symbol i will be reserved for $\sqrt{-1}$.)

Let $\tilde{e}_A$ be a local orthonormal frame field on X, and let $\tilde{\theta}_A$ be the coframe dual to $\tilde{e}_A$. Then the structure equations of X are

$$(3)\quad d\tilde{\theta}_A = \sum_B \tilde{\omega}_{AB} \wedge \tilde{\theta}_B \quad \text{and} \quad d\tilde{\omega}_{AB} = \sum_C \tilde{\omega}_{AC} \wedge \tilde{\omega}_{CB} - c\tilde{\theta}_A \wedge \tilde{\theta}_B$$

where the $\tilde{\omega}_{AB}$ $(= -\tilde{\omega}_{BA})$ are the connection forms on X. If $\langle \cdot , \cdot \rangle$ is the Riemannian metric on X, then $\tilde{\omega}_{AB} = \langle d\tilde{e}_A , \tilde{e}_B \rangle$.

Suppose that e_1, e_2 is a frame on M as described in the previous section and that the frame $\tilde{e}_A$ is chosen so that on M, $e_j = \tilde{e}_j$, and the $\tilde{e}_\alpha$ are normal to M. Then differential forms on M can be identified with differential forms on X restricted to M:

$$\theta_j = \tilde{\theta}_j|_M \quad \text{and} \quad \omega_{12} = \tilde{\omega}_{12}|_M.$$

To simplify the notation, when forms and vectors on X are restricted to M, let them be denoted by the same symbol without tilde:

$$\begin{aligned} \theta_A &\ \text{ will denote } \tilde{\theta}_A|_M , \\ \omega_{AB} &\ \text{ will denote } \tilde{\omega}_{AB}|_M , \quad \text{and} \\ e_A &\ \text{ will denote } \tilde{e}_A|_M . \end{aligned}$$

Then $\theta_\alpha = 0$ on M since the e_α are normal to M. When the relation $d\theta_\alpha = 0$ is expanded using the structure equations (3), Cartan's Lemma can be applied to show that there are functions $h_{\alpha jk}$ such that

$$(4)\qquad \omega_{\alpha j} = \sum_k h_{\alpha jk}\theta_k , \qquad h_{\alpha jk} = h_{\alpha kj}.$$

The $h_{\alpha jk}$ are the coefficients of the second fundamental form. The assumption that M is a minimal surface is equivalent to assuming that the second fundamental form has zero trace:

$$(5)\qquad h_{\alpha 11} + h_{\alpha 22} = 0.$$

Let T_xM and T_xX denote the tangent space to M and X, respectively, at a point x. Curves on M through x have their first derivatives at x in T_xM, but higher order derivatives will have components normal to M. The space spanned by the derivatives of order up to r is called the *rth osculating space of* M at x, denoted $T_x^{(r)}M$. If for all r, $T_x^{(r)}M \subsetneqq T_xX$ at all x in a neighborhood of M, then that neighborhood would lie in a totally geodesic submanifold of X (**[4]**, p. 241). The assumption that M lies fully in X means that for some r, $T_x^{(r)}M = T_xX$ at generic points of M.

The *rth normal space* of M at x, denoted $\mathrm{Nor}_x^{(r)}M$, is the orthogonal complement of $T_x^{(r)}M$ in $T_x^{(r+1)}M$, so

$$T_x^{(r+1)}M = T_x^{(r)}M \oplus \mathrm{Nor}_x^{(r)}M.$$

The results in §2 and §3 of [2] show that at generic points of M, the dimension of $\mathrm{Nor}_x^{(r)}M$ is 2 when $1 \le r \le n-1$, and the dimension of $\mathrm{Nor}_x^{(n)}M$ is 1 or 2, depending on whether N is odd or even. Those normal spaces that have dimension 2 will be called the *normal planes* of M. Let β_N denote the number of normal planes possessed by M at a generic point:

$$\beta_N = \begin{cases} n, & \text{if } N = 2n+2, \\ n-1, & \text{if } N = 2n+1. \end{cases}$$

