# THE COCHRAN SEQUENCES OF SEMI-BOUNDARY LINKS

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For a 1-dimensional semi-boundary link, Cochran constructed a sequence of Sato-Levine invariants of successively derived links. This is a linear recurrence sequence and conversely any linear recurrence sequence can be constructed in this way. An upper bound for the growth of this sequence is obtained.

**1. Introduction.** An ordered pair $L = (M, K)$ of oriented circles which are disjointly and smoothly embedded in $S^3$ is called a *semi-boundary link* if the linking number $\mathrm{lk}(M, K)$ is equal to zero [Sa]. If $L = (M, K)$ is a semi-boundary link, then there exist Seifert surfaces $V$ of $M$ and $W$ of $K$ which intersect only in the interiors and transversely. We call $(V, W)$ a *Seifert pair* for $L$.

The orientations on $V$ and $W$, which induce the orientation of $L$ on the boundaries, determine a normal 2-frame field on $V \cap W$. The *Sato-Levine invariant* $\beta(L)$ of the link $L$ is the homotopy class in $\pi_3 S^2 \cong \mathbf{Z}$ which is represented by the Thom-Pontryagin construction on the 2-frame field. It is proven in [Sa] that this integer is independent of the choice of the Seifert surfaces and that it is a link-concordance invariant. The following well-known proposition is included for completeness.

PROPOSITION 1.1. *If $L = (M, K)$ is a semi-boundary link, then*
(a) $\beta(L) = 0$ *if $L$ is a boundary link.*
(b) $\beta(\overline{L}) = \beta(L)$ *where $\overline{L} = (K, M)$.*
(c) $\beta(L^\times) = \beta(L)$ *where $L^\times$ is the same as $L$ with the orientation of either $M$ or $K$ reversed.*
(d) $\beta(-L) = -\beta(L)$ *where $-L$ is a mirror image of $L$.*

*Proof.* (a) is obvious by the definition of the Sato-Levine invariant. (b), (c) and (d) easily follows fromt he fact that

$$[g \circ f \circ h] = \{\deg(g)\}^2 \deg(h)[f]$$

in $\pi_3 S^2$ where $f: S^3 \to S^2$, $g: S^2 \to S^2$ and $h: S^3 \to S^3$ are continuous maps [Hu]. □

Let $(V, W)$ be a Seifert pair for a semi-boundary link $L = (M, K)$. We may assume that the intersection $F = V \cap W$ is connected, by a surgery on the Seifert surfaces. In this case, we call $(V, W)$ a *special Seifert pair* for $L$. Since $W$ is oriented, we can push a small tubular neighborhood of $F$ in $W$ off $F$ in a normal direction of $W$. Call this surface $W'$. Then $W'$ is a Seifert surface for $K$ such that $F \cap W' = \varnothing$. Therefore $D(L) = (F, K)$ is a new semi-boundary link *derived* from $L = (M, K)$. We follow Cochran in calling $D(L)$ a *derived link* or a *derivative* of $L$. $D(L)$ is well defined only up to an equivalence relation called *weak-cobordism* which is weaker than the link-concordance [C]. Since $D(L)$ is a semi-boundary link, its Sato-Levine invariant is defined. We can iterate this procedure to get a sequence of integers as follows:

(i) $\beta_1(L) = \beta(L)$,
(ii) $\beta_i(L) = \beta(D^{i-1}(L))$, for $i \geq 2$.

This sequence $\{\beta_i(L)\}_{i=1}^{\infty}$ is well defined and depends only on the weak-corbordism class of $L$. We call it the *Cochran sequence* of $L$.

By Proposition 1.1(a), $\{\beta_i(L)\}_{i=1}^{\infty}$ is identically zero if $L$ is a boundary link. One of the most important properties of this sequence is that it is additive under componentwise oriented connected sum of semi-boundary links [C].

