TORUS ORBITS IN G/P

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Let G be a complex semisimple Lie group of rank l, with fixed Borel subgroup B and maximal torus H. Let P be a standard parabolic subgroup. The torus H acts on G/P by $gP \mapsto hgP$. The closure X in G/P of an orbit $\{hgP | h \in H\}$ is called a *torus orbit* if it is l-dimensional and satisfies a certain genericity condition; it is a rational algebraic variety whose structure is intimately related to Lie theory, symplectic geometry, and the theory of convex bodies. This paper presents: (1) an abstract description of the torus orbit X by means of a rational polyhedral fan; (2) a description of the torus-invariant divisor whose linear system provides a natural embedding (the *Plücker embedding*) of X into a projective space; (3) a discussion of the correspondence between this divisor and the momentum mapping associated to the action on X of the compact torus $T \subset H$; (4) a list of generators of the ideal defining the Plücker embedding; (5) a formula for the intersection multiplicity of certain important torus invariant divisors on X.

We have encountered torus orbits in several problems, and the calculations just mentioned have proved useful in those other studies. In work with N. Ercolani, we find torus orbits as compactified (complex) level varieties in a certain integrable Hamiltonian system, the so-called Toda lattice. A. Bloch, T. Ratiu, and Flaschka use torus orbits in the compact setting, K/T rather than G/B, to prove a convexity theorem for a "Hermitian" Toda lattice (Duke Math. J., to appear). In collaboration with R. Cushman, we study Gröbner bases for projective embeddings of torus orbits; these are simpler than, but in some model cases dual to, the standard monomials on G/P itself. Finally, the theory of integrable systems suggests that a detailed understanding of torus orbits in loop groups might be useful and interesting.

Because the summary of necessary definitions from the theory of toric varieties takes several pages (cf. §2), we devote this Introduction mostly to a description of results that can be stated without much specialized apparatus. Just a few words about items (1), (2), and (3) above. In §3, we establish some properties of the image of the momentum map referred to in point (3); it is a convex polytope with vertices in the weight lattice. The fan Δ defining X as toric variety is

then constructed in Theorem 1, §4. A "Plücker" embedding is defined and studied in §5. As one knows from Borel-Weil-Bott theory, G/P can be embedded in a projective space by the sections of a certain line bundle L_ω, where ω (a sum of fundamental weights) characterizes the parabolic subgroup P. The pullback L_ω^X of this bundle to $X \subset G/P$ embeds X in a (generally different) projective space. The corresponding divisor on X is computed in Theorem 2, and the dimension of the projective space in Theorem 3: it is equal to the number of distinct weights in the representation of G with highest weight ω.

Some of this material appears, in one form or other, in the literature, e.g. [1], [3], [9]. We have not, however, seen the complete picture spelled out in a way that makes it possible to do computations using the extensive theory of toric varieties. The results provide a simple and elegant illustration of toric varieties, and should be better known.

We now summarize the content of §§6 and 7. As mentioned already, one may associate a weight $\omega = \omega_{i_1} + \cdots + \omega_{i_s}$, to the parabolic P. Correspondingly, there is a representation (with highest weight ω) of G on a vector space V^ω with highest weight vector v^ω. The stabilizer of v^ω is precisely P. Furthermore, the projectivization $\mathbf{P}(\mathscr{O}^\omega)$ of the orbit of G through v^ω is isomorphic to G/P. Let $\mathscr{A}$ be the set of all weights, listed with multiplicity if necessary, and choose a weight vector v^μ for each $\mu \in \mathscr{A}$. Then one may write $v \in V^\omega$ as

$$v = \sum_{\mu \in \mathscr{A}} \pi_\mu v^\mu.$$

The π_μ are called *Plücker coordinates*. Kostant found a set of quadratic equations in the π_μ which generates the ideal of $\mathbf{P}(\mathscr{O}^\omega)$ in $\mathbf{P}(V^\omega)$. In §6, we rewrite his equations, and extract an ideal for the Plücker embedding of the torus orbit X.

THEOREM. *The variety* $\mathbf{P}(\mathscr{O}^\omega)$ *is defined by equations of the form*

$$\pi_\mu \pi_{\mu'} = \sum_{\nu+\nu'=\mu+\mu'} c_{\mu\mu'}^{\nu\nu'} \pi_\nu \pi_{\nu'}.$$

The generic torus orbit is defined by

$$(*) \qquad \pi_\mu \pi_{\mu'} = k_{\mu\mu'}^{\nu\nu'} \pi_\nu \pi_{\nu'}, \qquad \nu + \nu' = \mu + \mu'.$$

Some of the equations $(*)$ may degenerate to linear equations. Remember that weights with multiplicity > 1 are listed repeatedly; if

μ', ν' both label the weight β, then $\mu + \mu' = \mu + \beta = \mu + \nu'$, and the factor π_μ cancels from

$$\pi_\mu \pi_{\mu'} = k \pi_\mu \pi_{\nu'},$$

leaving $\pi_{\mu'} = k \pi_{\nu'}$. In this way, the dimension of the projective space in which X is naturally embedded can often be decreased; Theorem 4 gives the precise statement.

As already mentioned, this result is used elsewhere in a study of Gröbner bases of the ideals defining projective embeddings of torus orbits.

Our final Theorem, in §7, is important for the analysis of the complex Toda lattice. Let X be a torus orbit in G/B. Let D_j be the torus invariant divisor defining the line bundle L_{ω_j}.

THEOREM. *The intersection number* $(D_1 \cdots D_l)$ *is given by*

$$(**) \qquad\qquad (D_1 \cdots D_l) = |W| / \det C,$$

where $|W|$ *is the order of the Weyl group of* G *and* C *is the Cartan matrix of* G.

There is a similar formula for the intersection $(D_{i_1} \cdots D_{i_s} ; V(\tau))$, where $V(\tau)$ is a suitable slice transverse to the intersection of the D_{i_j}. This computation uses all the formulas derived in the preparatory §§3, 4, and 5.

In the nonperiodic Toda lattice, the interest is in the cohomological *and* set-theoretic intersection multiplicity of divisors linearly equivalent to the D_j. These are the so-called "balances" of Painlevé analysis. Empirical formulas were found by one of us (H.F.) in 1986, and stimulated much of our subsequent work. Formulas like $(**)$ were announced by M. Adler and P. van Moerbeke at a conference at MSRI in June 1989; in their setting, X is an *additive* torus orbit, i.e., abelian variety, in a loop group, and the D_j are translates of the theta-divisor; this is relevant to the *periodic* Toda lattice. It is not clear at present why results about $(\mathbb{C}^*)^l$-invariant divisors should carry over to a quite different situation with barely any change; a generalization of the present paper to loop groups may provide some interesting answers.

He was a member of the MSRI in Berkeley during the spring of 1989, when portions of this work were done. L. H. is a Research Associate of C.N.R.S., Belgium.

2. Notation and basic facts. In this section, we set down some Lie theory notation and certain facts about torus orbits in homogeneous spaces. The material will be used routinely in later sections, so the reader might want to skim this part in order to become acquainted with our conventions.

2.1. *Lie theory.* G is a simply connected complex semisimple Lie group of rank l. Fix a Borel subgroup B and its torus H. The Lie algebras are $\mathscr{G}$, $\mathscr{B}$, and $\mathscr{H}$. The Weyl group is

$$W = \mathrm{Normalizer}(H)/H.$$

We denote its elements by w, and we do not distinguish between the class w and a representative of that class unless, of course, the choice of representative makes a difference.

The positive simple roots are $\alpha_1, \ldots, \alpha_l$. The root system is R, and the set of positive (resp. negative) roots is R^+ (resp. $-R^+$). The root lattice will be denoted by M, and the Euclidean space spanned by M is called $M_{\mathbf{R}}$ (this notation conforms to [8]). There is a natural inner product $(\cdot, \cdot)$ on $M_{\mathbf{R}}$. The Weyl group W acts on $M_{\mathbf{R}}$, preserving the inner product. The reflection in a root β will be denoted by s_β. For $\alpha \in R$, we fix a root vector e_α in the root space $\mathscr{G}_\alpha$; thus $[\xi, e_\alpha] = \alpha(\xi)e_\alpha$, $\xi \in \mathscr{H}$. The homomorphism $H \to \mathbf{C}^*$ induced by α is denoted by exponent α:

$$(\exp \xi)^\alpha = \exp(\alpha(\xi)).$$

The fundamental weights are called $\omega_1, \ldots, \omega_l$. The *coweights* are defined by

$$\check{\omega}_j = \frac{2\omega_j}{(\alpha_j, \alpha_j)};$$

the **Z**-lattice generated by the $\check{\omega}_j$ is called N, and the corresponding Euclidean space is $N_{\mathbf{R}}$. M and N are dual; we denote the pairing by $\langle \cdot, \cdot \rangle$, so that $\langle \check{\omega}_i, \alpha_j \rangle = \delta_{ij}$. The Weyl group W also acts on $N_{\mathbf{R}}$.

Abstractly, one may identify $N_{\mathbf{R}}$ with the real part $\mathscr{H}_{\mathbf{R}}$ of the Lie algebra of H, and $M_{\mathbf{R}}$ with its dual $(\mathscr{H}_{\mathbf{R}})^*$. The pairing $\langle \cdot, \cdot \rangle$ is given by the Killing form.

A subgroup P of G containing B is called *parabolic*. Every parabolic P is associated with certain data. There is a subset $S \subset \{1, \ldots, l\}$ determined by P; let $\widetilde{S}$ be the complement of S. Let $-R^+(\widetilde{S})$ be

the set of negative roots in the root subsystem of R generated by α_k, $k \in \widetilde{S}$, and let $-R^+(S) = -R^+ \backslash -R^+(\widetilde{S})$. The Lie algebra $\mathscr{P}$ of P has the direct sum decomposition

$$\mathscr{P} = \sum_{\alpha \in R^+} \mathbf{C}\mathscr{G}_\alpha + \sum_{\alpha \in -R^+(\widetilde{S})} \mathbf{C}\mathscr{G}_\alpha.$$

Set

$$\mathscr{N}_P = \sum_{\alpha \in -R^+(S)} \mathbf{C}\mathscr{G}_\alpha;$$

this is a nilpotent Lie algebra, and $\mathscr{G} = \mathscr{N}_P \oplus \mathscr{P}$. Let N_P be the Lie group $N_P = \exp \mathscr{N}_P$.

Let ω_P, or ω for short, be the weight $\sum_{j \in S} \omega_j$. This weight is stabilized by the subgroup W_S of W generated by the reflections s_{α_k}, $k \in \widetilde{S}$. There is an irreducible representation ρ^ω of G, with highest weight ω; call the representation space V^ω and the highest weight vector v^ω. We use the same notation for the infinitesimal representation of $\mathscr{G}$. P is the subgroup of G that stabilizes the one-dimensional complex vector space spanned by v^ω. The projectivization of the orbit $\mathscr{O}^\omega$ of G through v^ω can be identified with G/P:

$$\mathbf{P}(\mathscr{O}^\omega) = \mathbf{P}(\{\rho^\omega(g)v^\omega | g \in G\}) \cong G/P.$$

This identification amounts to an embedding of the abstract algebraic variety G/P into the projective space $\mathbf{P}(V^\omega)$.