Choose the normal vectors e_α so that $\mathrm{Nor}_x^{(r)}M$ is spanned by $\{e_{2r+1}, e_{2r+2}\}$, $1 \le r \le \beta_N$. When $N = 2n+1$, $\mathrm{Nor}_x^{(n)}M$ will be spanned by $\{e_{2n+1}\}$. The derivatives of vector fields in $T_x^{(r)}M$ must lie in $T_x^{(r+1)}M$, so de_{2r-1} and de_{2r} cannot have any e_α components for $\alpha > 2r+2$. Since $\omega_{AB} = \langle de_A, e_B\rangle$ and $\omega_{AB} = -\omega_{BA}$,

$$\omega_{2r-1,A} = \omega_{2r,A} = 0 \quad \text{when } A > 2r+2 \text{ and when } A < 2r-3.$$

When $r = 2$, these relations imply that

$$h_{\alpha jk} = 0 \quad \text{when } \alpha > 4. \tag{6}$$

Set $H_\alpha = h_{\alpha 11} + ih_{\alpha 12}$ for $\alpha = 3$ and 4. Then using (5), (6), and the form $\varphi = \theta_1 + i\theta_2$, equations (4) can be written

$$\omega_{\alpha 1} + i\omega_{\alpha 2} = H_\alpha \overline{\varphi} \quad \text{for } \alpha = 3, 4,$$

$$\omega_{\alpha j} = 0 \quad \text{for } \alpha > 4.$$

Now applying the structure equations (3) to the relation $\omega_{51} + i\omega_{52} = 0$ leads to

$$(H_3\omega_{53} + H_4\omega_{54}) \wedge \overline{\varphi} = 0.$$

So for some H_5,

$$H_3\omega_{53} + H_4\omega_{54} = H_5\overline{\varphi}.$$

Applying this argument inductively to the relations

$$\omega_{2r+1,2r-3} + i\omega_{2r+1,2r-2} = 0,$$

$$\omega_{2r+2,2r-3} + i\omega_{2r+2,2r-2} = 0,$$

for $r = 2, \dots, n$, produces H_α such that

$$H_{2r-1}\omega_{\alpha,2r-1} + H_{2r}\omega_{\alpha,2r} = H_\alpha\overline{\varphi}, \tag{7}$$

for $\alpha = 2r+1$ and $2r+2$. This relation extends to the case where $r = 1$ by setting $H_1 = 1$, and $H_2 = i$.

The rth normal plane, $\mathrm{Nor}_x^{(r)} M$, of M will be called *exceptional* if $H_{2r+2} = \pm i H_{2r+1}$. (Note that the sign can be reversed by reversing the orientation of $\mathrm{Nor}_x^{(r)} M$. Note also that when $N = 2n+1$, $\mathrm{Nor}_x^{(n)} M$ is a line, not a plane, and the notion of exceptionality does not apply.) The minimal surface M will be called *exceptional* if all of its normal planes are exceptional. Minimal immersions of the 2-sphere S^2 into $X^{2n+2}(c)$ are always exceptional [**2**]. (These surfaces are called "superminimal" by Bryant [**1**].)

For the remainder of this paper, assume that M is an exceptional minimal surface. The orientations of the normal planes of such a surface can be chosen so that $H_{2r+2} = +iH_{2r+1}$ for $1 \leq r \leq \beta_N$. Then equations (7) become

$$(8) \qquad \begin{aligned} H_{2r-1}(\omega_{2r+1,2r-1} + i\omega_{2r+1,2r}) &= H_{2r+1}\overline{\varphi}, \\ H_{2r-1}(\omega_{2r+2,2r-1} + i\omega_{2r+2,2r}) &= iH_{2r+1}\overline{\varphi}, \end{aligned}$$

for $r = 1, \ldots, \beta_N$, together with

$$(9) \qquad H_{2n-1}(\omega_{2n+1,2n-1} + i\omega_{2n+1,2n}) = H_{2n+1}\overline{\varphi}$$

when $N = 2n+1$.

