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MULTIPLICATION FORMULAE FOR PERIODIC FUNCTIONS

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Carlitz and others have proved that if f is a polynomial such that it satisfies the formula

$$(1.1)\qquad \sum_{j=0}^{d-1} f\left(x+\frac{j}{d}\right) = d^{1-k} f(dx)$$

then f is (essentially) the k^{th} degree Bernoullli polynomial. The purpose of this paper is to discuss the slightly more general formula

$$(1.2)\qquad \sum_{n(d)} f\left(x+\frac{n}{d}\right) = \theta(d) f(xd).$$

when f is periodic with period 1. The notation $n(d)$ under the summation sign indicates that n runs through a complete system of residues $\bmod d$. Formulae like (1.1) and (1.2) occur also in theories of Franel's formula and in the elementary theory of Dedekind sums.

In this paper we will pretty much characterize the periodic bounded variation solutions of (1.2). Then we provide a weak generalization of the current best form of Franel's theorem, and use this result to provide a method for constructing new solutions of (1.2) from old ones, at least in principle.

1. Introduction. D. H. Lehmer discussed (1.1) and (1.2) at the 1986 West coast number theory conference in Tucson, Arizona and called for examples of (1.1) and (1.2). He gave examples including polynomials, step functions, bounded variation periodic functions, and periodic meromorphic functions.

See [**2**, **3**] for discussions of (1.1) and Bernoulli polynomials, [**4**, **6**, **8**, **11**] for discussions of Franel's formula and [**7**] for discussions of Dedekind sums. Since there is some measure of diversity in our theorems, a section will be devoted to each one. Section 2 contains a discussion of two new examples that arise from the fact that its main theorem tells us where not to look for examples of (1.2). Section 4 contains as many examples of (1.2) that are bounded variation that I know of.

2. Periodic bounded variation solutions of (1.2). In this section our bounded variation functions will be identical to their Fourier series.

One might note that the set of f that satisfies (1.2) is a real vector space. Does (1.2) have non-zero solutions for a given θ? The next two simple theorems give some information about this question.

THEOREM 2.1. *If f is a non-zero solution of*

$$\sum_{n=0}^{d-1} f\left(x+\frac{n}{d}\right) = \theta(d)f(dx) \tag{2.1}$$

then θ is completely multiplicative.

The earliest form of our proof seems to be due to Carlitz. Note that if f is a solution of (2.1), then

$$\begin{aligned} \theta(rs)f(rsx) &= \sum_{j=0}^{rs-1} f\left(x+\frac{j}{rs}\right) \\ &= \sum_{m=0}^{r-1}\sum_{n=0}^{s-1} f\left(x+\frac{m+nr}{rs}\right) \\ &= \sum_{m=0}^{r-1}\sum_{n=0}^{s-1} f\left(x+\frac{m}{rs}+\frac{n}{s}\right) \\ &= \sum_{m=0}^{r-1} \theta(s) f\left(sx+\frac{m}{r}\right) \\ &= \theta(s)\theta(r)f(rsx). \end{aligned} \tag{2.2}$$

The conclusion $\theta(rs) = \theta(r)\theta(s)$ follows when x is chosen to be a value so that $f(rsx) \neq 0$.

Having noted that θ must be completely multiplicative for (1.1) or (1.2) or (2.1) to have non-zero solutions, we argue that many formulae (1.2) do have solutions when θ is completely multiplicative. We use the notation $e(t) = e^{2\pi i t}$.

THEOREM 2.2. *If θ is completely multiplicative and*

$$f_\theta(x) = \sum_{n\neq 0} \frac{\theta(n)}{n} e(nx) \tag{2.3}$$

where the series always converges and where the domain of θ is extended so that $\theta(-n) = -\theta(n)$, then f_θ is a solution of (1.2). *The summation in* (2.3) *is over all non-zero integers.*

The proof is easy.