Let Π^ω be the set of weights of the representation ρ^ω (weights that have multiplicity are thought of as being listed several times). For each $\mu \in \Pi^\omega$, choose a weight vector v^μ. Every $x \in \mathbf{P}(\mathscr{O}^\omega) \cong G/P$ can be written as

$$x = c \sum_{\mu \in \Pi^\omega} \pi_\mu(x)v^\mu;$$

the (nonzero) scalar c is arbitrary, since we are dealing with the projectivized orbit. The homogeneous coordinates $[\pi_\mu]$ on $\mathbf{P}(V^\omega)$ are called *Plücker coordinates*. It is possible to introduce a Hermitian inner product in V^ω so that the weight vectors form an orthonormal basis; moreover, the operators $\rho^\omega(e_\alpha)$ and $\rho^\omega(e_{-\alpha})$ are adjoints of each other, and elements of the compact real form of G are represented by unitary operators. The orbit $\mathbf{P}(\mathscr{O}^\omega)$ is defined by a set of equations in the $[\pi_\mu]$; we explain those later. For a (projectivized) weight vector v^μ to lie on $\mathbf{P}(\mathscr{O}^\omega)$, it is necessary and sufficient that $\mu \in W \cdot \omega = $ the orbit of the Weyl group through ω. Actually, since

W_S stabilizes ω, the w's range only over coset representatives from W/W_S; we write $w \in W/W_S$ for short.

Both the description of P in terms of the root subsystem $-R^+(\widetilde{S})$ and as stabilizer of v^ω will be useful later.

EXAMPLE 1. Let $G = \mathrm{SL}(3, \mathbf{C})$. Take $S = \{2\} \subset \{1, 2\}$. The set $-R^+(\widetilde{S})$ is $\{-\alpha_1\}$, and $-R^+(S) = \{-\alpha_2, -\alpha_1 - \alpha_2\}$. The subgroups P and N_P consist of matrices of the form

$$p = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

The weight ω_P is the fundamental weight ω_2. The corresponding representation ρ^{ω_2} acts on $\bigwedge^2 \mathbf{C}^3$ by $g : v \wedge w \mapsto gv \wedge gw$. Evidently, the parabolic P stabilizes the line through the highest weight vector $v_1 \wedge v_2$, or—equivalently—the plane spanned by v_1 and v_2 ($\{v_1, v_2, v_3\}$ is the standard basis of $\mathbf{C}^3$). The orbit $\mathscr{O}^{\omega_2}$ can be identified with the set of all 2-planes in $\mathbf{C}^3$; this is the dual projective plane $(\mathbf{CP}^2)^*$. The Plücker coordinates are the standard homogeneous coordinates $[z_1^* : z_2^* : z_3^*]$.   □

2.2. *Torus orbits.* The complex torus $H \subset G$ acts on G/P in the obvious way:

$$H \ni h : gP \mapsto hgP = hgh^{-1}P.$$

The orbits of this action have dimension d, $0 \le d \le l$. We are interested primarily in l-dimensional orbits that are generic in a sense we now define.

DEFINITION 1. For $w \in W/W_S$, let P^w be the group wPw^{-1}, and let $N_P^w = wN_Pw^{-1}$. Let $\widetilde{Z}^w$ be the big cell of G/P^w, i.e. the set of cosets gP^w for which g has a factorization $g = n^w p^w$, with $n^w \in N_P^w$ and $p^w \in P^w$.

One knows that $\widetilde{Z}^w$ is a Zariski open subset of G/P^w. It can also be identified with a Zariski open subset of G/P.

LEMMA 1. *There is a natural isomorphism* $G/P^w \cong G/P$.

Proof. Send gP^w to gwP. This is bijective:

$$gP^w = g'P^w \Leftrightarrow g^{-1}g' \in wPw^{-1} \Leftrightarrow w^{-1}g^{-1}g'w \in P$$
$$\Leftrightarrow (gw)^{-1}(g'w) \in P \Leftrightarrow gwP = g'wP.$$

One should note that the argument does not depend on a choice of representative for w.   □

LEMMA 2. *The $\widetilde{Z}^w$ cover G/P.*

Proof. This follows from the Bruhat decomposition [2].   □

Terminology. A set $Y = \{hgP | h \in H\}$ is called an open torus orbit (through gP). "Open" refers to the fact that Y is open in its closure; it is almost never open in G/P. If $\dim Y = \dim H = l = \operatorname{rank} G$, we say that Y is *maximal.* We say that an open torus orbit Y is *generic* if it is maximal and

$$Y \subset \bigcap_{w \in W/W_S} \widetilde{Z}^w.$$

The closure X of a generic open torus orbit it called a *torus orbit.* The closure of an open torus orbit of dimension $d < l$ will be called a torus orbit, and there will be some qualifier like "codimension k" or "lower-dimensional".

REMARK 1. (i) Since we are interested primarily in the closures of generic open torus orbits, it seems reasonable to minimize the number of adjectives in that case.

(ii) Suppose an open torus orbit Y meets $\widetilde{Z}^w$ in one point $n^w P^w$. Then all points of the form $h n^w h^{-1} P^w$ are in the big cell of G/P^w, so that $Y \subset \widetilde{Z}^w$. Thus, as soon as an open torus orbit meets $\bigcap_{w \in W/W_S} \widetilde{Z}^w$, it is generic.

(iii) A point $x \in G/P$ belongs to $\widetilde{Z}^w$ if and only if the Plücker coordinate $\pi_{w \cdot \omega}(x)$ is nonzero. Therefore, an open torus orbit is generic if and only if all Plücker coordinates $\pi_{w \cdot x}(x)$, $w \in W/W_S$, are nonzero. This is the type of definition of "generic" used in [3].

(iv) One can prove that generic open torus orbits must have the maximal dimension l; see below.

Fix a torus orbit X in G/P. We now list certain properties of X. The first result is proved in [3].

Fact 1. X contains the points wP, $w \in W/W_S$; they are invariant under the action of H.

When G/P is realized as the projective variety $\mathscr{P}^\omega$, the torus orbit $X \subset G/P$ also becomes a projective variety and so acquires a Kähler form whose imaginary part, Ω, is symplectic. Let K be the compact real form of G, and let $T \subset K$ be the (compact) maximal torus whose complexification is H. The action of T on G/P is Hamiltonian with respect to Ω, and has a momentum mapping $J: G/P \to \mathscr{T}^*$. Here $\mathscr{T}^*$ is the dual of the Lie algebra $\mathscr{T}$ of T (which is naturally

identified with $M_{\mathbf{R}}$.) We are concerned only with the restriction of J to a torus orbit X; it will again be denoted by J.

Fact 2. The image $J(X)$ of X under J is a closed convex polytope in $M_{\mathbf{R}}$ with vertices $w \cdot \omega$, $w \in W/W_S$. Furthermore, $J(wP) = w \cdot \omega$.

The proof is in [1] and [3]. It is important to note that Ω, J and $J(X)$ depend on a choice of projective embedding (into $\mathbf{P}(V^\omega)$, in the present case); furthermore, the momentum map J is determined only up to translation in $M_{\mathbf{R}}$. The normalization in Fact 2 is the customary one; we return to this matter later. A fairly explicit description of $J(X)$ in terms of roots and weights is worked out in §3.

Fact 3. The set of regular values of the momentum map $J: X \to M_{\mathbf{R}}$ is the interior of $J(X)$. The singular values of J lie on the boundary of $J(X)$. Each lower-dimensional torus orbit in X maps onto a closed, lower-dimensional face of $J(X)$, and each face is the image of exactly one lower-dimensional torus orbit.

REMARK 2. Since $\dim J(X) = l$ for a generic torus orbit, it follows from Fact 3 that $\dim X = l$. A generic torus orbit is necessarily of maximal dimension.   □

The final result is a special case of Theorem 5.5 in [7].

Fact 4. Let τ be a codimension 1 face of $J(X)$, and let Y_τ be the open torus orbit, of codimension 1 in X, whose image under J is the relative interior of τ. Let $\mu \in M_{\mathbf{R}}$ be normal to τ, and let $\check{\mu} \in N_{\mathbf{R}} \cong \mathscr{H}_{\mathbf{R}}$ be the element that is dual to μ via the Killing form. Then:

   (i) the torus $\exp t\check{\mu}$ fixes the points of Y_τ;
   (ii) the Weyl group stabilizer W_μ of μ permutes the vertices of τ;
   (iii) μ may be taken to be a Weyl group conjugate of one of the fundamental weights $\omega_1, \ldots, \omega_l$.

We indicate briefly how this follows from Lerman's theorem. Lerman works with the compact group K and realizes G/P as a coadjoint orbit, but this is a minor point. He is interested in singular values of the momentum map from all of G/P to $\mathscr{T}^*$. He shows that the irreducible components of the set of singular values have the form

$$\text{convex hull of } W_\mu \cdot \eta,$$

where η is a vertex of $J(X)$ and μ is Weyl group conjugate to some ω_j. In general, such a set intersects the interior of $J(X)$. Atiyah's
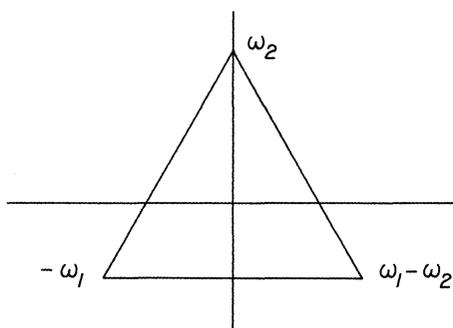
FIGURE 1

result, Fact 3, guarantees that the singular values of J on X lie on the faces of $J(X)$. If now η is a vertex of τ, then by Fact 2 the set $W \cdot \eta$ consists of all vertices of $J(X)$. To get exactly the vertices of τ, one must cut down to a subgroup of W that leaves τ invariant. According to Lerman, this subgroup is W_μ for some μ. It is generated by reflections in roots orthogonal to μ, and since the face τ has codimension 1, μ must be orthogonal to it.

EXAMPLE 1, CONTINUED. The torus $\mathrm{diag}(h_1, h_2, h_3)$ (with $h_1 h_2 h_3 = 1$) acts according to:

$$[z_1^* : z_2^* : z_3^*] \mapsto [h_1^{-1} z_1^* : h_2^{-1} z_2^* : h_3^{-1} z_3^*].$$

The fixed points are $[1 : 0 : 0]$, etc. One checks that $[0 : 0 : 1]$ represents the plane $v_1 \wedge v_2$, which is the coset $\mathrm{id}\, P$; $[0 : 1 : 0]$ is the plane $v_1 \wedge v_3 =$ the coset

$$wP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} P,$$

and so on. There are three one-dimensional open torus orbits: $[* : * : 0]$, $[* : 0 : *]$, and $[0 : * : *]$ ("$*$" means a nonzero entry.) The generic open torus orbit Y is $[* : * : *]$. In particular, $\overline{Y} = (\mathbf{CP}^2)^*$, so that there is only one torus orbit, G/P itself.

One can find the Kähler form of G/P in [10], for example; the polytope $J(X)$ has vertices at the weights

$$\omega_2, \quad \omega_2 - \alpha_2 = \omega_1 - \omega_2, \quad \omega_2 - \alpha_1 - \alpha_2 = -\omega_1$$

of the representation ρ^{ω_2}, see Figure 1. The weight vectors are $v_1 \wedge v_2$, $v_1 \wedge v_3$, and $v_2 \wedge v_3$, which correspond to the fixed points, as just explained. $\qquad \square$

2.3. *Toric varieties.* We collect some basic properties of toric varieties. Our notation follows [8].

The theory of toric varieties deals with abstract manifolds equipped with a torus action. The torus is isomporphic to $(\mathbf{C}^*)^l$, just like H in the preceding subsection, but it is not given quite so explicitly: one starts with dual lattices N and M, and thinks of the torus as a group of characters,

$$T_N = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{C}^*) = N \otimes_{\mathbf{Z}} \mathbf{C}^*.$$

Thus, an element of T_N is a function $t \colon M \to \mathbf{C}^*$ satisfying $t(m + m') = t(m)t(m')$, $t(0) = 1$. Points of the lattice N give rise to algebraic one-parameter subgroups of T_N according to

$$t_c(\cdot) = c^{\langle n, \cdot \rangle}, \qquad c \in \mathbf{C}^*.$$

In Lie group notation, one would set $s = \log c$ and write the one-parameter subgroup as $\exp sn$. The corresponding character is determined by the values of the roots on the subgroup: $t(m) = (\exp sn)^m$.