3. The theorems. For each real number c, the quantities A_p^c are defined as follows:

$$\begin{aligned} A_0^c &= 1/2 \text{ for all } c, \\ A_1^c &= c - K, \\ A_{p+1}^c &= \begin{cases} A_p^c[\Delta \log(A_p^c) + A_p^c/A_{p-1}^c - 2(p+1)K], & \text{if } A_p^c > 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

THEOREM A. *Suppose M is an exceptional minimal surface lying fully in $X^N(c)$. Let K denote the Gauss curvature of M, ds^2 its Riemannian metric, and let n be given by $N = 2n+1$ or $2n+2$. Then $A_p^c \geq 0$ for $1 \leq p \leq n$, with equality only at isolated points. If $N = 2n+2$, then $A_{n+1}^c \equiv 0$, and if $N = 2n+1$, then at points where each $A_p^c > 0$, the metric $d\hat{s}^2 = (A_n^c)^{1/(n+1)} ds^2$ is flat (has Gauss curvature $\hat{K} \equiv 0$).*

THEOREM B. *Suppose M is a smooth Riemannian surface with Gauss curvature K and Riemannian metric ds^2. Suppose $A_p^c > 0$ for $1 \leq p \leq n$ in some neighborhood of $x_0 \in M$. If $A_{n=1}^c \equiv 0$, set*

$N = 2n+2$. *If the metric* $d\hat{s}^2 = (A_n^c)^{1/(n+1)}\,ds^2$ *is flat, set* $N = 2n+1$. *Then there is a neighborhood* U *of* x_0 *and an isometric immersion* $f\colon U \to X^N(c)$ *such that* $f(U)$ *is an exceptional minimal surface lying fully in* $X^N(c)$.

The theorems will be proved in §4 and §5. Both proofs will use the following:

LEMMA. *The condition that* $d\hat{s}^2 = (A_n^c)^{1/(n+1)}\,ds^2$ *should be flat is equivalent to*

$$\Delta \log(A_n^c) - 2(n+1)K \equiv 0. \tag{10}$$

Proof. Recall that if $ds^2 = \lambda^2|dz|^2$ for isothermal coordinates $z = x+iy$, then the Gauss curvature is given by $K = -\Delta\log\lambda$. If $d\hat{s}^2 = (A_n^c)^{1/(n+1)}\,ds^2 = \lambda^2(A_n^c)^{1/(n+1)}|dz|^2$, then $\hat{K} = K - [\Delta\log(A_n^c)]/2(n+1)$ so $\hat{K} \equiv 0$ if and only if (10) holds. □

4. Proof of Theorem A.

LEMMA 4.1. *If* $M \subset X^N(c)$ *is an exceptional minimal surface, then*

$$d\omega_{2r-1,2r} = 2[|H_{2r+1}|^2/|H_{2r-1}|^2 - |H_{2r-1}|^2/|H_{2r-3}|^2]\theta_1 \wedge \theta_2 \tag{11}$$

for $r = 2, \dots, \beta_N$, *whenever the denominators are not zero.*

Proof. From the first equation in (8),

$$H_{2r-1}(\omega_{2r+1,2r-1} + i\omega_{2r+1,2r}) \wedge \overline{H}_{2r-1}(\omega_{2r+1,2r-1} - i\omega_{2r+1,2r}) = -H_{2r+1}\overline{H}_{2r+1}\varphi \wedge \overline{\varphi}.$$

Since $\varphi \wedge \overline{\varphi} = -2i\theta_1 \wedge \theta_2$, this implies that

$$\omega_{2r-1,2r+1} \wedge \omega_{2r+1,2r} = [|H_{2r+1}|^2/|H_{2r-1}|^2]\theta_1 \wedge \theta_2. \tag{12}$$

Similarly,

$$\omega_{2r-1,2r+2} \wedge \omega_{2r+2,2r} = [|H_{2r+1}|^2/|H_{2r-1}|^2]\theta_1 \wedge \theta_2. \tag{13}$$