**2. Kojima's function.** Recall the definition of Kojima's function first given in [KY]. Let $L = (M, K)$ be a semi-boundary link and let $\widetilde{X}_K$ be the infinite cyclic cover of $S^3 \backslash K$ whose covering transformation group is generated by $t$. Denote by $\ell$ a zero-push-off of $M$, i.e., $\ell$ is isotopic to $M$ in a small tubular neighborhood of $M$ and $\text{lk}(\ell, M) = 0$. Let $\tilde{\ell}$ and $\widetilde{M}$ be near-by lifts of $\ell$ and $M$ in $\widetilde{X}_K$. Since the Alexander polynomial $\Delta_K(t)$ of $K$ annihilates the $Z[t, t^{-1}]$-module $H_1(\widetilde{X}_K)$, $\Delta_K(t)\tilde{\ell}$ bounds a 2-chain $\zeta$ in $\widetilde{X}_K$.

DEFINITION.

$$\eta_L(t) = \frac{\sum_{n=-\infty}^{\infty} \text{Int}(\zeta, t^n \widetilde{M}) t^n}{\Delta_K(t)}$$

where $\text{Int}(\ ,\ )$ stands for the usual intersection number in $\widetilde{X}_K$.

For any semi-boundary link $L = (M, K)$, $\eta_L(t)$ is an invariant of $I$-equivalence class of $L$ satisfying the following properties:

(iii) $\eta_L(t^{-1}) = \eta_L(t)$,
(iv) $\eta_L(1) = 0$.

Since $\Delta_K(t)$ can be normalized to satisfy $\Delta_K(t^{-1}) = \Delta_K(t)$, one can change the variable by $x = (1 - t)(1 - 1/t)$ to get a rational function

$h_L(x)$ from $\eta_L(t)$. Cochran proved in [C] that $\eta_L(t)$ is equivalent to the sequence $\{\beta_i(L)\}_{i=1}^{\infty}$. The relation is given by the Maclaurin series expansion.

(v) $h_L(x) = \sum_{i=1}^{\infty} \beta_i(L)x^i$.

In other words, $h_L(x)$ is the generating function for the Cochran sequence $\{\beta_i(L)\}_{i=1}^{\infty}$.

## 3. Linear recurrence sequences.

DEFINITION. An infinite sequence $\{n_i\}_{i=1}^{\infty}$ of integers is a *linear recurrence sequence* if there exist integers $N > 0$, $a_1, \ldots, a_d$ such that

$$n_{i+d} + a_1 n_{i+d-1} + \cdots + a_d n_i = 0$$

for all $i > N$.

PROPOSITION 3.1. (a) *$\{n_i\}_{i=1}^{\infty}$ is a linear recurrence sequence if and only if its generating function is of the form*

$$\frac{b_1 x + b_2 x^2 + \cdots + b_m x^m}{1 + a_1 x + a_2 x^2 + \cdots + a_d x^d}$$

*for some integers $d > 0$, $m > 0$, $a_1, \ldots, a_d$ and $b_1, \ldots, b_m$.*

(b) *If $\{n_i\}_{i=1}^{\infty}$ is a linear recurrence sequence whose generating function is as in* (a), *then for all sufficently large $i$,*

$$n_i = \sum_{j=1}^{k} P_j(i)\gamma_j^i$$

*where $1 + a_1 x + a_2 x^2 + \cdots + a_d x^d = \prod_{j=1}^{k}(1 - \gamma_j x)^{d_j}$, the $\gamma_j$'s are distinct, and $P_j(i)$ is a polynomial in $i$ of degree $d_j - 1$.*

Proposition 3.1 is a version of Theorem 4.1 in [St].

THEOREM 3.2. (a) *For any semi-boundary link $L$, $\{\beta_i(L)\}_{i=1}^{\infty}$ is a linear recurrence sequence.*

(b) *For any linear recurrence sequence $\{n_i\}_{i=1}^{\infty}$, there is a semi-boundary link $L$ such that $\{\beta_i(L)\}_{i=1}^{\infty} = \{n_i\}_{i=1}^{\infty}$.*

*Proof.* (a) Using the fact that $\Delta_K(1) = \pm 1$, it is easy to see that $h_L(x)$ is of the form

$$\frac{b_1 x + b_2 x^2 + \cdots + b_m x^m}{1 + a_1 x + a_2 x^2 + \cdots + a_d x^d}$$

for some integers $d > 0$, $m > 0$, $a_1, \ldots, a_d$ and $b_1, \ldots, b_m$. Then (a) follows from Proposition 3.2 (a) and the identity (v).