$$\begin{aligned}(2.4)\qquad \sum_{j(d)} f_\theta\left(x+\frac{j}{d}\right) &= \sum_{j(d)}\sum_{n\neq 0}\frac{\theta(n)}{n}e\left(n\left(x+\frac{j}{d}\right)\right)\\ &= \sum_{n\neq 0}\frac{\theta(n)}{n}e(nx)\sum_{j(d)}e\left(\frac{nj}{d}\right)\\ &= \sum_{\substack{n\neq 0\\ d\mid n}} d\frac{\theta(n)}{n}e(nx)\\ &= \sum_{m\neq 0} d\frac{\theta(md)}{md}e(mdx)\\ &= \theta(d)\sum_{m\neq 0}\frac{\theta(m)e(mdx)}{m} = \theta(d)f(dx).\end{aligned}$$

Theorem 2.1 suggests asking for solutions of (1.2) when θ is a familiar completely multiplicative function, for example θ is a Dirichlet character or given by the formula $\theta(d) = d^n$ or the Liouville-λ function. Theorem 2.3 gives a solution in case θ is a Dirichlet character, but in case θ is λ, Theorem 2.2 gives information only at some cost. For example, the convergence of the series for f_λ is not trivial. This example will be discussed in greater detail in §4.3.

What kind of converse of Theorem 2.2 can we arrange?

THEOREM 2.3. *If f is* (a) *periodic with period* 1 *and satisfies* (1.2) *with a completely multiplicative θ,* (b) *real and* (c) *bounded variation and identical to its Fourier series, then $f = cf_\theta$ where c is a constant.*

This result is also easy to prove. Since f is bounded variation, period 1 we can write

$$f(x) = \sum_n a_n e(nx)$$

for all values of x. Then, using (1.2), the series for x, interchanging summations and finally using the formula for the sum of a geometric progression

$$\begin{aligned}(2.5)\qquad \sum_m \theta(d)a_m e(mdx) &= \theta(d)f(dx)\\ &= \sum_{j(d)} f\left(x+\frac{j}{d}\right) = \sum_{j(d)}\sum_n a_n e\left(n\left(x+\frac{j}{d}\right)\right)\\ &= \sum_n a_n e(nx)\sum_{j(d)} e\left(\frac{jn}{d}\right) = \sum_{d|n} d a_n e(nx)\\ &= \sum_m d a_{md} e(mdx).\end{aligned}$$

By the identity theorem for Fourier series, $\theta(d)a_m = da_{md}$ for all m and d. Taking $m = 1$ gives $a_d = a_1\frac{\theta(d)}{d}$. Noting that since f is real, $a_{-d} = d_d$ the result follows.

Where else would one look for solutions of (1.2)? We offer two suggesions.

First, letting Q and R be the rationals and reals as usual, let V be a rational vector space such that $Q \subset V \subset R$. Let $p_V(v) = 1$ when $v \in V$ and $p_V(v) = 0$ when $v \notin V$. Now x, $x + \frac{j}{d}$, dx (where j is in a complete residue system $\bmod d$) are all in V or none are in V. From this fact $p_V(x)f(x)$ is a solution of (1.2) when f is a solution of (1.2).

Next, let F be a real arithmetic function such that $F(dn, dr) = F(n, r)$ and $F(n+r, r) = F(n, r)$. Define $f(p/q) = F(p, q)$. It is easy to see that

$$(2.6)\qquad \sum_{j(d)} F(n+jd, rd) = \theta(d)F(n, r)$$

is sufficient to derive (1.2) for f. I would think that characterizing the set of F that satisfies the above three conditions would be interesting. I offer one class of examples for F as just described. Let P be a set of primes and let A be the set of all natural numbers all of whose prime factors are in P. Define $\gamma_P(n)$ to be the largest factor of n that is in A, and let $F_P(x, r)$ be 1 if $\gamma_P(r)$ divides x and 0 otherwise. Since $\gamma_P(r)|r$, it follows that $\gamma_P(r)|x$ is equivalent to $\gamma_P(r)|x+r$ and hence $F_P(x+r, r) = F_P(x, r)$. It is also clear that γ_P is completely multiplicative and that $\gamma_P(dr)|dx$ is equivalent to $\gamma_P(r)|x$. Hence, $F_P(dx, dr) = F_P(x, r)$. Finally,

$$\begin{aligned}1 = F_P(x+jr, dr) &\Leftrightarrow \gamma_P(dr)|x+jr \Leftrightarrow \gamma_P(d)\gamma_P(r)|x+jr\\ &\Rightarrow \gamma_P(r)|x \Rightarrow F(x, r) = 1.\end{aligned}$$