From (8) with r replaced by $r-1$,

$$\overline{H}_{2r-3}(\omega_{2r-1,2r-3} - i\omega_{2r-1,2r-2}) \wedge H_{2r-3}(\omega_{2r,2r-3} + i\omega_{2r,2r-2}) = \overline{H}_{2r-1}\varphi \wedge iH_{2r-1}\overline{\varphi}.$$

The real part of this expression simplifies to

$$\begin{aligned} &\omega_{2r-1,2r-3} \wedge \omega_{2r-3,2r} + \omega_{2r-1,2r-2} \wedge \omega_{2r-2,2r} \\ &\quad = -2[|H_{2r-1}|^2/|H_{2r-3}|^2]\theta_1 \wedge \theta_2. \end{aligned} \tag{14}$$

Now (11) follows from the structure equation (3) for $d\omega_{2r-1,2r}$ and from (12), (13), and (14). □

LEMMA 4.2. *For each* $r = 1, \ldots, \beta_N + 1$, *there is a* G_{2r-1} *such that*

$$dH_{2r-1} + iH_{2r-1}(r\omega_{12} - \omega_{2r-1,2r}) = G_{2r-1}\overline{\varphi}. \tag{15}$$

Proof. When $r = 1$, $H_1 = 1$, so $G_1 = 0$. Suppose (15) holds when $r = p$. Set $r = p$ in the two equations in (8) and take their exterior derivatives using the structure equations (3). (Note that $d\overline{\varphi} = i\omega_{12} \wedge \overline{\varphi}$.) In both cases, the result is

$$\{dH_{2p+1} + iH_{2p+1}[(p+1)\omega_{12} - \omega_{2p+1,2p+2}]\} \wedge \overline{\varphi} = 0,$$

so for some G_{2p+1},

$$dH_{2p+1} + iH_{2p+1}[(p+1)\omega_{12} - \omega_{2p+1,2p+2}] = G_{2p+1}\overline{\varphi}$$

and the lemma follows by induction. □

COROLLARY. *If* H_{2r-1} *is not identically zero, then its zeros are isolated.*

Proof. By (15),

$$d\overline{H}_{2r-1} - i\overline{H}_{2r-1}(r\omega_{12} - \omega_{2r-1,2r}) \equiv 0 \pmod{\varphi},$$

and the corollary follows from the theorem in §4 in [2]. □

LEMMA 4.3. *If* $M \subset X^N(c)$ *is an exceptional minimal surface, then*

$$A_r^c = 2^{2r-1}|H_{2r+1}|^2, \qquad r = 0, \ldots, \beta_N, \tag{16}$$

and the zeros of each A_r^c *are isolated.*

Proof. The preceding corollary shows that the zeros of A_r^c are isolated whenever (16) holds. Clearly (16) holds when $r = 0$. To show (16) when $r = 1$, note that the structure equations (1) and (3) give two ways of computing $d\omega_{12}$:

$$d\omega_{12} = -K\theta_1 \wedge \theta_2 = \sum_{\alpha=3}^{4} \omega_{1\lambda} \wedge \omega_{\alpha 2} - c\theta_1 \wedge \theta_2.$$

Expanding the summation using equations (8) with $r = 1$ and then extracting the coefficients of $\theta_1 \wedge \theta_2$ yields

$$A_1^c = c - K = 2|H_3|^2.$$

Now proceed inductively: suppose that for some p, (16) holds for $r < p$. If $A_{p-1}^c = 0$, then $A_p^c = 0$ by definition. Also, $H_{2p-1} = 0$ by (16), so $H_{2p+1} = 0$ by (8), showing that (16) holds for $r = p$. Since the zeros of the A's propagate along the sequence, the only other case to consider is $A^c_{p-2} \neq 0$ and $A^c_{p-1} \neq 0$. By Lemma 4.1,

$$(17) \qquad d\omega_{2p-1,2p} = \tfrac{1}{2}[2^{2p-1}|H_{2p+1}|^2/A^c_{p-1} - A^c_{p-1}/A^c_{p-2}]\theta_1 \wedge \theta_2.$$