(b) Let

$$h(x) = \frac{b_1 x + b_2 x^2 + \cdots + b_m x^m}{1 + a_1 x + a_2 x^2 + \cdots + a_d x^d}$$

be the generating function for a linear recurrence sequence $\{n_i\}_{i=1}^{\infty}$. Substituting $(1 - t)(1 - 1/t)$ for $x$ in $h(x)$, we get

$$\eta(t) = \frac{\sum_{j=1}^{m} d_j (2 - t^j - t^{-j})}{1 + \sum_{i=1}^{d} c_i (2 - t^i - t^{-i})}$$

for some integer $c_i$'s and $d_j$'s. We may assume that $d \geq m$ by adding terms with coefficient zero, if necessary. We will construct a semi-boundary link $L_j$, for each $j$ with $\varepsilon_j \neq 0$, satisfying

$$\eta_{L_j}(t) = \frac{\varepsilon_j (2 - t^j - t^{-j})}{1 + \sum_{i=1}^{d} c_i (2 - t^i - t^{-i})}$$

where, for each $k = 1, \ldots, m$,

$$\varepsilon_k = \begin{cases} 1 & \text{if } d_k > 0, \\ 0 & \text{if } d_k = 0, \\ -1 & \text{if } d_k < 0. \end{cases}$$

First we will construct a knot whose Alexander polynomial is

$$\Delta(t) = \varepsilon_j \left( 1 + \sum_{i=1}^{d} c_i (2 - t^i - t^{-1}) \right)$$

as follows [**L**, **R1**, **R2**].

Let $K$ be an unknot. Embed a solid torus $T$ in $\mathbf{S}^3 \backslash K$ so that
(1) the centerline $C(T)$ of $T$ is unknotted,
(2) $\mathrm{lk}(K, C(T)) = 0$,
(3) there are $\varepsilon_j c_1, \ldots, \varepsilon_j c_d$ twists in $T$ separated by strands of $T$, each of which links once around $K$ as shown in Figure 1.

Let $\lambda$ be a longitude of $T$ such that $\mathrm{lk}(\lambda, C(T)) = \varepsilon_j$ when $\lambda$ and $C(T)$ are oriented in the same direction. Since $\mathbf{S}^3 \backslash \mathrm{int}(T)$ is another solid torus, there is a homeomorphism

$$h: \mathbf{S}^3 \backslash \mathrm{int}(T) \to \mathbf{S}^3 \backslash \mathrm{int}(T)$$

sending $\lambda$ into a meridian curve $\mu$ of $T$. Let $K_h = h(K) \subset \mathbf{S}^3$ and let $\widetilde{X}$ be the infinite cyclic covering space of $\mathbf{S}^3 \backslash K_h$ obtained by attaching solid tori to the infinite cyclic covering space of $\mathbf{S}^3 \backslash (\mathrm{int}(T) \cup K)$. Then $\Delta(t)$ is the Alexander polynomial of $K_h$.

Let $M$ be another unknot in $\mathbf{S}^3 \backslash (\mathrm{int}(T) \cup K)$ which goes once around the $j$-th twist of $T$ in Figure 1. Let $M_h = h(M) \subset \mathbf{S}^3$. Then
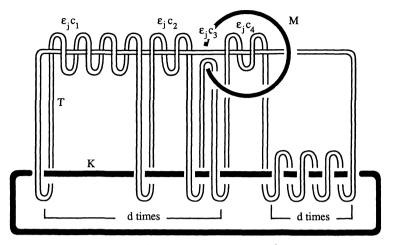
FIGURE 1. This is the case when $d = 4$, $\varepsilon_j c_1 = 4$, $\varepsilon_j c_2 = -2$, $\varepsilon_j c_3 = 0$, $\varepsilon_j c_4 = -2$ and $j = 3$.