Thus, there is a j so that $F_P(x+jr, dr) = 1$ only when $f_P(x, r) = 1$. Thus to prove (2.6) we only need consider the case $F_P(x, r) = 1$ since $F_P(x, r) = 0$ implies that all the terms in (2.6) are 0. Thus, assume $F_P(x, r) = 1$ or that $\gamma_P(r)|x$. Then

$$(2.7) \quad 1 = F_P(x + jr, dr) \Leftrightarrow j(r/\gamma_P(r)) \equiv (-x/\gamma_P(r)) \pmod{\gamma_P(d)}.$$

This last congruence has exactly $d\gamma_P(d)$ solutions as j runs through a complete system of residue mod d. It follows that F_P satisfies (2.6) with $\theta(d) = d/\gamma_P(d)$, i.e., $\theta(d)$ is the part of d that is made from prime factors of d where the primes are not in P.

3. Franel's formula. Franel's formula is

$$(3.1) \qquad \int_0^1 \psi(mt)\psi(nt) = \frac{(m, n)^2}{12mn}$$

where $\psi(t) = t - [t] - \frac{1}{2}$. It was made famous by Landau in his "Vorlesungen über die Zahlentheorie," Mikolas [**9**, **10**] has offered various generalizations of (3.1), and here we prove, using standard methods,

THEOREM 3.1. *Let f and g be periodic with period* 1 *and satisfy*

$$(3.2) \qquad \sum_{j(d)} f\left(x + \frac{j}{d}\right) = \theta_f(d)f(xd),$$

$$(3.2) \qquad \sum_{j(d)} g\left(x + \frac{j}{d}\right) = \theta_g(d)g(xd)$$

for all d and x so that the quantities $x+\frac{j}{d}$ and xd are in the domains of f and g. Then if the integral on the left-hand side of

$$(3.3) \qquad \int_0^1 f(\alpha + mt)g(\beta + nt)\,dt = \frac{(m, n)^2}{mn}\theta_f\left(\frac{n}{(m, n)}\right)\theta_g\left(\frac{m}{(m, n)}\right) \cdot \int_0^1 f\left(\frac{n\alpha}{(m, n)} + t\right) g\left(\frac{m\beta}{(m, n)} + t\right) dt$$

exists for m and n integers and α and β real numbers, the integral on the right-hand side of (3.3) *exists and* (3.3) *is correct.*

The integrals in (3.3) must satisfy the following properties

$$(3.1.1) \qquad \int_a^b [c_1 f_1(t) + c_2 f_2(t)]\,dt = c_1 \int_a^b f_1(t)\,dt + c_2 \int_a^b f_2(t)\,dt,$$

$$\int_{l(a)}^{l(b)} f(t)\,dt = \int_a^b f(l(x))l^1(x)\,dx, \tag{3.1.2}$$

where $l(x) = cx + d$ and

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^d f(t)\,dt \tag{3.1.3}$$

so long as all the integrals exist. In case the integrals are Cauchy mean values (we will use Cauchy mean values in §4) (3.13) is false, but Theorem 3.1 is true so long as $p\alpha + q$ and $p\beta + q$ are finite points of f and g respectively where p and q are integers.

The proof uses familiar principles. We first prove under the hypothesis of the theorem that the formulae

$$\begin{aligned} &\int_0^1 f(\alpha + mt)g(\beta + nt)\,dt \\ &\quad = \frac{(m, n)}{m}\theta_g\left(\frac{m}{(m, n)}\right)\int_0^1 g\left(\frac{m\beta}{(m, n)}t\right) f(\alpha + t)\,dt \end{aligned} \tag{3.4}$$

is true for all integers m and n. The integral on the right-hand side of (3.4) has the same form as the integral on the left-hand side of (3.4). Applying (3.4) to itself, so to speak, will give the Theorem. Let $d = (m, n)$, $m = dM$, $n = dN$. Then