Using this, the exterior derivative of (15) when $r = p$ is

$$(18) \quad \begin{aligned} &\{dG_{2p-1} + iG_{2p-1}[(p+1)\omega_{12} - \omega_{2p-1,2p}]\} \wedge \overline{\varphi} \\ &\quad = -\frac{i}{2}H_{2p-1}[2^{2p-1}|H_{2p+1}|^2/A^c_{p-1} \\ &\qquad\qquad - A^c_{p-1}/A^c_{p-2} + 2pK]\theta_1 \wedge \theta_2. \end{aligned}$$

By Lemma 4.2, the exterior derivative of $A^c_{p-1} = 2^{2p-3}H_{2p-1}\overline{H}_{2p-1}$ is

$$(A^c_{p-1})_1\theta_1 + (A^c_{p-1})_2\theta_2 = 2^{2p-3}(H_{2p-1}\overline{G}_{2p-1}\varphi + \overline{H}_{2p-1}G_{2p-1}\overline{\varphi}).$$

Wedging this with φ and comparing coefficients of $i\theta_1 \wedge \theta_2$ yields

$$(19) \qquad \overline{\partial}A^c_{p-1} = 2^{2p-3}\overline{H}_{2p-1}G_{2p-1}.$$

By (2), (15), and (19),

$$(20) \quad \begin{aligned} \Delta A^c_{p-1}\varphi \wedge \overline{\varphi} = 2^{2p-1}&\overline{H}_{2p-1} \\ &\times \{dG_{2p-1} + iG_{2p-1}[(p+1)\omega_{12} - \omega_{2p-1,2p}]\} \\ &\wedge\ \overline{\varphi} + 2^{2p-1}|G_{2p-1}|^2\varphi \wedge \overline{\varphi}. \end{aligned}$$

Note that by (19) and the inductive hypothesis,

$$|G_{2p-1}|^2 = |\overline{\partial}A^c_{p-1}|^2/2^{4p-6}|H_{2p-1}|^2 = \|dA^c_{p-1}\|^2/2^{2p-1}A^c_{p-1}.$$

Combining this with (20) and (18) shows that

$$\begin{aligned} \Delta A^c_{p-1} = A^c_{p-1}&[2^{2p-1}|H_{2p+1}|^2/A^c_{p-1} - A^c_{p-1}/A^c_{p-2} + 2pK] \\ &+ \|dA^c_{p-1}\|^2/2A^c_{p-1}. \end{aligned}$$

Thus,

$$\begin{aligned} 2^{2p-1}|H_{2p+1}|^2 &= A^c_{p-1}[\Delta \log(A^c_{p-1}) + A^c_{p-1}/A^c_{p-2} - 2pK] \\ &= A^c_p, \end{aligned}$$

completing the induction. □

To finish the proof of Theorem A, consider first the case $N = 2n+2$. Calculations analogous to those in the proof of Lemma 4.1 show that

$$d\omega_{2n+1,2n+2} = -2[|H_{2n+1}|^2/|H_{2n-1}|^2]\theta_1 \wedge \theta_2 = -[A^c_n/2A^c_{n-1}]\theta_1 \wedge \theta_2.$$

Using this in the proof of Lemma 4.3 with $p = n + 1$ shows that

$$A^c_{n+1} = A^c_n[\Delta \log(A^c_n) + A^c_n/A^c_{n-1} - 2(n+1)K] \equiv 0.$$

The proof of Theorem A when $N = 2n + 1$ follows by modifying the arguments in the proofs of the lemmas to apply to (9) instead of (8). Using (9) in the proof of Lemma 4.1 when $r = n$ yields

$$d\omega_{2n-1,2n} = 2[|H_{2n+1}|^2/2|H_{2n-1}|^2 - |H_{2n-1}|^2/|H_{2n-3}|^2]\theta_1 \wedge \theta_2.$$