$L_j = (M_h, K_h)$ is a semi-boundary link. Now we compute $\eta_{L_j}(t)$. Choose a nonsingular disc $E$ bounded by $M$ which intersects $\partial T$ in two circles. Let $\alpha$ be one of the circles in $E \cap \partial T$. Choose a slight push-off $M'$ of $M$ in $E$. Let $E'$ be the disc in $E$ such that $\partial E' = M'$. Define $\theta = E \backslash \mathrm{int}(T)$, and $\theta' = E' \backslash \mathrm{int}(T)$. Let

$$
\begin{array}{ccc}
\widetilde{M'} & & \widetilde{M} \\
\cap & & \cap \\
\tilde{\alpha} \subset \tilde{\theta'} & \subset & \tilde{\theta} \\
\cap & & \cap \\
\widetilde{E'} & \subset & \widetilde{E}
\end{array}
$$

be a diagram of lifts of $\alpha$, $\theta$, $\theta'$, $E$, $E'$, $M$, and $M'$ in $\widetilde{X}$. Choose a generator $t$ of the covering transformation so that

$$\partial \tilde{\theta'} = \widetilde{M'} - (1 - t^j)\tilde{\alpha} + \partial \gamma$$

as 1-cycles in $\widetilde{X}$, where $\gamma$ is a 2-chain in $t^j \partial \widetilde{T}$ and $\widetilde{T}$ is a lift of $T$ containing $\tilde{\alpha}$. Since $\mathrm{lk}(K, C(T)) = 0$ and $K$ is unknotted, there is a singular disc $\delta$ in $\mathbf{S}^3 \backslash K$ bounded by a 0-longitude $\lambda_0$ of $T$. We may assume that $M$ and $\delta$ intersect at two points. Then there exists a lift $\tilde{\delta}$ of $\delta \backslash \mathrm{int}(T)$ in $\widetilde{X}$ satisfying

$$\mathrm{Int}(\tilde{\delta}, t^n \widetilde{M}) = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = -j, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is not hard to see that

$$\Delta(t)\tilde{\alpha} = \partial \tilde{\delta} + \partial \sigma$$

where $\sigma$ is a 2-chain in $\bigcup_{k=-\infty}^{\infty} t^k \partial \widetilde{T}$. Let

$$\zeta = (1 - t^j)(\tilde{\delta} + \sigma) + \Delta(t)(\tilde{\theta}' - \gamma).$$

Then $\partial \zeta = \Delta(t)\widetilde{M}'$. It is clear that, for any $n$,

$$\text{Int}(\sigma, t^n \widetilde{M}) = \text{Int}(\tilde{\theta}', t^n \widetilde{M}) = \text{Int}(\gamma, t^n \widetilde{M}) = 0.$$

Therefore,

$$\eta_L(t) = \frac{\sum_{n=-\infty}^{\infty} \text{Int}((1 - t^j)\tilde{\delta}, t^n \widetilde{M})t^n}{\Delta(t)} = \frac{\varepsilon_j(2 - t^j - t^{-j})}{1 + \sum_{i=1}^{d} c_i(2 - t^i - t^{-i})}$$

Then, by the identity (v) and the additivity of the Cochran sequence,

$$L = \underbrace{(L_1 \natural \cdots \natural L_1)}_{|d_1|} \natural \cdots \natural \underbrace{(L_m \natural \cdots \natural L_m)}_{|d_m|}$$

satisfies $\eta_L(t) = \eta(t)$. Finally, $\{\beta_i(L)\}_{i=1}^{\infty} = \{n_i\}_{i=1}^{\infty}$.      $\square$

COROLLARY 3.3. (a) *Any sequence of integers which eventually becomes a geometric progression is a Cochran sequence.*

(b) *Any sequence of integers which eventually becomes an arithmetic progresssion is a Cochran sequence.*

(c) *The Fibonacci sequence* $\{1, 1, 2, 3, 5, 8, \ldots\}$ *is a Cochran sequence.*

(d) *For any positive integer* $d$, $\{n^d\}_{n=1}^{\infty}$ *is a Cochran sequence.*

*Proof.* (a) $n_{i+1} - rn_i = 0$ is the linear recurrence relation for a geometric progression with common ratio $r$.