$$\begin{aligned} \int_0^1 f(\alpha + mt)g(\beta + nt)\,dt &= \frac{1}{m}\int_0^m f(\alpha + t)g\left(\beta + \frac{n}{m}t\right)\,dt \\ &= \frac{1}{m}\sum_{j=0}^{m-1}\int_j^{j+1} f(\alpha + t)g\left(\beta + \frac{N}{M}t\right)\,dt \\ &= \frac{1}{m}\int_0^1 f(\alpha + t)\sum_{j=0}^{m-1} g\left(\beta + \frac{N}{M}(t + j)\right)\,dt \\ &= \frac{1}{m}\int_0^1 f(\alpha + t)\sum_{u=0}^{d-1}\sum_{v=0}^{M-1} g\left(\beta + \frac{Nt}{M} + \frac{N}{M}(v + uM)\right)\,dt \\ &= \frac{d}{m}\int_0^1 f(\alpha + t)\sum_{N(M)} g\left(\beta + \frac{Nt}{M} + \frac{Nv}{M}\right)\,dt. \end{aligned}$$

Since $(N, M) = 1$, as N runs through a complete system of residues mod m so does Nv. Thus, we may replace Nv by v and obtain for the left-hand side of the above expression

$$\frac{d}{m}\int_0^1 f(\alpha+t)\sum_{v(M)} g\left(\beta+\frac{Nt}{M}+\frac{v}{M}\right)dt$$

$$=\frac{d}{m}\int_0^1 f(\alpha+t)\theta_g(M)g(M\beta+Nt)\,dt$$

and (3.4) is deduced.

4. The construction of new examples from old. Parseval type formulae. Examples. Let f and g satisfy

$$f(x)=\sum_{m\neq 0}\frac{\theta_f(m)}{m}e(mx), \qquad g(x)=\sum_{n\neq 0}\frac{\theta_g(n)}{n}e(nx), \tag{4.1}$$

where we temporarily assume the series are absolutely convergent. If we form

$$h(x)=\int_0^1 f(x+t)g(t)\,dt \tag{4.2}$$

then $h(x)$ is equal to

$$\int_0^1 \sum_{m\neq 0}\frac{\theta_f(m)}{m}e(m(x+t))\sum_{n\neq 0}\frac{\theta_g(n)}{n}e(nt)\,dt$$

$$=\sum_{m,n\neq 0}\frac{\theta_f(m)\theta_g(n)}{mn}e(mx)\int_0^1 e((n+m)t)\,dt$$

$$=\sum_{m\neq 0}\frac{(\theta_f(m)\theta_g(-m)/(-m))}{m}e(mx)$$

and $h(x)$ will satisfy (1.2) with

$$\theta(d)=\theta_f(d)\theta_g(d)/d. \tag{4.3}$$

Theorem 4.1 will obtain the same conclusions on weaker hypotheses and will be deduced from Theorem 3.1 instead of Fourier analysis.

THEOREM 4.1. *Let f and g be periodic with period* 1 *and satisfy* (3.2). *Let h be defined by equation* (4.2). *Then h satisfies* (1.2) *with* (4.3) *being true.*

From (4.2) we obtain

$$\sum_{j(d)} h\left(x+\frac{j}{d}\right)=\int_0^1 g(t)\sum_{j(d)} f\left(x+t+\frac{j}{d}\right)dt$$

$$=\theta_f(d)\int_0^1 g(t)f(dx+dt)\,dt.$$

By Theorem (3.1), the left-hand side of the above formula is

$$\theta_f(d)\theta_f\left(\frac{1}{(1,d)}\right)\theta_g\left(\frac{d}{(1,d)}\right)\frac{(1,d)^2}{1\cdot d}\int_0^1 f\left(dx\cdot\frac{1}{(1,d)}+t\right)g(0+t)\,dt$$

and the result follows so long as the integration method satisfies (3.1.1), (3.1.2) and (3.1.3). If the integrals are Cauchy mean values, then we must assume that $px+q$ is a finite point of f and g when p and q are integers.

We now proceed to a discussion of examples of (1.2).