DEFINITION. A subset σ of $N_{\mathbf{R}}$ is a *strongly convex rational polyhedral cone* if there exist $n_1, \ldots, n_r \in N$ such that

$$\sigma = \{a_1 n_1 + \cdots + a_r n_r | a_i \geq 0, \forall i\}$$

and $\sigma \cap (-\sigma) = \{0\}$.

DEFINITION. A *fan* in $N_{\mathbf{R}}$ is a finite collection Δ of strongly convex rational polyhedral cones satisfying the following conditions: (i) Every face of any $\sigma \in \Delta$ is in Δ; (ii) for all $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ again belongs to Δ; (iii) $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbf{R}}$.

REMARK. Oda calls this a *finite complete fan*, but since we consider no other kind we drop the adjectives. See Figure 2 below for a picture of a fan.   □

To a fan Δ, one associates an abstract variety with torus action. We review the steps.

Let $\sigma \in \Delta$. The *dual cone* is the set

$$\check{\sigma} = \{x \in M_{\mathbf{R}} | \langle x, y \rangle \geq 0, \ \forall y \in \sigma\}.$$

Set $\mathscr{S}_\sigma = M \cap \check{\sigma}$. $\mathscr{S}_\sigma$ is an additive semigroup. A *character* of $\mathscr{S}_\sigma$ is a function $u \colon \mathscr{S}_\sigma \mapsto \mathbf{C}$ satisfying

$$u(0) = 1, \quad u(m + m') = u(m)u(m'), \qquad m, m' \in \mathscr{S}_\sigma.$$

Note that u is allowed to be zero. Let U_σ be the set of all characters of $\mathscr{S}_\sigma$.

PROPOSITION [8, *Prop. 1.2, abbreviated*]. *Let* $m_1, \ldots, m_p \in \mathscr{S}_\sigma$ *be elements such that*

$$\mathscr{S}_\sigma = \mathbf{Z}_{\geq 0} m_1 + \cdots + \mathbf{Z}_{\geq 0} m_p.$$

Define $\mathbf{e}(m)(u) = u(m)$. *The map*

$$(\mathbf{e}(m_1), \ldots, \mathbf{e}(m_p)) \colon U_\sigma \hookrightarrow \mathbf{C}^p$$

is one-to-one, and if U_σ *is identified with its image, it becomes an irreducible normal affine variety.*

The U_σ, $\sigma \in \Delta$, are patched together as follows. Let $\tau = \sigma \cap \sigma'$. Its dual $\check{\tau}$ contains both $\check{\sigma}$ and $\check{\sigma}'$. There exist [8, Proposition 1.3] $m_0 \in \mathscr{S}_\sigma$ and $m_0' \in \mathscr{S}_{\sigma'}$ such that

$$\langle m_0, \tau \rangle = \langle m_0', \tau \rangle = 0$$

and

$$\check{\tau} = \check{\sigma} \oplus \mathbf{R}_{\geq 0}(-m_0) = \check{\sigma}' \oplus \mathbf{R}_{\geq 0}(-m_0').$$

Note that if u is a character on $\check{\tau}$, it must satisfy

$$u(m_0)u(-m_0) = u(m_0 + (-m_0)) = u(0) = 1,$$

so that $u(m_0) \neq 0$. Likewise, $u(m_0') \neq 0$. This characterizes the intersection $U_\sigma \cap U_{\sigma'}$. If $u \in U_\sigma$, extend it to $\check{\tau} \cap M$ and restrict it to $U_{\sigma'}$; that defines the transition functions.

PROPOSITION [8, *Theorem 1.4, abbreviated*]. *The* U_σ, $\sigma \in \Delta$, *glue together to define a complete (but not necessarily projective) algebraic variety denoted by* $T_N \operatorname{emb}(\Delta)$. *Such a variety is called a toric variety.*

EXAMPLE 2. Because we will use this example to illustrate certain points later on, we go against our conventions this one time and let N be the *coroot* lattice and M the *weight* lattice of the Lie algebra $A_2 = \operatorname{sl}(3, \mathbf{C})$. The computations that follow are related to Example 1, but that won't be clear until the end of this section.

Consider the fan Δ and the dual cones depicted in Figure 2.

Look at the dual cone $\check{\sigma}_1$. The lattice semigroup $S_{\check{\sigma}_1} = \check{\sigma}_1 \cap M$ is generated over $\mathbf{Z}_{\geq 0}$ by *three* elements, $\omega_1 + \omega_2$, $2\omega_1 - \omega_2$, and ω_1. Call them m_1, m_2 and m_3. Since

$$(\omega_1 + \omega_2) + (2\omega_1 - \omega_2) = 3\omega_1,$$

a character u must satisfy

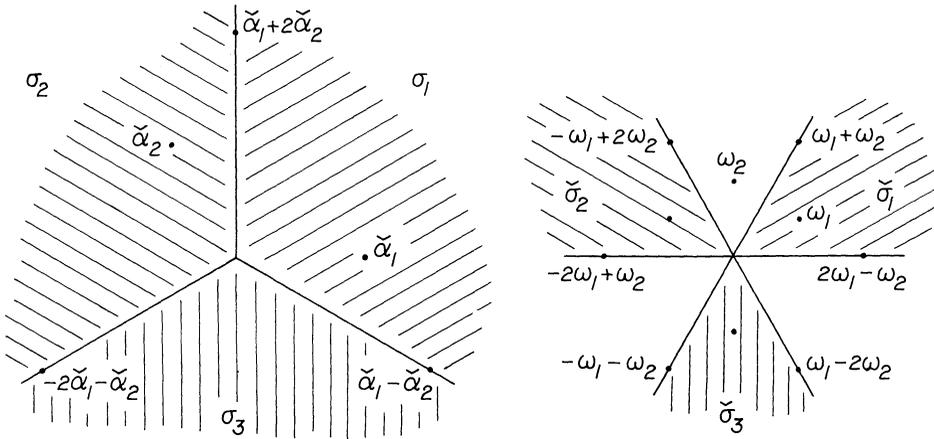$$u(\omega_1 + \omega_2)u(2\omega_1 - \omega_2) = u(\omega_1)^3.$$

FIGURE 2

Therefore we have, identically on U_{σ_1},

$$\mathbf{e}(m_1)\mathbf{e}(m_2) = \mathbf{e}(m_3)^3.$$

Set $\mathbf{e}(m_i) = x_i$. U_{σ_1} is identified with its image in $\mathbf{C}^3$ under $(\mathbf{e}(m_1),$ $\mathbf{e}(m_2), \mathbf{e}(m_3))$: it is the affine variety $x_1 x_2 = x_3^3$.

Likewise, for the cones σ_2 and σ_3, we set

$$(\mathbf{e}(-\omega_1 + 2\omega_2), \mathbf{e}(-2\omega_1 + \omega_2), \mathbf{e}(\omega_2 - \omega_1)) = (y_1, y_2, y_3)$$

and

$$(\mathbf{e}(\omega_1 - 2\omega_2), \mathbf{e}(-\omega_1 - \omega_2), \mathbf{e}(-\omega_2)) = (z_1, z_2, z_3).$$

The equations defining the affine varieties are the same:

$$y_1 y_2 = y_3^3, \quad z_1 z_2 = z_3^3.$$

We show how to find the transition functions. Consider $\sigma_1 \cap \sigma_2$. This is the one-dimensional cone τ through $\check{\alpha}_1 + 2\check{\alpha}_2$, with dual $\check{\tau} =$ the upper half plane in $M_{\mathbf{R}}$. The annihilators of τ are

$$m_0, \, m_0' = \pm(2\omega_1 - \omega_2).$$

The intersection $U_{\sigma_1} \cap U_{\sigma_2}$ is defined by $x_2 \neq 0$, $y_2 \neq 0$. To get the transition functions, compute as follows:

$$u(\omega_1 + \omega_2)u(-2\omega_1 + \omega_2) = u(-\omega_1 + 2\omega_2),$$

which implies

$$x_1 y_2 = y_1.$$

In this way, one finds

$$y_1 = x_1/x_2, \quad y_2 = 1/x_2, \quad y_3 = x_3/x_2,$$
$$z_1 = x_2/x_1, \quad z_2 = 1/x_1, \quad z_3 = x_3/x_1.$$

These three affine varieties, together with the transition functions, define the variety $T_N \operatorname{emb}(\Delta)$. It is easy to check that $T_N \operatorname{emb}(\Delta)$ can be defined by a single equation in homogeneous coordinates $[w_0 : w_1 : w_2 : w_3]$ in $\mathbf{CP}^3$,

$$w_0 w_1 w_2 = w_3^3.$$

The surface has three singularities of type A_2, and is birational to $\mathbf{CP}^2$. It contains the torus T_N as open dense subset: indeed, $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3}$ is the set of all *nonzero* characters, which is identified with T_N.   □

We now explain what polytopes in $M_{\mathbf{R}}$ have to do with fans in $N_{\mathbf{R}}$.

$T_N \operatorname{emb}(\Delta)$ is an *abstract* algebraic variety. One tries to embed it in a projective space. To this end, one needs an ample divisor. By [8, Corollary 2.5], it is enough to look for ample *torus-invariant* divisors. Such divisors are unions of codimension 1 torus suborbits in $T_N \operatorname{emb}(\Delta)$, which can be described in terms of the fan Δ.

For each edge ρ (= one-dimensional face in Δ), the characters u on $M \cap \rho^{\perp}$ define a codimension 1 toric subvariety $V(\rho)$ of $T_n \operatorname{emb}(\Delta)$ [8, Proposition 1.6], and all such subvarieties are obtained in this way. A torus-invariant divisor (and hence every divisor, up to linear equivalence) is determined by assigning an integer (the multiplicity) to each edge in Δ. The multiplicities are encoded in a function h which carries additional information about the divisor.

DEFINITION. Let $h\colon N_{\mathbf{R}} \to \mathbf{R}$ be a continuous function which assumes integer values on N and is linear on each $\sigma \in \Delta$. Such a function is called a Δ-*linear support function*; we write $h \in \operatorname{SF}(N, \Delta)$. If $h(x + x') \geq h(x) + h(x')$ for all $x, x' \in N_{\mathbf{R}}$, we say that h is *upper convex*, and if the inequality is strict when x, x' belong to different cones σ, σ' of Δ, h is said to be *strictly upper convex*.

THEOREM [8, *Proposition* 2.1, *Corollaries* 2.14, 2.15]. *If ρ is an edge in Δ, let $n(\rho) \in N$ be the minimal lattice generator. Let $h \in \operatorname{SF}(N, \Delta)$ be strictly upper convex. Then the divisor*

$$D_h = -\sum_{\rho} h(n(\rho)) V(\rho)$$

is ample (and very ample if T_N emb(Δ) is nonsingular). On each cone $\sigma \in \Delta$, h is defined by

$$h(x) = \langle l_\sigma , x \rangle$$

for a certain $l_\sigma \in M$. The convex hull of the l_σ is a polytope $\square_h$ in $M_{\mathbf{R}}$. The dimension of $H^0(X, \mathscr{O}_X(D_h))$ equals the number of lattice points on and inside $\square_h$.

REMARK. As is pointed out in [8], the sign convention is peculiar. To get *positive* multiplicities, one must assign *negative* values to $h(n(\rho))$. $\square$

This beautiful theorem opens the door to the detailed study of toric varieties. As explained in [8, Chapter 2], there is a 1-1 correspondence between torus-invariant divisors and Δ-linear support functions. Furthermore, every divisor is linearly equivalent to a torus-invariant one. For each h, there is a (possibly empty) polytope $\square_h$, whose definition is different from the one in the theorem when h is not upper convex. The number of lattice points in $\square_h$ always gives $\dim H^0(X, \mathscr{O}_X(D_h))$.

One of our goals is to find the $\square_h$ corresponding to the V^ω embedding of a torus orbit in G/P, and to relate it to the momentum polytope $J(X)$.