(b) $n_{i+2} - 2n_{i+1} + n_i = 0$ is the linear recurrence relation for an arithmetic progression.

(c) $n_{i+2} - n_{i+1} - n_i = 0$ is the linear recurrence relation for the Fibonacci sequence. Figure 2 shows a link whose Cochran sequence is the Fibonacci sequence.

(d) The following identity

$$\sum_{k=0}^{d+1}(-1)^k \binom{d+1}{k}(n+k)^d = 0$$

is a linear recurrence relation for the sequence $\{n^d\}_{n=1}^{\infty}$.      $\square$

It is obvious that Cochran sequences can be unbounded. But they cannot grow arbitrarily fast.
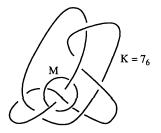
FIGURE 2. Fibonacci link

COROLLARY 3.4. *For a semi-boundary link $L = (M, K)$, there exist constants $C > 0$ and $\gamma > 0$ such that*

$$|\beta_n(L)| \leq Cn^s\gamma^n$$

*for all $n$, where $s$ is a nonnegative integer less than a half of the degree of the Alexander polynomial of $K$.*

*Proof.* This is an easy consequence of Proposition 3.1(b).          □

For example, $\{n^n\}_{n=1}^{\infty}$ and $\{n!\}_{n=1}^{\infty}$ cannot be Cochran sequences since, for any $C, s, \gamma > 0$,

$$n^n > Cn^s\gamma^n \quad \text{and} \quad n! > Cn^s\gamma^n$$

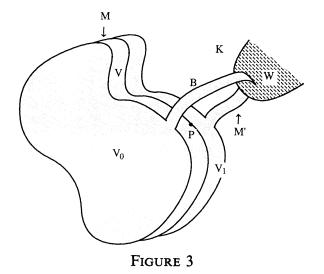for all sufficiently large $n$.

For a semi-boundary link $L = (M, K)$, $\overline{L} = (K, M)$ is also a semi-boundary link satisfying $\beta(\overline{L}) = \beta(L)$ as in Proposition 1.1 (b). Therefore the two sequences $\{\beta_i(L)\}_{i=1}^{\infty}$ and $\{\beta_i(\overline{L})\}_{i=1}^{\infty}$ have the same first entries. By the following theorem, this is the only relation between them.

THEOREM 3.5. *For any two linear recurrence sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ such that $n_1 = m_1$, there exists a semi-boundary link $L$ satisfying $\{\beta_i(L)\}_{i=1}^{\infty} = \{n_i\}_{i=1}^{\infty}$ and $\{\beta_i(\overline{L})\}_{i=1}^{\infty} = \{m_i\}_{i=1}^{\infty}$.*

For a semi-boundary link $L$, and for any integer $m$, Cochran constructed a new semi-boundary link $\int_m L$ called an *m-antiderivative* of $L$, which is well defined up to weak-cobordism satisfying

(vi)  $\beta_1(\int_m L) = m$,

(vii)  $D(\int_m L)$ is weakly-cobordant to $L$ and hence

$$\beta_i\left(\int_m L\right) = \beta_{i-1}(L) \quad \text{for } i \geq 2.$$

FIGURE 3

LEMMA 3.6. $D(\overline{\int_0 L})$ *is weakly-cobordant to a boundary link for any semi-boundary link* $L$.

*Proof.* Here we use Cochran's construction of antiderivatives. Let $L = (M, K)$ be a semi-boundary link with a special Seifert pair $(V, W)$. Let $c\colon V \times [0, 1] \hookrightarrow \mathbf{S}^3$ be a collar of $V$ such that

$$c(V \times \{\tfrac{1}{2}\}) = V \quad \text{and} \quad c(V \times [0, 1]) \cap K = \varnothing.$$