In order to illustrate the above theorems, we will discuss examples of solutions of (1.2) in case $\theta(d) = d^n$, $\theta(d) = \chi(d)d^n$ and $\theta(d) = \lambda(d)d^n$. In these examples, χ is a Dirichlet character, and λ is Lioville's function, i.e., $\lambda(p) = -1$ for p a prime, and λ is completely multiplicative. Thus, we will have discussed (1.2) in the cases when θ is one of the common completely multiplicative functions. In the case $\theta(d) = d^n$ and $\theta(d) = \chi(d)d^n$ we will encounter familiar examples, when $\theta(d) = \lambda(d)d^n$, less so.

4.1. $\theta(d) = d^n$.

Since the series

(4.1.1) $$\psi_m(x) = (2\pi i)^{-m}\sum_{k\neq 0} k^{-m}e(nk)$$

converges for $m \geq 1$ and differs from the periodic extension of the Bernoulli polynomial of degree m by a constant factor, we have an example with $\theta(d) = d^n$ when $n = 1-m$ or when n is not a positive integer. In fact, when $n = 0$, $m = 1$, $\psi_1(x) = ((x))$, a familiar function from the theory of Dedekind sums [7, p. 1]. This same Carus monograph just referred to also provides an example for n a positive integer. In fact, in the second variation of the third proof of the reciprocity formula [see (27) page 18 of 7] for Dedekind sums, Rademacher-Grosswald give $(x = ky)$

(4.1.2) $$\sum_{m=1}^{k}\cot\left(y+\frac{m}{k}\right)\pi = k\cot ky\pi$$

and thus $\cot y\pi$ is an example of (1.2) with $n = 1$. Since $\cot \pi y$ is meromorphic one can easily see that $\pi^{1-n}d^{n-1}/dy^{n-1}\cot\pi y$ is a solution when $n \geq 2$. It can easily be directly verified that

(4.1.3) $$M_n(z) = \sum_n \frac{(-1)^n(n!)}{(z+m)^n} = \frac{d^{n-1}}{dx^n}\pi\cot\pi z$$

is also a solution when $n \geq 1$ (see **[10]** alsc).

I know of no other periodic meromorphic solutions of (1.2) when $n \geq 1$. Of course, there can be no bounded variation periodic solutions of (1.2) for $n \geq 1$, if these solutions are exactly equal to their Fourier series, by Theorem 2.2. Naturally if ρ_V is the vector space characteristic function from §2, $\rho_V(x)\psi_m(x)$ and $\rho_V(x)M_n(z)$ provide infinitely many solutions to (1.2) in the case $\theta(d) = d^n$. Naturally, this last class of examples cannot be bounded variation.

(4.2.1) $\theta(d) = \chi(d)d^n$

This section could be considered to be a generalization of 4.1. Here, χ is a Dirichlet character. Define

$$\psi(\chi, m, x) = (2\pi i)^{-m} \sum_{j\neq 0} \chi(j) j^{-m} e(jx). \tag{4.2.1}$$

If r is the modulus of χ, then

$$\psi(\chi, m, x) = \sum_{k(r)} \chi(k)(2\pi i)^{-m} \sum_{\substack{j\equiv k(r)\\ j\neq 0}} j^{-m} e(jx). \tag{4.2.2}$$

Next, consider

$$\begin{aligned}(4.2.3)\quad & \frac{1}{r}\sum_{s(r)} \psi_m\left(x+\frac{s}{r}\right) e\left(\frac{-sk}{r}\right) \\ &= (2\pi i)^{-m} \sum_{j\neq 0} j^{-m}\left(\frac{1}{r}\sum_{s(r)} e\left(j\left(x+\frac{s}{r}\right)-\frac{sk}{r}\right)\right) \\ &= (2\pi i)^{-m} \sum_{j\neq 0} j^{-m} e(jx)\left(\frac{1}{r}\sum_{s(r)} e\left(\frac{s(j-k)}{r}\right)\right) \\ &= (2\pi i)^{-m} \sum_{\substack{j\equiv k(r)\\ j\neq 0}} j^{-m} e(jx).\end{aligned}$$