EXAMPLE 2, CONTINUED. Thus far, an abstract toric variety T_N emb(Δ) was realized as $w_0 w_1 w_2 = w_3^3$ in $\mathbf{CP}^3$. We compute the hyperplane divisor and the corresponding function h.

Take the edge ρ through $\check{\alpha}_1 + 2\check{\alpha}_2$ (see Figure 2). In the cone $\check{\sigma}_1$, $\rho^\perp$ is the line through $2\omega_1 - \omega_2$. A character on $M \cap \rho^\perp$ is one that vanishes on the other two generators of $\check{\sigma}_1 \cap M$. Under the map $\vec{e}(\cdot)$, this corresponds to $x_1 = x_3 = 0$. Similarly, one sees that the edge through $\check{\alpha}_1 - \check{\alpha}_2$ corresponds to $x_2 = 0$ and the edge through $-2\check{\alpha}_1 - \check{\alpha}_2$ to $y_2 = 0$.

Now, a hyperplane section of the embedding into $\mathbf{CP}^3$ is $w_3 = 0$. Its divisor on $w_0 w_1 w_2 = w_3^3$ is torus invariant. It has three irreducible components: $w_1 = 0$, $w_2 = 0$, and $w_0 = 0$, which correspond, respectively, to $x_1 = 0$, $x_2 = 0$, and $y_2 = 0$. Thus, the divisor can be written

$$V(\check{\alpha}_1 + 2\check{\alpha}_2) + V(\check{\alpha}_1 - \check{\alpha}_2) + V(-2\check{\alpha}_1 - \check{\alpha}_2).$$

These multiplicities are provided by the function h defined by

$$\langle -\omega_1 , \cdot \rangle \text{ on } \sigma_1, \quad \langle \omega_1 - \omega_2 , \cdot \rangle \text{ on } \sigma_2, \quad \langle \omega_2 , \cdot \rangle \text{ on } \sigma_3.$$

The polytope $\square_h$ is just the triangle in Figure 1. $\square$

We can now explain the point of this example.

(1) Associated to a projective embedding of a toric variety $T_N \operatorname{emb}(\Delta)$, there is a polytope $\square_h$ with vertices in M (the root lattice, in our setting);

(2) associated to a projective embedding of a torus orbit X in G/P, there is the momentum polytope $J(X)$ with vertices in the weight lattice (Fact 2).

[8, Theorem 2.22] also describes an inverse to the construction 1):

(3) Given a convex polytope $\square$ with vertices in M, there is a fan $\Delta(\square)$ and an $h = h_\square \in \operatorname{SF}(N, \Delta(\square))$ such that $\square = \square_h$.

In Example 1, we computed $J(X)$ for the torus orbit $X = (\mathbf{CP}^2)^*$. The fan in Example 2 corresponds to this polytope under the inverse construction 3); this emerges at the end of the example, when it is seen that $\square_h = J(X)$. Obviously, the singular variety $T_N \operatorname{emb}(\Delta)$ found in Example 2 is not the original X. Thus, the sequence of constructions

$$X \to J(X) \to \Delta(J(X)) \to T_N \operatorname{emb}(\Delta(J(X)))$$

does not return to the starting point. In the next two sections, we compute the correct fans and polytopes that describe torus orbits and their projective embeddings.

3. The momentum polytope. According to Fact 2, §2, the image $J(X)$ of X under the momentum map of the action of the compact torus is the convex hull of the Weyl group orbit $W \cdot \omega$. We shall need quite a bit of information about this polytope. Some of the results are probably known, but since there seems to be no convenient reference, we supply the (short) proofs.

DEFINITION 2. The convex polyhedral cone, with vertex at ω, generated by the edges of $J(X)$ emanating from ω, is denoted by $\omega + B$.

First we describe the edges that generate the cone $\omega + B$. The geometry is fairly clear. The weight ω lies in the intersection of $l - |S|$ walls of the positive Weyl chamber, and in the closure of $|W_S|$ many Weyl group translates of that chamber. To get the edges leaving ω, one must reflect ω in the $|S|$ "opposite" walls of each chamber containing ω. In general, not all of the images will be distinct.

LEMMA 3. $\omega + B = \omega + \sum_{j \in S, w \in W_S} \mathbf{R}_{\geq 0} w(-\alpha_j)$.

Proof. Let K be the cone defined by the sum on the right. Since the generators of $\omega + K$ are $w s_{\alpha_j} \cdot \omega$ $(j \in S, w \in W_S)$, which are

vertices of $J(X)$, it is clear that $\omega + B$ contains $\omega + K$. We must show the other inclusion.

The proof goes by induction on the *level* of weights in the orbit $W \cdot \omega$. If $\mu = \omega - \sum_{j=1}^{l} n_j \alpha_j$ is a weight, its level $L(\mu)$ is defined to be $\sum_{j=1}^{l} n_j$. We make the induction hypothesis:

$$\text{when } L(w \cdot \omega) \leq r, \quad \text{then } w \cdot \omega \in \omega + K.$$

This is clearly true for $r = 0$ (the only weight being ω itself) and for $r = 1$, where the weights are $s_{\alpha_j}(\omega) = \omega - \alpha_j$, $j \in S$. Suppose it true up to level r. We must show that if $\mu \in W \cdot \omega$ is a weight with $L(\mu) \leq r$ (assumed to lie in $\omega + K$) and s_{α_i} is a reflection in a simple root, then $\mu' = s_{\alpha_i}(\mu) \in \omega + K$. Of course, we need only consider α_i for which μ' has level $r + 1$. If $i \notin S$, then $s_{\alpha_i} \in W_S$; this reflection stabilizes both ω and K, and so $\mu' \in \omega + K$. If $i \in S$, then $s_{\alpha_i}(\mu) = \mu - n\alpha_i$ (for an integer n), and this is clearly in $\omega + K$. □

COROLLARY 1. *Let X be a torus orbit in G/B. The weight ω is $\delta = \omega_1 + \cdots + \omega_l$. There are precisely l edges emanating from each vertex $w \cdot \delta$ of $J(X)$: they connect $w \cdot \delta$ to $w \cdot \delta + w(-\alpha_j)$, $j = 1, \ldots, l$.*

Proof. At the vertex δ, this follows from the Lemma, since $W_S = \{\text{id}\}$. Apply $w \in W$ to get the conclusion for the other vertices. □

COROLLARY 2. *If μ and μ' are two vertices of $J(X)$ connected by an edge, then $\mu - \mu'$ is a root.*

REMARK. This result improves on the characterization of " (G, P)-hypersimplex" in [3, §7], at least for generic torus orbits. There, it is shown that every edge $\mu - \mu'$ of $J(X)$ (for possibly nongeneric X) is a real multiple of a root. (The argument is not quite correct.) Compare also Lemma 4 in [6]. □

Proof. For edges issuing from ω, this is just the definition of K. Apply the Weyl group to get the result at the other vertices. □

DEFINITION 3. Let $\check{C}^-$ be the cone in $M_{\mathbf{R}}$ spanned by $-\alpha_1, \ldots, -\alpha_l$. The dual cone C^- is spanned by the negatives of the fundamental coweights. It might be called the "negative co-Weyl chamber."

LEMMA 4. $\check{B} = \bigcup_{w \in W_S} w \cdot C^-$.

Proof. If $w \in W_S$, then $w \cdot \alpha_j$, $j \in S$, always contains α_j, so it is a positive root. Thus, for $1 \leq i \leq l$, $j \in S$, and $w, w' \in W_S$, we have

$$\langle w(-\check{\omega}_i), w'(-\alpha_j) \rangle = \langle \check{\omega}_i, w^{-1}w' \cdot \alpha_j \rangle$$
$$= \langle \check{\omega}_i, \text{ positive root} \rangle \geq 0,$$

which, by Lemma 3, shows that $w \cdot C^- \subset \check{B}$, for all $w \in W_S$.

To establish the other inclusion, we need only show that

(∗)   if $x \in N_{\mathbf{R}}$, there exists some $w \in W_S$ such that
$$\langle x, w(-\alpha_j) \rangle \geq 0 \text{ for all } j \notin S.$$

Indeed, if $x \in \check{B}$, then for this choice of $w \in W_S$, we see from Lemma 3 that

$$\langle x, w(-\alpha_j) \rangle \geq 0 \quad \text{for all } j \in S,$$

which shows that $x \in w \cdot C^-$.

Let us prove (∗). Introduce the Lie algebra $\widetilde{\mathscr{G}}$ generated by the α_j, $j \notin S$. Let $\widetilde{N}$ be the coweight lattice of $\widetilde{\mathscr{G}}$, considered as sublattice of N. Every $x \in N_{\mathbf{R}}$ has a decomposition

$$x = y + \sum_{i \in S} \lambda_i \check{\omega}_i, \quad y \in \widetilde{N}_{\mathbf{R}}.$$

Since W_S is the Weyl group of $\widetilde{\mathscr{G}}$, there exists some $w \in W_S$ such that $w^{-1} \cdot y$ belongs to the "negative co-Weyl chamber" of $\widetilde{\mathscr{G}}$, i.e.

$$\langle y, w(-\alpha_j) \rangle \geq 0 \quad \text{for all } j \notin S.$$

Because w stabilizes $\check{\omega}_i$, $i \in S$, we have

$$\left\langle \sum_{i \in S} \lambda_i \check{\omega}_i, w(-\alpha_j) \right\rangle = \left\langle \sum_{i \in S} \lambda_i \check{\omega}_i, -\alpha_j \right\rangle = 0 \quad \text{for all } j \notin S.$$

Hence

$$\langle x, w(-\alpha_j) \rangle \geq 0 \quad \text{for all } j \notin S$$

as desired.   □

COROLLARY 3. $B = \bigcap_{w \in W_S} w \cdot \check{C}^-$.

Proof. This follows from a simple property of duality: for any finite collection $\mathscr{A}$ of closed convex cones,

$$\left(\bigcup_{\sigma \in \mathscr{A}} \sigma \right)^{\vee} = \bigcap_{\sigma \in \mathscr{A}} \check{\sigma},$$

if $\bigcup_{\sigma \in \mathscr{A}} \sigma$ is convex.   □

DEFINITION 4. Let $\check{\sigma}_{\mathrm{id}}$ be the cone in $M_{\mathbf{R}}$ generated by the roots in $-R^+(S)$.

LEMMA 5. $\check{\sigma}_{\mathrm{id}} = B$.

Proof. If $w \in W_S$ and $j \in S$, then $w(-\alpha_j) \in -R^+(S)$. Thus, by Lemma 3, $B \subset \check{\sigma}_{\mathrm{id}}$. By Corollary 3, the other inclusion is established if we show that

$$-R^+(S) \subset \bigcap_{w \in W_S} w \cdot \check{C}^-.$$

This, however, is obvious. If $\alpha \in -R^+(S)$, it has a simple root $-\alpha_j$ as a summand, for some $j \in S$. Then for $w \in W_S$, $w \cdot \alpha$ still contains $-\alpha_j$, so it is a negative root and $w \cdot \alpha \in \check{C}^-$. $\square$

LEMMA 6. $\Pi^\omega \subset J(X)$.

REMARK. In [3], this is attributed to [1], but we could not find the result in Atiyah's paper. It is probably standard. $\square$

Proof. Let $\mu \in \Pi^\omega$. It has the form

$$\mu = \omega - \sum_{j=1}^{l} n_j \alpha_j,$$

or $\mu = \omega + \alpha$ with $\alpha \in \check{C}^-$. For all $w \in W$, one has $w \cdot \mu \prec \omega$ [4]. If $w \in W_S$, then $w \cdot \omega = \omega$, and $\omega \succ w \cdot \mu = \omega + w \cdot \alpha$ implies that $w \cdot \alpha \in \check{C}^-$. It follows that

$$\alpha \in \bigcap_{w \in W_S} w \cdot \check{C}^-,$$

whence by Corollary 3,

$$\Pi^\omega \subset \omega + B.$$

Now choose another point $w \cdot \omega$ in the orbit $W \cdot \omega$. Declare the Weyl chamber containing this weight to be positive, and repeat the argument above with the corresponding new set of simple roots. It follows that Π^ω is contained in the intersection of all the cones with vertices in $W \cdot \omega$ and generated by the edges of $J(X)$ at those vertices, and that set is precisely $J(X)$. $\square$

LEMMA 7. *If* $\alpha \in -R^+(S)$, *then* $\omega + \alpha \in \Pi^\omega$.