The proof of Lemma 4.2 applied to (9) implies that for some G_{2n+1},

$$dH_{2n+1} + iH_{2n+1}(n+1)\omega_{12} = G_{2n+1}\overline{\varphi}.$$

The proof of Lemma 4.3 when $p = n$ shows that

$$A^c_n = 2^{2n-2}|H_{2n+1}|^2$$

so that

$$d\omega_{2n-1,2n} = \tfrac{1}{2}[A^c_n/A^c_{n-1} - A^c_{n-1}/A^c_{n-2}]\theta_1 \wedge \theta_2.$$

Using this, the proof of Lemma 4.3 can be repeated again with appropriate modifications when $p = n + 1$ to show that $\Delta \log(A^c_n) - 2(n+1)K = 0$. By the lemma in §3, $d\hat{s}^2 = (A^c_n)^{1/(n+1)} ds^2$ is flat. □

5. Proof of Theorem B. Let $F(M)$ and $F(X)$ denote the bundles of orthonormal frames on M and X, respectively. Consider the manifold

$$P = F(M) \times F(X) \times \mathbb{C}^{2n}$$

where $\mathbb{C}^{2n}$ has coordinates $(H_3, \dots, H_{2n+1}, G_3, \dots, G_{2n+1})$. Use the projections $\pi_1\colon P \to F(M)$ and $\pi_2\colon P \to F(X)$ to pull the forms θ_j, ω_{12}, $\tilde{\theta}_A$, and $\tilde{\omega}_{AB}$ back to P, and let the pulled-back forms be denoted by the same symbols. For example, the pull-back $\pi_1^*(\theta_j)$ will be denoted simply θ_j. Let I denote the ideal of differential forms on P generated by the following 1-forms:

$\tilde{\theta}_j - \theta_j$, $\tilde{\theta}_\alpha$, $\tilde{\omega}_{12} - \omega_{12}$,

$\tilde{\omega}_{2r-1,A}$ and $\tilde{\omega}_{2r,A}$ for $1 \le r \le \beta_N$, and $A > 2r + 2$ or $A < 2r - 3$,

$H_{2r-1}(\tilde{\omega}_{2r+1,2r-1} + i\tilde{\omega}_{2r+1,2r}) - H_{2r+1}\overline{\varphi}$, $1 \le r \le \beta_N$,

$H_{2r-1}(\tilde{\omega}_{2r+2,2r-1} + i\tilde{\omega}_{2r+2,2r}) - iH_{2r+1}\overline{\varphi}$, $1 \le r \le \beta_N$,

$dH_{2r+1} + iH_{2r+1}[(r+1)\omega_{12} - \tilde{\omega}_{2r+1,2r+2}] - G_{2r+1}\overline{\varphi}$, $0 \le r \le \beta_N$, together with the forms

$$H_{2n-1}(\tilde{\omega}_{2n+1,2n-1} + i\tilde{\omega}_{2n+1,2n}) - H_{2n+1}\overline{\varphi},$$
$$dH_{2n+1} + iH_{2n+1}(n+1)\omega_{12} - G_{2n+1}\overline{\varphi},$$

when $N = 2n+1$. Let Q be the submanifold of P determined by the relations

$$(21)\quad A_r^c = 2^{2r-1}|H_{2r+1}|^2 \quad\text{and}\quad \overline{\partial}A_r^c = 2^{2r-1}\overline{H}_{2r+1}G_{2r+1} \qquad \text{for } 1 \le r \le \beta_N,$$

together with

$$(22)\qquad A_n^c = 2^{2n-2}|H_{2n+1}|^2 \quad\text{and}\quad \overline{\partial}A_n^c = 2^{2n-2}\overline{H}_{2n+1}G_{2n+1}$$

when $N = 2n+1$. Lengthy calculations (most of which are outlined in §2 and §4) show that if $N = 2n+2$ and $A_{n+1}^c \equiv 0$, or if $N = 2n+1$ and $\Delta\log(A_n^c) - 2(n+1)K \equiv 0$, then I is a closed ideal in Q, i.e., if $\eta \in I$, then $d\eta \in I$. Choose an initial point of Q by choosing initial points $x_0 \in M$, $y_0 \in X$, initial frames e_i and $\tilde{e}_A$, and initial values of the H's and G's that satisfy (21) and (22) at x_0. By the Frobenius Theorem, there is a submanifold $S \subset Q$ passing through this initial point such that all forms in I are zero on S.