Thus, if we combine (4.2.3) and (4.2.2) we obtain

$$\begin{aligned}(4.2.4)\quad \psi(\chi, m, x) &= \frac{1}{r}\sum_{k(r)}\sum_{s(r)} \chi(k) e\left(\frac{-sk}{r}\right)\psi_m\left(x+\frac{s}{r}\right) \\ &= \frac{1}{r}\sum_{s(r)} \psi_m\left(x+\frac{s}{r}\right)\sum_{k(r)} \chi(k) e\left(\frac{-sk}{r}\right) \\ &= \frac{1}{r}\sum_{s(r)} \psi_m\left(x+\frac{s}{r}\right) G_{-s}(\chi),\end{aligned}$$

where $G_t(\chi)$ is the Gauss sum

$$G_t(\chi) = \sum_{k(r)} \chi(k) e\left(\frac{tk}{r}\right). \tag{4.2.5}$$

Thus, we have expressed $\psi(\chi, m, x)$ in terms of more familiar functions. The reader might compare (4.2.4) to (4.9) of **[9]**. Also note Proposition 9.7 of that same paper, and also the elegant formulae of §10.

From (4.2.1) it is obvious that $\psi(\chi, m, x)$ is a solution of (1.2) with $\theta(d) = \chi(d)d^{1-n}$ for $n \geq 1$. We use Theorem 4.1 to construct solutions of (1.2) with $\theta(d) = \chi(d)d^n$ for n positive. Consider

$$f(x) = \int_0^1 \psi(\chi, 1, t) M_3(t+x)\, dt \tag{4.2.6}$$

by Theorem 4.1, f satisfies (1.2) with $\theta(d) = \chi(d)d^3/d = d^2\chi(d)$ if the integral (4.2.6) converges. We will see that (4.2.6) converges as a Cauchy mean value in the process of computing (4.2.6).

We start with the indefinite integral, for n, a positive integer,

$$\int \left(t - \frac{1}{2}\right) M_{n+2}(x+t)\, dt = \left(t - \frac{1}{2}\right) M_{n+1}(x+t) - M_n(x+t)$$

which is valid for t in any interval not containing $m - x$ for m an integer. Taking $x \in (0, 1)$,

$$\begin{aligned}\int_0^{1-x-\varepsilon} \psi_1(t) M_{n+2}(x+t)\, dt + \int_{1-x+\varepsilon}^1 \psi(t) M_{n+2}(x+t)\, dt \\ = M_{n+1}(x) + \left(\frac{1}{2} - x\right)(M_{n+1}(-\varepsilon) - M_{n+1}(\varepsilon)) \\ - \varepsilon(M_{n+1}(-\varepsilon) + M_{n+1}(\varepsilon)) + (M_n(\varepsilon) - M_n(-\varepsilon)).\end{aligned} \tag{4.2.7}$$

The limit, as ε tends to zero in (4.2.7) exists when n is odd. For odd n, the Cauchy mean value

$$\int_0^1 \psi_1(t) M_{n+2}(t+x)$$

is

$$M_{n+1}(x) + \lim_{\varepsilon \to 0} (2M_n(\varepsilon) - 2\varepsilon M_{n+1}(\varepsilon)).$$

By (4.1.3), the limit is $-4n!\zeta(n)$ and hence,

$$\int_0^1 \psi_1(t) M_{n+2}(x+t)\, dt = M_{n+1}(x) - 4n!\zeta(n) \tag{4.2.8}$$

for n odd and positive.

Keeping n odd and positive, and working backwards from a formal calculation,

$$
\begin{aligned}
(4.2.9)\quad \frac{1}{r}\sum_{s(r)} G_s(\chi)M_{n+1}\left(x+\frac{s}{r}\right)
&= \frac{1}{r}\sum_{s(r)} G_s(\chi)\left(M_{n+1}\left(x-\frac{s}{r}\right) - 4n!\zeta(n)\right) \\
&= \frac{1}{r}\sum_{s(r)} G_x(\chi)\int_0^1 \psi_1(t)M_{n+2}\left(t+x-\frac{s}{r}\right)dt \\
&= \frac{1}{r}\sum_{s(r)} G_x(\chi)\int_0^1 \psi_1\left(t+\frac{s}{r}\right)M_{n+2}(t+x)\,dt \\
&= \int_0^1 \psi(\chi\,,\,1\,,\,t)M_{n+2}(t+x)\,dt \overset{\text{def}}{=} M(\chi\,,\,n+1\,,\,x).
\end{aligned}
$$