Proof. Let $\alpha \in -R^+(S)$. Write

$$\alpha = -\sum_{j \in S} n_j \alpha_j - \sum_{k \notin S} n_k \alpha_k.$$

Since $\omega = \sum_{j \in S} \omega_j$, we have

$$(\omega, \alpha) < 0.$$

Because $\omega - \alpha$ is not a weight, it follows that $\omega + \alpha$ is a weight. □

4. The fan of a torus orbit. We will show that a torus orbit in G/P is a toric variety $T_N \operatorname{emb}(\Delta)$; according to our conventions, the fan Δ will lie in the space $N_\mathbf{R}$ generated by the coweight lattice, while the dual cones lie in $M_\mathbf{R}$, the Euclidean space containing the root lattice.

Let X be a fixed torus orbit. Set $Z^w = \tilde{Z}^w \cap X$. The Z^w cover X by Lemma 2, and because X is generic, the Z^w are nonempty. They will be shown to correspond to the affine varieties U_σ for the maximal-dimensional cones σ of a certain fan Δ.

LEMMA 8. *Fix an ordering of the roots in* $-R^+(S)$. *Every* $n \in N_P$ *has a unique factorization*

$$n = \prod_{\alpha \in -R^+(S)} \exp \xi_\alpha,$$

where $\xi_\alpha \in \mathscr{G}_\alpha$, *and the product is taken in the chosen ordering of the* α.

This is standard; see, for example, [2, Ch. 14]. We write $\xi_\alpha = c_\alpha e_\alpha$, and refer to the $c_\alpha \in \mathbf{C}$ as *the coordinates of* $n \in N_P$.

REMARK. A similar factorization holds for elements of N_P^w. □

REMARK 3. Let X_0 be the generic open torus orbit in X (so that $X = \overline{X}_0$). By definition, $X_0 \subset \bigcap_{w \in W/W_S} Z^w$. On the other hand, if $x \in \bigcap Z^w$, then by Remark 2, the open torus orbit through x is generic, so it has dimension l. But X contains only one l-dimensional open torus orbit, so that

$$X_0 = \bigcap_{w \in W/W_S} Z^w.$$ □

When X_0 is thought of as subset of Z^w, we will call it Z_0^w. The goal now is to define a cone $\mathscr{S}_{\sigma_w}$ of lattice points in M, and to identify Z_0^w with a certain set of characters on S_{σ_w}. Then, the lower-dimensional open torus orbits in Z^w are identified with characters

supported on the lower-dimensional faces of $\mathscr{S}_{\sigma_w}$. In this way, we build the dual cones $\breve{\sigma}_w$ and piece together the affine varieties U_σ. It will suffice to do the construction for $w = \mathrm{id}$; the results for Z^w follow upon conjugation by w. For simplicity, write Z and Z_0 for Z^{id} and Z_0^{id}.

Once and for all, pick a reference point

$$n^0 P \in X_0 = \bigcap_{w \in W/W_S} Z^w,$$

and write its coordinates with respect to G/P as c_α^0. The torus H acts on $n^0 P$ as follows:

$$(1) \quad h n^0 h^{-1} P = \prod_{\alpha \in -R^+(S)} h \exp(c_\alpha^0 e_\alpha) h^{-1} P = \prod_{\alpha \in -R^+(S)} \exp(h^\alpha c_\alpha^0 e_\alpha) P.$$

DEFINITION 5. Let $x = h n^0 h^{-1} P \in Z_0$. Define the character u_x on $\mathscr{S}_{\sigma_{\mathrm{id}}} = \breve{\sigma}_{\mathrm{id}} \cap M_{\mathbf{R}}$ by

$$(2) \quad u_x(\alpha) = h^\alpha, \quad \alpha \in -R^+(S)$$

(for σ_{id} see Definition 4). Extend u_x to all of $\mathscr{S}_{\sigma_{\mathrm{id}}}$ by the rules

$$u_x(0) = 1, \quad u_x(m + m') = u_x(m)u_x(m').$$

Let $U_{\sigma_{\mathrm{id}}}^0$ be the set of characters obtained in this way.

REMARK. Note that by definition, u_x is never zero on $\mathscr{S}_{\sigma_{\mathrm{id}}}$. Therefore, $U_{\sigma_{\mathrm{id}}}^0$ is a proper subset of the set $U_{\sigma_{\mathrm{id}}}$ of all characters. □

We now associate a character of $\mathscr{S}_{\sigma_{\mathrm{id}}}$ to the remaining points of Z. The argument is carried out in detail only for characters corresponding to the points of (nongeneric) codimension 1 open torus orbits in Z. It will be clear that the extension to smaller nongeneric open torus orbits in Z requires more complicated notation but no new ideas.

Let Y be a codimension 1 open torus orbit in X. By Facts 3 and 4, §2, the one-dimensional stabilizer torus $\exp t\breve{\mu}$ of Y is generated by a $\mu \in M_{\mathbf{R}}$ that is normal to a codimension 1 face of the momentum polytope $J(X)$. Assume first that this face contains the vertex ω of $J(X)$. Then by Lemma 5, μ is normal to a codimension 1 face $\breve{\tau}$ of $\breve{\sigma}_{\mathrm{id}}$. In the expression

$$(3) \quad \exp(t\breve{\mu}) n^0 \exp(-t\breve{\mu}) = \prod_{\alpha \in -R^+(S)} \exp(e^{\alpha(\breve{\mu})t} c_\alpha^0 e_\alpha)$$

there are factors with $\alpha(\breve{\mu}) = 0$ (when $\alpha \in \breve{\tau}$) and factors with $\alpha(\breve{\mu}) \neq 0$. Change the sign of μ, if necessary, to make $\alpha(\breve{\mu}) \leq 0$ for all

$\alpha \in -R^+(S)$. This is possible because the whole cone $\check{\sigma}_{id}$ lies on one side of $\check{\tau}$. As $t \to \infty$, the factors with $\alpha(\check{\mu}) < 0$ tend to the identity element, and the limit is

$$n^0(\tau) = \prod_{\alpha \in \check{\tau}} \exp(c_\alpha^0 e_\alpha).$$

The point $n^0(\tau)P$ still belongs to the big cell of G/P (it has an $N_P P$ factorization); hence it lies in Z. The $(l-1)$-dimensional torus generated by the annihilator of μ in $\mathscr{H} = N_R$ acts in the usual way:

$$(4) \qquad hn^0(\tau)h^{-1}P = \prod_{\alpha \in \check{\tau}} \exp(h^\alpha c_\alpha^0 e_\alpha).$$

It follows from (4) that the whole orbit Y lies in Z. Now define a character on points of the form (4) by setting

$$u(\alpha) = \begin{cases} h^\alpha, & \text{if } \alpha \in \check{\tau}, \\ 0, & \text{if } \alpha \notin \check{\tau}. \end{cases}$$

Extend it to $\check{\sigma}_{id} \cap M_R$ according to the usual rule.

The codimension 1 open torus orbits associated to faces of $\check{\sigma}_{id}$ are in fact the only ones that meet Z. Take any codimension 1 open torus orbit Y. Let

$$(5) \qquad n(Y)P = \prod_{\alpha \in -R^+(S)} \exp(c_\alpha^Y e_\alpha)P$$

be a point in $Y \cap Z$. Since $\Pi^\omega \subset J(X)$, there exists a $\xi \in N_R \cong \mathscr{H}_R$ such that $\omega(\xi) > \mu(\xi)$ for all $\mu \in \Pi^\omega$, $\mu \neq \omega$. In particular, since (by Lemma 7) $\omega + \alpha \in \Pi^\omega$ for all $\alpha \in -R^+(S)$,

$$(6) \qquad \omega(\xi) > (\omega + \alpha)(\xi) \Leftrightarrow \alpha(\xi) < 0 \quad \text{for all } \alpha \in -R^+(S),$$

and therefore

$$(7) \qquad \lim_{t \to \infty} e^{t\xi} n(Y)P e^{-t\xi} = \lim_{t \to \infty} \prod_{\alpha \in -R^+(S)} \exp(e^{t\alpha(\xi)} c_\alpha^Y e_\alpha) = \text{id}\, P.$$

This shows that $\text{id}\, P \in \overline{Y}$, and since $J(\text{id}\, P) = \omega$ (see Fact 2 in §2), the face of $J(X)$ associated to $\overline{Y}$ contains ω.

Entirely similar arguments establish a correspondence between codimension k open torus orbits in Z and the relative interiors of codimension k faces of $\check{\sigma}_{id}$. The only difference is that the stabilizers are now generated by k elements $\check{\mu}_1, \ldots, \check{\mu}_k$ in N_R, all of which are normal to a codimension k face of $\check{\sigma}_{id}$.

FIGURE 3

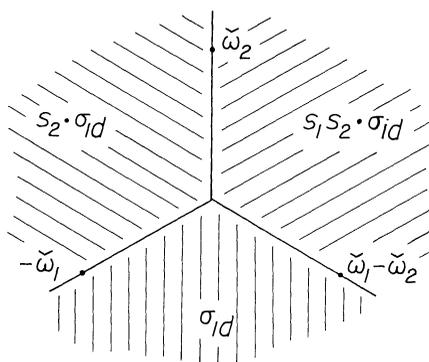We have now associated a character u_x on $\mathscr{S}_{\sigma_{\mathrm{id}}}$ to each point $x \in Z$. There are, in fact no other characters: if u is a character, define $x \in Z$ by

$$x = \prod_{\alpha \in -R^+(S)} \exp(u(\alpha)c_\alpha^0 e_\alpha).$$

It is easy to see that this definition is consistent.

To complete the identification of the torus orbit X with an abstract toric variety $T_N \operatorname{emb}(\Delta)$, one must construct the fan Δ in $N_{\mathbf{R}}$ and verify that the patches U_σ glue together as described in 2.3.

THEOREM 1. *Set*

$$\sigma_{\mathrm{id}} = \bigcup_{w \in W_S} w \cdot C^-.$$

The fan Δ of X consists of the cones $w \cdot \sigma_{\mathrm{id}}$, for $w \in W/W_S$, and all their faces.

Proof. It follows from Lemmas 4 and 5 that the cones σ_{id} and $\check{\sigma}_{\mathrm{id}}$ are dual. The verification of the patching conditions is tedious but straightforward, and is omitted.   □

EXAMPLE 1, CONTINUED. Consider again the torus orbit $X = (\mathbf{CP}^2)^* \cong G/P$. The fan described in Theorem 1 is drawn in Figure 3.

In the next section, we show how to compute the polytope $\square_h$ that determines the embedding of this toric variety into $\mathbf{P}(V^{\omega_2})$. It is related to, but not quite the same as, the momentum polytope $J(X)$.   □

5. The Plücker embedding of a torus orbit. The Plücker embedding of G/P into $\mathbf{P}(V^\omega)$ can be described in terms of a certain very ample

line bundle L_ω on G/P. The weight ω defines a character χ_ω on the parabolic subgroup P,

$$\chi_\omega(p) = \langle \rho^\omega(p)v^\omega, v^\omega \rangle^{-1},$$

and L_ω is the quotient of $G \times \mathbf{C}$ by the equivalence relation

$$(g, c) \sim (gp, \chi_\omega^{-1}(p)c)$$

with $g \in G$, $c \in \mathbf{C}$, and $p \in P$. The space $H^0(G/P, L_\omega)$ of global sections of L_ω may be identified with the set of holomorphic functions $s: G \to \mathbf{C}$ satisfying

$$s(gp) = \chi_\omega^{-1}(p)s(g).$$

One possible basis for this space is provided by the Plücker coordinates

$$\pi_\mu(g) = \langle \rho^\omega(g)v^\omega, v^\mu \rangle.$$

In this section, we compute the pullback L_ω^X (necessarily very ample on X) of L_ω to a generic torus orbit $X \subset G/P$. The projective embedding defined by L_ω^X will be called the *Plücker embedding of the torus orbit*.