Let $V_i = c(V \times \{i\})$ for $i \in [0, 1]$. Choose orientations on $V_0$, $V_1$, and $c(M \times [0, 1])$ so that they agree on boundaries. In the procedure we smooth all the corners whenever necessary. Let $\gamma$ be a smooth nonsingular path joining two points $P \in M$ and $Q \in K$ such that

$$\mathrm{int}(\gamma) \cap (c(V \times [0, 1]) \cup W) = \varnothing.$$

Let $J$ be a small closed interval on $M$ centered at $P$. Push a copy of $c(J \times (0, 1))$ off $c(M \times [0, 1])$ along $\gamma$ passing $Q$ slightly, so that the image of $c(J \times [0, 1])$ becomes a band $B$ connecting $V_0$ and $V_1$, which intersects $\mathrm{int}(W)$ transversely on the image of $c(J \times \{\tfrac{1}{2}\})$ in a small neighborhood of $Q$. See Figure 3. Then

$$M' = \partial(c(M \times [0, 1]) \cup B)$$

is a knot intersecting $W$ in two points. Replacing a small tubular neighborhood of $M' \cap W$ in $W$ by a tube that goes along the lower half of $M'$, we get a Seifert surface $W_0$ for $K$ such that $W_0 \cap M' = \varnothing$. Then
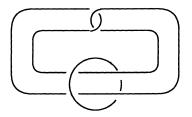
$$(c(M \times [0, 1]) \cup B, W_0)$$

FIGURE 4. Whitehead link

is a special Seifert pair for $(M', K)$. It is easy to see that

$$((c(M \times [0, 1]) \cup B) \cap W_0, K)$$

is ambient isotopic to $(M, K)$. Since $c(M \times [0, 1])$ is untwisted, we have $\beta(M', K) = 0$. Therefore $(M', K) = \int_0 L$. Using the same Seifert surfaces for $\overline{\int_0 L} = (K, M')$, we get $D(\overline{\int_0 L}) = (M, M')$. Since $V \cap (V_0 \cup V_1 \cup B) = \varnothing$,

$$(M, M') = (\partial V, \partial(V_0 \cup V_1 \cup B))$$

is a boundary link. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 3.7. $\{\beta_i(\overline{\int_0 L})\}_{i=1}^{\infty}$ is identically zero.

*Proof of Theorem* 3.5. By Theorem 3.2, there are semi-boundary links $L_1$ and $L_2$ such that $\{\beta_i(L_1)\}_{i=1}^{\infty} = \{n_j\}_{j=2}^{\infty}$ and $\{\beta_i(L_2)\}_{i=1}^{\infty} = \{m_j\}_{j=2}^{\infty}$.

Let $W$ be the Whitehead link as in Figure 4. Then it is easy to see that

$$\{\beta_i(W)\}_{i=1}^{\infty} = \{\beta_i(\overline{W})\}_{i=1}^{\infty} = \{1, 0, 0, \ldots\}$$

and

$$\{\beta_i(-W)\}_{i=1}^{\infty} = \{\beta_i(\overline{-W})\}_{i=1}^{\infty} = \{-1, 0, 0, \ldots\}$$

where $-W$ is the mirror image of $W$. Let

$$H = \begin{cases} W & \text{if } n_1 \geq 0, \\ -W & \text{if } n_1 < 0. \end{cases}$$

Define

$$L = \underbrace{H \natural \cdots \natural H}_{|n_1|} \natural \int_0 L_1 \natural \overline{\int_0 L_2}.$$

Then, by the additivity of the Cochran sequence and Corollary 3.7, we have

$$\beta_i(L) = |n_1|\beta_i(H) + \beta_i\left(\int_0 L_1\right) + \beta_i\left(\overline{\int_0 L_2}\right)$$
$$= \begin{cases} n_1 + 0 + 0 & \text{if } i = 1, \\ 0 + n_i + 0 & \text{if } i > 1 \end{cases}$$
$$= n_i \quad \text{for all } i.$$

Similarly,

$$\beta_i(\overline{L}) = |n_1|\beta_i(\overline{H}) + \beta_i\left(\overline{\int_0 L_1}\right) + \beta_i\left(\int_0 L_2\right)$$
$$= \begin{cases} m_1 + 0 + 0 & \text{if } i = 1, \\ 0 + 0 + m_i & \text{if } i > 1 \end{cases}$$
$$= m_i, \quad \text{for all } i. \qquad \square$$

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