By (4.2.8), this last integral exists as a Cauchy mean value and by Theorem 4.1 satisfies (1.2) with

$$(4.2.10)\qquad \theta(d) = \chi(d)d^{n+2}/d = \chi(d)d^{n+1}.$$

Thus, we have an example of (1.2) with $\theta(d) = \chi(d)d^n$ for all integers n except negative odd integers. Differentiating $M(\chi\,,\,n\,,\,x)$ when n is a negative even integer gives examples of (1.2) with $\theta(d) = \chi(d)d^{n+1}$. Thus, we have examples for all exponents except -1. Presumably $M(\chi\,,\,1\,,\,x)$ should do the job, but the present theorems only suggest that it does.

A word here about the domain of $M(\chi\,,\,n+1\,,\,x) = M$. By (4.2.9), the domain of M is the reals (or complexes) excluding the integers. Thus, $M(\chi\,,\,n\,,\,x)$ satisfies (1.2) for all d and some x. Perhaps, however, they are the best, or only, meromorphic solutions of (1.2).

$$(4.3)\qquad \theta(d) = -\lambda(d)d^n$$

By Theorem 2.1,

$$(4.3.1)\qquad L_n(x) = \sum_{k\neq 0}\frac{\lambda(k)}{k^{n+1}}e(kx)$$

satisfies (1.2) with $\theta(d) = \lambda(d)d^{-n}$ for $n \geq 1$. The convergence domain for (4.3.1) when $n = 0$ seems unknown, although it follows from a theorem of Davenport's.

The theorem of Davenport [5] is

THEOREM 4.3.1. *If k is a positive number, and t is real, then*

$$\sum_{n\le x}\mu(n)e(nt)=O(x\log^{-k}x) \tag{4.3.2}$$

uniformly in t as x tends to infinity.

A standard lattice point argument enables us to deduce the next theorem from (4.3.2).

THEOREM 4.3.2. *If k is a positive number and t is real, then*

$$\sum_{n\le x}\lambda(n)e(nt)=O(x\log^{k}x) \tag{4.3.3}$$

uniformly in t as x tends to infinity.

Now, since

$$\lambda(n)=\sum_{u^2v=n}\mu(v) \tag{4.3.4}$$

we have

$$\sum_{n\le x}\lambda(n)e(nt)=\sum_{u^2v\le x}\mu(v)e(u^2vt). \tag{4.3.5}$$

The sum on the right-hand side of (4.3.5) may be regarded as a sum over the lattice points satisfying $1\le u$, $1\le v$, $u^2v\le x$. This region may be broken into three parts by a point (H, K) on $u^2v=x$ in the standard way one finds in elementary theories of the divisor problem. One obtains

$$\begin{aligned}\sum_{n\le x}\lambda(n)e(nt)&=\sum_{u\le H}\sum_{v\le x/u^2}\mu(v)e(v(u^2t))\\&\quad+\sum_{v\le K}\mu(v)\sum_{u\le\sqrt{x/v}}e(u^2(vt))\\&\quad-\sum_{\substack{u\le H\\ v\le K}}\mu(v)e(u^2vt)=S_1+S_2-S_3.\end{aligned} \tag{4.3.6}$$

The terms in these sums are one in absolute value, so S_2 and S_3 may be estimated by their number of terms. Thus $S_2 = O(\sqrt{xK})$ and $S_3 = O(HK)$. Also,

$$S_1 = \sum_{u \le H} O\left(\frac{x}{u^2(\log(x/u^2))^k}\right) = O\left(\frac{x}{(\log x)^k}\right) \tag{4.3.7}$$

if we take $H = x^{1/4}$, $K = \sqrt{x}$ and thus we obtain (4.3.3).

Now partial summation applied to

$$\sum_{n=P}^{Q} \frac{\lambda(n)}{n} e(nt) \tag{4.3.8}$$

using (4.3.3) gives (4.3.8) to be $O(P/\log P)^{K-1})$ and thus by standard theorems in the theory of uniform convergence

$$f_\lambda(t) = \sum_{n \neq 0} \frac{\lambda(n)}{n} e(nt) \tag{4.3.9}$$

is uniformly convergent and bounded for all t. Thus, also g_λ is bounded and continuous.

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