Let $h \in \mathrm{SF}(N, \Delta)$. It defines a torus-invariant divisor D_h and an equivariant line bundle L_h (an invertible sheaf, if the torus orbit X is singular). We compute its transition functions. Each $m \in M$ determines a holomorphic function $\mathbf{e}(m)$ on the torus T_N. This function extends to a rational function, still called $\mathbf{e}(m)$, on $X = T_N\mathrm{emb}(\Delta)$. The divisor of $\mathbf{e}(m)$ is a principal torus invariant divisor on X. One has $\mathrm{div}(\mathbf{e}(-m)) = D_h$ with $h = \langle m, \cdot \rangle$ on each cone $\sigma \in \Delta$ [**8**, Proposition 2.1]. Furthermore, a codimension one toric subvariety $V(\rho)$ of X meets an affine set U_σ only when ρ is an edge of σ [**8**, Proposition 1.6]. Now D_h, by its very definition, coincides on U_σ with $\mathrm{div}(\mathbf{e}(-l_\sigma))$. It then follows that the transition functions are given by

$$\varphi_{\sigma'\sigma}(u) = \mathbf{e}(l_\sigma - l_{\sigma'})(u) \quad \text{on } U_\sigma \cap U_{\sigma'}.$$

Notice that since $l_\sigma = l_{\sigma'}$ on $\sigma \cap \sigma'$, one has

$$l_\sigma - l_{\sigma'} \in M \cap (\sigma \cap \sigma')^\perp \subset \mathscr{S}_{\sigma \cap \sigma'},$$

whence both functions $\mathbf{e}(l_\sigma - l_{\sigma'})$ and $\mathbf{e}(l_{\sigma'} - l_\sigma)$ are holomorphic and nonzero on $U_\sigma \cap U_{\sigma'}$.

THEOREM 2. *Let X be a torus orbit in G/P. The pullback L_ω^X of the line bundle L_ω from G/P to X can be defined by*

$$h = \langle w \cdot \omega - \omega, \cdot \rangle \quad \text{on } w \cdot \sigma_{\mathrm{id}}, \ w \in W/W_S.$$

The proof is based on a sequence of simple lemmas.

LEMMA 9. *The image $\widetilde{Z}^w$ of the big cell of G/P^w under the isomorphism $G/P^w \cong G/P$ in Lemma 1 can be identified as*

$$\widetilde{Z}^w = \{gP | w^{-1}gP \in \text{ big cell of } G/P\}.$$

Proof. Remembering that the isomorphism in question sends gP^w to gwP, we have

$$gP \in \widetilde{Z}^w \Leftrightarrow gP = wnP \text{ for some } n \in N_P$$
$$\Leftrightarrow w^{-1}gP = nP \Leftrightarrow w^{-1}gP \in \text{ big cell of } G/P. \qquad \square$$

LEMMA 10. *The line bundle L_ω can be trivialized over the covering $\widetilde{Z}^w$ of G/P. For fixed choices of the coset representatives of the Weyl group elements, the transition functions*

$$\psi_{w'w} : \widetilde{Z}^w \cap \widetilde{Z}^{w'} \to \mathbf{C}^*$$

of L_ω are given by

$$\psi_{w'w}(gP) = \chi_\omega(p')/\chi_\omega(p),$$

where $p, p' \in P$ are defined by

$$g = wnp = w'n'p', \qquad n, n' \in N_P.$$

Proof. Let $[(g, c)] \in L_\omega$ with $gP \in \widetilde{Z}^w$. By Lemma 9, we can factor

$$w^{-1}g = np, \qquad n \in N_P, p \in P,$$

and then

$$(g, c) = (wnp, c) \sim (wn, \chi_\omega(p)c).$$

Define a trivialization ψ_w on $\widetilde{Z}^w$ by

$$\psi_w([g, c]) = (gP, \chi_\omega(p)c).$$

The rest is clear. If a different coset representative for w is chosen, then ψ_w is multiplied by a nonzero constant of the form $\chi_\omega(h)$, $h \in H$, and the corresponding cocycles differ by a coboundary. $\qquad \square$

LEMMA 11. *Let $\beta \in -R^+(S)$ and assume that there is an edge of $J(X)$ from ω to $\omega + \beta$. Let*

$$n = \prod_{\alpha \in -R^+(S)} \exp(c_\alpha e_\alpha) \in N_P.$$

Then

$$\langle \rho^\omega(n)v^\omega, v^{\omega+\beta} \rangle = k_\beta c_\beta,$$

with k_β a nonzero constant independent of n.

Proof. If there is an edge of $J(X)$ from ω to $\omega + \beta$, $\omega + \beta$ is a vertex of $J(X)$ and so lies in the orbit $W \cdot \omega$. It is a weight of multiplicity 1. Hence $\rho^\omega(e_\beta)v^\omega = k_\beta v^{\omega+\beta}$ with $k_\beta \neq 0$. One easily computes that

$$\rho^\omega(n)v^\omega = v^\omega + k_\beta c_\beta v^{\omega+\beta} + \sum_{\substack{\alpha \in -R^+(S) \\ \alpha \neq \beta}} k_\alpha c_\alpha v^{\omega+\alpha} + \cdots .$$

The "$\cdots$" represent terms involving weight vectors v^μ for weights μ of the following form:

$$\mu = \omega + \sum_{j=1}^s n_j \gamma_j, \quad n_j \in \mathbf{Z}_{\geq 0}, \, \gamma_j \in -R^+(S), \text{ and } n_j \geq 2 \text{ if } s = 1.$$

If $s = 1$, then $\mu \neq \omega + \beta$ since $n\gamma$ is not a root when $n \geq 2$. If $s \geq 2$, then $\sum_{j=1}^s n_j \gamma_j$ is a sum of at least two lattice points in $\check{\sigma}_{\mathrm{id}}$, and there can be no edge from ω to μ in $\omega + \check{\sigma}_{\mathrm{id}} \supset J(X)$. It follows that the coefficient of $v^{\omega+\beta}$ in $\rho^\omega(n)v^\omega$ is precisely $k_\beta c_\beta$, as was to be shown. $\quad\square$

REMARK. Lemma 11 does not deal specifically with torus orbits. If nP lies on a torus orbit through some generic reference point $n^0 P$, then the conclusion remains true even if there is no edge from ω to $\omega + \beta$. (Remember that, by Lemma 7, $\omega + \beta \in \Pi^\omega$ when $\beta \in -R^+(S)$). The argument, however, seems to require the Plücker equations for the torus orbit (cf. §6). $\quad\square$

Proof of Theorem 2. First we need to show that h is integral and continuous. Integrality is clear: $w \cdot \omega$ is a weight, so $w \cdot \omega - \omega$ is a root and belongs to the lattice M. To prove continuity, it suffices to show that if $\rho = \mathbf{R}_{\geq 0} n_\rho$, $n_\rho \in N$, is an edge common to $w \cdot \sigma_{\mathrm{id}}$ and $w' \cdot \sigma_{\mathrm{id}}$, then $\langle (w - w') \cdot \omega, n_\rho \rangle = 0$. The edge ρ defines a codimension one torus suborbit $V(\rho)$ of X [8, Proposition 1.6]. According to Fact 4, §2, the corresponding codimension 1 face τ of $J(X)$ annihilates n_ρ. Furthermore, since ρ is common to $w \cdot \sigma_{\mathrm{id}}$ and $w' \cdot \sigma_{\mathrm{id}}$, $V(\rho)$ meets $U_{w \cdot \sigma_{\mathrm{id}}}$ and $U_{w' \cdot \sigma_{\mathrm{id}}}$. As we saw in §4 (cf. computation of transition functions between the U_σ), τ contains both $w \cdot \omega$ and $w' \cdot \omega$, which gives $\langle (w - w') \cdot \omega, n_\rho \rangle = 0$ as desired.

Let us denote by $\psi_{w'w}$ (respectively, $\varphi_{w'w}$) the cocycles defining L_ω^X (respectively, L_h) over the covering $Z^w = \tilde{Z}^w \cap X$ of X. The

rest of the argument consists in showing that

$$(8) \qquad \psi_{w'w} = \varphi_{w'w} \frac{c_{w'}}{c_w},$$

with c_w some nonzero constant. The cocycles ψ and φ therefore differ by a coboundary, and the line bundles are isomorphic.

We first observe that it suffices to establish (8) for adjacent cones $w \cdot \sigma_{\mathrm{id}}$, $w' \cdot \sigma_{\mathrm{id}}$ (i.e., the cones intersect in a codimension one face). Indeed, since $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbf{R}}$, any two cones $w \cdot \sigma_{\mathrm{id}}$, $w' \cdot \sigma_{\mathrm{id}}$ can be connected by a sequence

$$w \cdot \sigma_{\mathrm{id}} = w_1 \cdot \sigma_{\mathrm{id}}, \quad w_2 \cdot \sigma_{\mathrm{id}}, \dots, w_n \cdot \sigma_{\mathrm{id}} = w' \cdot \sigma_{\mathrm{id}}$$

with $w_j \cdot \sigma_{\mathrm{id}}$ and $w_{j+1} \cdot \sigma_{\mathrm{id}}$ adjacent. If now (8) is proved for adjacent cones, then by the cocycle condition we have

$$\psi_{w_1 w_3} = \psi_{w_1 w_2} \psi_{w_2 w_3} = \varphi_{w_1 w_2} \varphi_{w_2 w_3} \frac{c_{w_1}}{c_{w_2}} \frac{c_{w_2}}{c_{w_3}} = \varphi_{w_1 w_3} \frac{c_{w_1}}{c_{w_3}}$$

on $Z^{w_1} \cap Z^{w_2} \cap Z^{w_3}$. Because this intersection contains the unique open dense torus orbit, we have

$$\psi_{w_1 w_3} = \varphi_{w_1 w_3} \frac{c_{w_1}}{c_{w_3}} \quad \text{on } Z^{w_1} \cap Z^{w_3}$$

by continuity, and then (8) follows by induction.

To prove (8) for adjacent cones, we may assume that $w' = \mathrm{id}$. For $n \in N_P$, Lemma 10 gives

$$\psi_{\mathrm{id}\, w}(nP) = \chi_\omega^{-1}(p),$$

with p uniquely determined by $w^{-1}n = n'p$, $n' \in N_P$, $p \in P$. From this we obtain (omitting the symbol ρ^ω for ease of notation)

$$\begin{aligned}
(9) \qquad \psi_{\mathrm{id}\, w}(nP) &= \langle pv^\omega, v^\omega \rangle \\
&= \langle (n')^{-1} w^{-1} n v^\omega, v^\omega \rangle \\
&= \langle w^{-1} n v^\omega, v^\omega \rangle \\
&= a_w \langle n v^\omega, v^{w \cdot \omega} \rangle, \qquad a_w \neq 0.
\end{aligned}$$

In the last two steps, we used properties of $\langle \cdot, \cdot \rangle$ mentioned in §2.1.