When $N = 2n+2$, $\dim Q = 2n^2+6n+6$ and there are $2n^2+6n+3$ independent 1-forms in I, so $\dim S = 3$. By a standard argument ([**5**], pp. 73–77) the restricted projection $\pi_1|_S$ is locally a diffeomorphism from S to $F(M)$. More precisely, there is a neighborhood $V \subset S$ and a neighborhood $W \subset F(M)$ containing (x_0, e_1, e_2) such that $\pi_1|_V\colon V \to W$ is a diffeomorphism. Define $f_*\colon W \to F(X)$ by

$$f_* = \pi_2 \circ (\pi_1|_V)^{-1}.$$

Let $f_*(x, e_1, e_2) = (y, \tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{2n+2})$ for $(x, e_1, e_2) \in W$. If f_* is a bundle map, then it projects down to a map $f\colon U \to X$ where $U \subset M$ is a neighborhood of x_0. The forms in I were chosen so that f would be an isometry ($\theta_i = f^*(\tilde{\theta}_i)$, where f^* is the pull-back induced by f) and so that $f(U)$ would be an exceptional minimal surface. Note that since $A_r^c > 0$, $1 \le r \le n$, each $H_{2r+1} \ne 0$ by (21), so $f(U)$ lies fully in $X^{2n+2}(c)$.

To show that f_* is a bundle map, let a be a complex number with absolute value 1 and consider the action on P given by

$$e_1 + ie_2 \to a(e_1 + ie_2),$$
$$\tilde{e}_{2r-1} + i\tilde{e}_{2r} \to a^r(\tilde{e}_{2r-1} + i\tilde{e}_{2r}), \qquad 1 \le r \le n+1,$$
$$G_{2r+1} \to aG_{2r+1}, \qquad 1 \le r \le n,$$

with $x \in M$, $y \in X$, and the H's unchanged. This induces the following action on forms:

$$\begin{gathered}\theta_1 + i\theta_2 \to a(\theta_1 + i\theta_2), \\ \tilde{\theta}_{2r-1} + i\tilde{\theta}_{2r} \to a^r(\tilde{\theta}_{2r-1} + i\tilde{\theta}_{2r}), \qquad 1 \le r \le n+1, \\ \eta_{p,q} \to a^{p+q}\eta_{p,q}, \qquad 1 \le p, q \le n+1,\end{gathered}$$

where

$$\eta_{p,q} = \tilde{\omega}_{2p-1,2q-1} + i\tilde{\omega}_{2p-1,2q} + i(\tilde{\omega}_{2p,2q-1} + i\tilde{\omega}_{2p,2q}).$$

The forms ω_{12} and $\tilde{\omega}_{2r+1,2r+2}$ are unchanged. Also,

$$\overline{\partial} \to a\overline{\partial}.$$

The submanifold Q and the ideal I are invariant under this action, so the integral submanifolds of I are also invariant. It follows that f_* is a bundle map, which completes the proof when $N = 2n+2$.

When $N = 2n+1$, $\dim Q = 2n^2+4n+4$ and there are $2n^2+4n+2$ independent 1-forms in I, so $\dim S = 2$. The initial conditions are such that $\theta_1 \wedge \theta_2 \neq 0$ at the initial point and therefore in a neighborhood of the initial point. Thus, there are neighborhoods $V \subset S$ and $U \subset M$ such that projection from S to M is a diffeomorphism from V to U, and f can be defined as the inverse of this projection followed by the projection from S to X. As in the previous case, f is an isometry and $f(U)$ is an exceptional minimal surface lying fully in $X^{2n+1}(c)$. □