Recall that Z^{id} and $U_{\sigma_{\mathrm{id}}}$ are identified by

$$u \mapsto n = \prod_{\alpha \in -R^+(S)} \exp(u(\alpha) c_\alpha^0 e_\alpha)$$

and that the cocyle defining L_h is

$$(10) \quad \varphi_{\mathrm{id}\, w}(u) = \mathbf{e}(l_w - l_{\mathrm{id}})(u) = \mathbf{e}(w \cdot \omega - \omega)(u) = u(w \cdot \omega - \omega).$$

An argument similar to that used to prove continuity of h shows that the cones σ_{id} and $w \cdot \sigma_{\mathrm{id}}$ are adjacent if and only if ω and $w \cdot \omega$ are connected by an edge in $J(X)$. By Lemma 3, then, $w \cdot \omega = \omega + \beta$ with $\beta \in -R^+(S)$, and by Lemma 11,

$$\langle \rho^\omega(n)v^\omega, v^{\omega+\beta} \rangle = k_\beta c_\beta^0 u(\beta).$$

Note that $c_\beta^0 \neq 0$ because the reference point n^0 was chosen to be generic. Combining (9) and (10), we get

$$\psi_{\mathrm{id}\,w} = a_w k_\beta c_\beta^0 \varphi_{\mathrm{id}\,w} = c_w \varphi_{\mathrm{id}\,w},$$

where c_{id} is normalized to be 1. This concludes the proof. $\quad\square$

EXAMPLE 1, CONTINUED. For the fan in Figure 3, we define h to be 0 on σ_{id}, $\langle -\alpha_2, \cdot \rangle$ on $s_2 \cdot \sigma_{\mathrm{id}}$, and $\langle -\alpha_1 - \alpha_2, \cdot \rangle$ on $s_1 s_2 \cdot \sigma_{\mathrm{id}}$; here s_1, s_2 are the fundamental reflections in α_1, α_2. The divisor is $D_h = V(\check{\omega}_2)$. This is the hyperplane divisor on $(\mathbf{CP}^2)^*$, which gives the standard embedding into $\mathbf{CP}^2$. $\quad\square$

Notice in this example that $\square_h = J(X) - \omega_2$: the polytope $\square_h$ from toric variety theory is a translate of the momentum polytope. This is true in general.

COROLLARY 4. *In the setting of Theorem 2, $\square_h = J(X) - \omega$.*

Proof. Because L_ω^X is very ample, [8, Theorem 2.13] says that $\square_h$ is the convex hull of the l_σ for σ running over the maximal dimensional cones in Δ. Thus, $\square_h$ is the convex hull of the points $w \cdot \omega - \omega$, $w \in W/W_S$. $\quad\square$

We now compute the dimension of $H^0(X, L_\omega^X)$, which, according to [8, Corollary 2.9], is $\#(\square_h \cap M)$, the number of points of the root lattice on and inside $\square_h$.

THEOREM 3. *Let X be a torus orbit. Then $\dim H^0(X, L_\omega^X) =$ the number of distinct weights in the weight system Π^ω of the representation ρ^ω.*

Proof. From Lemma 6, the number of distinct weights in Π^ω is the number of weight lattice points congruent to ω (modulo the root lattice) in the convex hull of $W \cdot \omega$. By Corollary 4, that is precisely the number of points of the root lattice M on and inside $\square_h$. $\quad\square$

EXAMPLE 3. Let $G = G_2$, and take α_1 to be a long root. Let $S = \{1\}$. Then $\omega = \omega_1 = 2\alpha_1 + 3\alpha_2$, and the representation ρ^{ω_1} is

the adjoint representation, which is 14-dimensional. The flag variety G/P_1 is embedded (by L_{ω_1}) into $\mathbf{CP}^{13}$. Since 0 is a weight of multiplicity 2, and all other weights (= roots) have multiplicity 1, a torus orbit X is embedded (by $L_{\omega_1}^X$) into $\mathbf{CP}^{12}$. It is easy to check that the polygon $\square_h$ from Theorem 2 is the hexagon with vertices at

$$0, \ -\alpha_1, \ -3\alpha_1 - 3\alpha_2, \ -4\alpha_1 - 6\alpha_2, \ -3\alpha_1 - 6\alpha_2, \ -\alpha_1 - 3\alpha_2,$$

and that it contains 13 points of the root lattice.

The polygon $\square_h$ contains a lot of information. For example, there is the following criterion for nonsingularity of a toric variety:

THEOREM [8, *Theorem 2.22, abbreviated*]. *Let* $X = T_N \, \mathrm{emb}(\Delta)$ *be a toric variety, let* h *define a very ample divisor* D_h, *and let* $\square_h$ *be the corresponding polytope in* $M_{\mathbf{R}}$. *Then* X *is nonsingular if and only if the edges emanating from each vertex of* $\square_h$ *are generated by a* $\mathbf{Z}$-*basis of the lattice* M.

Now, the edges from the vertex 0 in our example go to $-\alpha_1$ and $-\alpha_1 - 3\alpha_2$, and these are certainly not a $\mathbf{Z}$-basis of the root lattice M. Hence, a torus orbit in G/P_1 is singular. (This provides a counterexample to Proposition 3.3.1 in [3].) $\square$

REMARK 4. Example 1 and Corollary 4 show that the polytopes $\square_h$ and $J(X)$ generally lie in different lattices, and can define different toric varieties when the construction in [8] is used.

We have already mentioned that usually (e.g. in [1], [3]) the momentum mapping on G/P is normalized so that the vertices of the image lie in the weight lattice. The weight lattice is the character group of the maximal torus H in the simply connected group G. Its center Z acts trivially on G/P but (in general) nontrivially on L_ω.

Other tori can act on G/P as well. A μ in the weight lattice defines a character χ_μ of H. Let $Z_0 = \{\zeta \in Z | \chi_\mu(\zeta) = 1\}$. The character lattice of H/Z_0 is a subgroup of the weight lattice. When $\mu = \omega$, it becomes the root lattice M. Since the momentum map of the T-action is defined only up to translation by a constant element of $\mathscr{T}^* \cong M_{\mathbf{R}}$, it is natural, when considering the action of H/Z_0, to perform a translation $J \mapsto J - \mu$. For the adjoint torus action of interest to us, this agrees with Corollary 4.

We should also note that a torus T_N always acts effectively on $T_N \, \mathrm{emb}(\Delta)$. (Indeed, a $t \in T_N$ is a homomorphism from M to $\mathbf{C}^*$, and it acts on $u \in \mathscr{S}_\sigma$ by $(tu)(m) = t(m)u(m)$. Thus, t acts

as the identity only if $t \equiv 1$.) It is impossible, therefore, to have a non-effective action of T_N on G/P and an effective action on a line bundle over G/P (as is the case for H). □

EXAMPLE 4. The results obtained so far allow us to decide which torus orbits in G/P are nonsingular, when $G = \mathrm{SL}(l+1, \mathbf{C})$. This is somewhat tangential to the main concerns of the paper, so we only give a brief summary.

PROPOSITION. *Let* $G = \mathrm{SL}(l+1, \mathbf{C})$. *A torus orbit in* G/P *is nonsingular if and only if the weight* ω_P *is either* ω_1 *or* ω_l, *or if it has the form*

$$\omega_P = \sum_{k=i}^{j} \omega_k$$

for some i, j *satisfying* $1 \le i < j \le l$.

The varieties G/P associated to ω_1 and ω_l are $\mathbf{CP}^l$ and $(\mathbf{CP}^l)^*$; they are themselves torus orbits (see Example 1), and are obviously nonsingular. In the general case, we use the criterion mentioned in Example 3.

Consider first a maximal parabolic P_k. Then $S = \{k\}$, and $-R^+(S)$ consists of roots of the form

$$-\sum_{m=i}^{j} \alpha_m,$$

with $i \le k \le j$. None of these is a positive linear combination of the others, so they all represent edges of the cone $\breve{\sigma}_{\mathrm{id}}$. When $2 \le k \le l-1$, we see via Lemma 5 that there are more than l edges at the vertex $\mathrm{id} \cdot \omega - \omega = 0$ of $\square_h$, and such a torus orbit must be singular.

Warning. Remember that "torus orbit" means "generic". There are nonsingular maximal orbits in G/P_k, but they are not generic. For example, the generic torus orbit in $G(2, 4)$ is singular, but there are non-generic ones isomorphic to $\mathbf{CP}^2$.

Now let $S = \{i_1, \ldots, i_s\}$, so that $\omega = \omega_{i_1} + \cdots + \omega_{i_s}$. For each maximal parabolic P_{i_j}, abbreviate $-R^+(\{i_j\}) = -R^+_{i_j}$. One has

$$-R^+(S) = \bigcup_{k=1}^{s} -R^+_{i_k}.$$

If

$$\beta \in -R^+_{i_j} \cap -R^+_{i_m},$$

then

$$\beta = -(\alpha_i + \cdots + \alpha_{i_j} + \cdots + \alpha_{i_m} + \cdots + \alpha_j)$$
$$= \underbrace{-(\alpha_i + \cdots + \alpha_{i_j})}_{\varepsilon - R_{i_j}^+} \underbrace{-(\alpha_{i_j+1} + \cdots + \alpha_{i_m} + \cdots + \alpha_j)}_{\varepsilon - R_{i_m}^+}.$$

Since β is a sum of two lattice points in $\check\sigma_{\mathrm{id}}$, it is not the generator of an edge of $\check\sigma_{\mathrm{id}}$. If we remove all the sets $-R_{i_j}^+ \cap -R_{i_m}^+$ from $-R^+(S)$, we are left with the edges of $\check\sigma_{\mathrm{id}}$. This is a property of $\mathrm{SL}(l+1, \mathbf{C})$, and fails even for other classical groups.

It is not hard to see that $-R^+(S)$ is generated (over $\mathbf{Z}$) by blocks $Q_{i_1}, \ldots, Q_{i_s}$ of roots; here Q_{i_j} contains all roots of the form

$$(11) \qquad\qquad -\sum_{k=r}^{t} \alpha_k, \qquad i_{j-1} < r \le t < i_{j+1},$$

with the convention $i_0 = 0$, $i_{s+1} = l + 1$. An illustration is given in Figure 4, for the weight $\omega_2 + \omega_4$ of A_5. The blocks Q_2 and Q_4 are marked by $\circ$ and $\diamond$, while the remaining roots in $-R^+(S)$ are indicated by $\bullet$. The roots of the parabolic are marked with $*$.
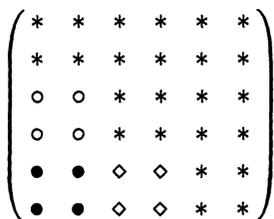
$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ \circ & \circ & * & * & * & * \\ \circ & \circ & * & * & * & * \\ \bullet & \bullet & \diamond & \diamond & * & * \\ \bullet & \bullet & \diamond & \diamond & * & * \end{pmatrix}$$

FIGURE 4

The roots in

$$(12) \qquad\qquad \bigcup_{k=1}^{s} Q_{i_k}$$

represent edges of $\square_h$ at the vertex 0. One sees from (11) that there are l edges precisely when $i_1, \ldots, i_s$ are consecutive integers. In that case, it is easy to verify that the l edges in (12) are a $\mathbf{Z}$-basis for M. $\square$

6. Plücker equations. We saw in Theorem 3 that a torus orbit X in G/P naturally embeds into a projective space whose dimension is the number of weights in the weight system Π^{ω_P} (minus 1). This embedding, which we called the Plücker embedding of X, is the restriction

to X of the familiar Plücker embedding of G/P. In this section, we derive the "Plücker equations" for this torus orbit embedding.

First, a short review of the Plücker equations for G/P realized as $\mathbf{P}(\mathscr{O}^\omega)$ is useful. Those equations are due to Kostant; his proof was apparently first published in [6]. We use the form given in [5] and in unpublished lecture notes by Dale Peterson:

THEOREM. *Let G/P be realized as projective highest weight orbit $\mathbf{P}(\mathscr{O}^\omega)$ in $\mathbf{P}(V^\omega)$. Pick a basis $\{u_i\}$ of $\mathscr{G}$ which is orthonormal with respect to the Killing form. Let $|\omega|^2$ denote the squared length of ω in the usual metric on the weight lattice. The quadratic equations*

(Pl)
$$|\omega|^2 x \otimes x = \sum_i \rho^\omega(u_i) x \otimes \rho^\omega(u_i) x$$

generate the ideal of $\mathbf{P}(\mathscr{O}^\omega)$ in $\mathbf{P}(V^\omega)$.