6. Isometric exceptional minimal surfaces.

COROLLARY 6.1. *Suppose M_1 and M_2 are exceptional minimal surfaces lying fully in $X^{2n+2}(c)$. If M_1 and M_2 are isometric, then they are congruent.*

Proof. The surfaces M_1 and M_2 are real analytic, so it suffices to show that they are locally congruent. By Theorem A, $A^c_{n+1} \equiv 0$ and there are isometric neighborhoods U_1, U_2 in M_1, M_2, respectively, on which $A^c_p > 0$, $1 \le p \le n$. Let $g\colon U_1 \to U_2$ be the isometry and let $\tilde{e}^j_A$ denote frame fields on X adapted to U_j such that $\tilde{e}^2_k = g_*(\tilde{e}^1_k)$. These frames determine H^j_{2r+1} and G^j_{2r+1} as in §2 and §4, and thus determine submanifolds of P that are integral manifolds for the ideal I. Since that ideal satisfies the Frobenius condition, its integral manifolds are completely determined by initial conditions. Let $x_1 \in U_1$ and $x_2 = g(x_1) \in U_2$ be the initial points. The H's and G's satisfy (21), so for each r there is a value of τ such that at the initial points, $H^2_{2r+1} = e^{i\tau}H^1_{2r+1}$ and $G^2_{2r+1} = e^{i\tau}G^1_{2r+1}$. By (8), rotating the vectors $\tilde{e}^1_{2r+1}, \tilde{e}^2_{2r+2}$ counterclockwise in $\mathrm{Nor}^{(r)}_{x_1}M_1$ through an angle $-\tau$ changes H^1_{2r+1} to $e^{-i\tau}H^1_{2r+1}$. By (15), this rotation changes G^1_{2r+1} to $e^{-i\tau}G^1_{2r+1}$. It follows that the normal vectors $\tilde{e}^1_\alpha$ can be chosen so that $H^1_{2r+1} = H^2_{2r+1}$ and $G^1_{2r+1} = G^2_{2r+1}$ at the initial points. With this choice of the frame $\tilde{e}^1_A$, an isometry of X that takes x_1 to x_2 and $\tilde{e}^1_A$ and $\tilde{e}^2_A$ will take U_1 to U_2 and therefore M_1 to M_2. □

COROLLARY 6.2. *Suppose M is an exceptional minimal surface lying fully in $X^{2n+1}(c)$. Then there is a 1-parameter family M_τ, $0 \le \tau < 2\pi$, of exceptional minimal surfaces in $X^{2n+1}(c)$ such that every exceptional minimal surface in $X^{2n+1}(c)$ that is isometric to M is congruent to some M_τ.*

Proof. As in the proof of the previous corollary, different exceptional minimal surfaces in $X^{2n+1}(c)$ that are isometric to M can only arise through a choice of different initial conditions, and choosing different initial values for $H_3, H_5, \dots, H_{2n-1}, G_3, G_5, \dots, G_{2n-1}$ is equivalent to choosing different initial normal vectors $\tilde{e}_3, \dots, \tilde{e}_{2n}$. As for H_{2n+1} and G_{2n+1}, they can be replaced by $e^{i\tau}H_{2n+1}$ and $e^{i\tau}G_{2n+1}$ in (22), but this is not equivalent to a rotation in $\mathrm{Nor}^{(n)}_x M$, which is 1-dimensional. Thus, an integral manifold for the ideal I is determined by initial points in M and $X^{2n+1}(c)$, initial frames e_j and $\tilde{e}_A$, and one value of τ to determine the initial values $e^{i\tau}H_{2n+1}$ and $e^{i\tau}G_{2n+1}$. Varying τ will produce a 1-parameter family M_τ of minimal surfaces and every exceptional minimal surface in $X^{2n+1}(c)$ that is isometric to M will be congruent to a member of this family. □

Added in proof. When N is even, my exceptional minimal surfaces are also known as *isotopic minimal surfaces.* See **[6]** for an alternate definition in this case.

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