Every $x \in V^\omega$ is a linear combination of weight vectors v^μ, where the μ run over the set $\mathscr{A}$ of all weights, counted with multiplicity:

(13)
$$x = \sum_{\mu \in \mathscr{A}} \pi_\mu(x) v^\mu.$$

The coordinate-free relations (Pl) produce many equations for the Plücker coordinates $\pi_\mu(x)$. These provide some perspective on our result, so we give a brief summary.

Substitute (13) into (Pl). The left side is

$$|\omega|^2 \sum_{\mu, \nu \in \mathscr{A}} \pi_\mu(x) \pi_\nu(x) v^\mu \otimes v^\nu.$$

For each pair $(\mu, \nu) \in \mathscr{A} \times \mathscr{A}$, we get an equation

(14)
$$|\omega|^2 \pi_\mu(x) \pi_\nu(x) = \sum_{\sigma, \tau \in \mathscr{A}} c_{\mu\nu}^{\sigma\tau} \pi_\sigma(x) \pi_\tau(x).$$

The coefficients in the right side of (14) are complicated: one must expand each $\rho^\omega(u_i) x$ in the basis $\{v^\mu\}$, collect terms, and so forth. We cannot, and do not need to, describe them explicitly. It is possible, however, to restrict the range of the summation on the right side of (14).

If $\xi \in \mathscr{H}$ and $x \in \mathbf{P}(\mathscr{O}^\omega)$, then $\exp(t\xi) \cdot x \in \mathbf{P}(\mathscr{O}^\omega)$ also. Insert this into (13): we get

$$\exp(t\xi) \cdot x = \sum_{\mu \in \mathscr{A}} \pi_\mu(x) e^{t\mu(\xi)} v^\mu,$$

and it follows that

$$\pi_\mu(\exp(t\xi) \cdot x) = e^{t\mu(\xi)}\pi_\mu(x).$$

If the same substitution is made in (14), one finds

$$|\omega|^2\pi_\mu(x)\pi_\nu(x)e^{t(\mu+\nu)(\xi)} = \sum_{\sigma,\tau\in\mathscr{A}} c_{\mu\nu}^{\sigma\tau}\pi_\sigma(x)\pi_\tau(x)e^{t(\sigma+\tau)(\xi)}.$$

It is now easy, by letting ξ range over the subspace of $N_\mathbf{R}$ annihilated by $\mu + \nu$, to extract a set of more restricted Plücker equations:

LEMMA 12. *The ideal of* $\mathbf{P}(\mathscr{O}^\omega)$ *is generated by equations of the form*

$$(15) \qquad |\omega|^2\pi_\mu(x)\pi_\nu(x) = \sum_{\sigma+\tau=\mu+\nu} c_{\mu\nu}^{\sigma\tau}\pi_\sigma(x)\pi_\tau(x),$$

where μ, ν, σ, τ *range over the set* $\mathscr{A}$ *of weights.*

We will show that the Plücker equations of a torus orbit X are closely related to (15). Basically, each equation in (15) is replaced by a set of equations

$$\pi_\mu\pi_\nu = c\pi_\sigma\pi_\tau = c'\pi_{\sigma'}\pi_{\tau'} = \cdots, \qquad \mu + \nu = \sigma + \tau = \sigma' + \tau' \cdots.$$

Some of these equations, however, may amount to $0 = 0$, and others become linear, so we cut down the dimension of the ambient projective space, from

$$\mathbf{CP}^{\dim V^\omega - 1} \text{ to } \mathbf{CP}^{\# \text{ of distinct weights in } \Pi^\omega - 1}.$$

Let $x_0 \in X$ be a generic point, with Plücker coordinates π_μ^0. Recall that "generic" means: $\pi_{w\cdot\omega}^0 \neq 0$ for $w \in W/W_S$. The following convention is useful: if a weight μ has multiplicity > 1, the Plücker coordinates associated to μ are denoted by $\pi_\mu, \pi_\mu', \pi_\mu'', \ldots$.

LEMMA 13. *For each distinct weight* μ, *there is at least one Plücker coordinate amongst the* $\pi_\mu^0, \pi_\mu'^0, \ldots$ *that does not vanish (at the point* x_0). *Pick one, and let* $\pi_\mu(\cdot)$ *be the coresponding coordinate function on* X. *Then: the set of these* π_μ's, *one for each distinct weight, is a basis of* $H^0(X, L_\omega^X)$.

Proof. Since L_ω^X is the pullback of L_ω from G/P to X, the set

$$\{\pi_\mu, \pi_\mu', \ldots | \mu \in \Pi^\omega\}$$

generates $H^0(X, L_\omega^X)$. If $\mu \in W \cdot \omega$, then $\pi_\mu^0 \neq 0$ by definition of "generic". Suppose now that $\mu \notin W \cdot \omega$. Because

$$\pi_\mu(h \cdot x_0) = h^\mu \pi_\mu^0,$$

it follows that either

$$\pi_\mu = \pi_\mu' = \pi_\mu'' = \cdots = 0 \quad \text{if } \pi_\mu^0 = \pi_\mu'^0 = \cdots = 0,$$

or

$$\pi_\mu' = \frac{\pi_\mu'^0}{\pi_\mu^0} \pi_\mu, \quad \pi_\mu'' = \frac{\pi_\mu''^0}{\pi_\mu^0} \pi_\mu, \ldots, \quad \text{if } \pi_\mu^0 \neq 0.$$

Thus, if $\pi_\mu^0 = \pi_\mu'^0 = \cdots = 0$ for some $\mu \notin W \cdot \omega$, we would have

$$\dim H^0(X, L^X)\omega) < \# \text{ of distinct weights of } \rho^\omega,$$

contradicting Theorem 3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK. From the theory of toric varieties, one knows that the space $L(D_h)$ of functions with at most a simple pole along D_h has the basis

$$\{\mathbf{e}(m) | m \in M \cap \square_h\}.$$

Since the map $m \mapsto \omega + m$ is an isomorphism between $M \cap \square_h$ and the weight system Π^ω (Corollary 4 to Theorem 2), one finds that

$$\mathbf{e}(m) = \mathbf{e}(m)(x_0) \frac{\pi_\omega^0}{\pi_{\omega+m}^0} \frac{\pi_{\omega+m}}{\pi_\omega}$$

for any one of the possible bases $\{\pi_\mu\}$ of $H^0(X, L_\omega^X)$ described in Lemma 13. In particular, this shows that

$$D_h = \mathrm{div}(\pi_\omega|_X),$$

for h as in Theorem 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next theorem describes the ideal for the Plücker embedding of a torus orbit X in G/P.

THEOREM 4. *Let $\{\pi_\mu\}$ be a basis of $H^0(X, L_\omega^X)$ as in Lemma 13. The set of quadratic equations*

(16) $$\frac{\pi_\mu \pi_\nu}{\pi_\mu^0 \pi_\nu^0} = \frac{\pi_0 \pi_\tau}{\pi_\sigma^0 \pi_\tau^0}, \qquad \mu + \nu = \sigma + \tau$$

generates the ideal of the Plücker embedding of X,

$$\pi: X \to \mathbf{CP}^s,$$

$$x \mapsto \pi(x) \overset{\text{def}}{=} [\pi_\mu(x)]_{\mu \in \Pi^\omega}.$$

Proof. Let $\mathscr{V}$ denote the zero locus of (16), and let X_0 be the (unique) open dense torus orbit $\{h \cdot x_0\}$ in X.

One inclusion is clear: if $x = h \cdot x_0$, then $\pi_\mu(x) = h^\mu \pi_\mu^0$, and the equations (16) are satisfied. Conversely, we want to show: if $y \in \mathbf{CP}^s$, with coordinates $[\pi_\mu]$, belongs to

$$\mathscr{V} \cap \{\pi_\mu \neq 0 \text{ for all } \mu\},$$

there exists an $h \in H$ such that

$$\pi_\mu = c h^\mu \pi_\mu^0;$$

here $c \in \mathbf{C}^*$ is a nonzero constant allowed when one compares projective coordinates.

The argument goes by (repeated) induction on the weights in the weight system Π^ω. When $\mu \in \Pi^\omega$, and

$$(17) \qquad \mu = \omega - \sum_{j=1}^{l} n_j(\mu)\alpha_j, \qquad n_j(\mu) \in \mathbf{Z}_{\geq 0},$$

recall that $L(\mu) = \sum_{j=1}^{l} n_j(\mu)$ is the *level* of the weight μ. We will define the desired $h \in H$ by prescribing the values h^{α_j} of the fundamental roots on h. For each j, there is a first level at which α_j appears in a weight (17), or else α_j never appears in a weight (this can happen, for instance, when G is semisimple but not simple). In the latter case, h^{α_j} remains arbitrary, and its value does not affect the subsequent argument. We therefore disregard such roots.

Suppose now that α_j does appear in (17); at the first level involving α_j, there is at least one weight of the form $\mu = \nu - \alpha_j$; we pick one of these weights and define

$$h^{\alpha_j} = \frac{\pi_\nu}{\pi_\nu^0} \Big/ \frac{\pi_\mu}{\pi_\mu^0}.$$

We now claim that for all $\mu \in \Pi^\omega$,

$$(18) \qquad \frac{\pi_\mu}{\pi_\mu^0} = \left(\frac{\pi_\omega}{\pi_\omega^0} h^{-\omega}\right) h^\mu.$$

This is evidently true at level 0, for the weight ω, and at level 1, where the weights have the form $\omega - \alpha_k$. Suppose that (18) has been established up to level $L(\mu) - 1$, and consider a weight μ at level $L(\mu)$. There are three cases.

Case 1. The weight μ is used to *define* a value h^{α_j}; in this case, (18) follows by definition and (for the compatibility with earlier steps) by induction hypothesis.

Case 2. A root α_j first appeared at level $L(\mu)$, the weight μ involves α_j, but another weight μ' was used to specify the value h^{α_j}. So, we have

$$\mu = \nu - \alpha_j, \quad \mu' = \nu' - \alpha_j, \quad L(\nu) = L(\nu') = L(\mu) - 1 = L(\mu') - 1.$$

The consistency condition (18) follows from

$$\frac{\pi_\mu \pi_{\nu'}}{\pi_\mu^0 \pi_{\nu'}^0} = \frac{\pi_{\mu'} \pi_\nu}{\pi_{\mu'}^0 \pi_\nu^0}, \qquad \mu + \nu' = \nu + \nu' - \alpha_j = \mu' + \nu,$$

the induction hypothesis, and Case 1.

Case 3. The weight μ could not have been used to define any of the numbers h^{α_j}. There is then an index j such that $\mu = \nu - \alpha_j$, and h^{α_j} was defined by a weight $\mu' = \nu' - \alpha_j$, with $L(\mu') \le L(\mu) - 1$. As in Case 2, we argue that (18) holds. □

REMARK. It is clear from the proof that one can use ratios like $h^\omega / h^{\omega - \alpha}$ only to determine the value h^α of *roots* on h. This again confirms that the root lattice, rather than the weight lattice, is relevant for the description of torus orbits. □

7. Intersection theory. Introduce the line bundles $L_j = L_{\omega_j}$ over G/B, defined (as in §5) by the characters

$$\chi_j(b) = \langle \rho^{\omega_j}(b) v^{\omega_j}, v^{\omega_j} \rangle^{-1}, \qquad b \in B.$$

They are not ample; the space $H^0(G/B, L_j)$ maps G/B to the highest weight orbit $\mathbf{P}(\mathcal{O}^{\omega_j})$ in $\mathbf{P}(V^{\omega_j})$, which is isomorphic to G/P_j where P_j is the maximal parabolic subgroup corresponding to $S = \{j\}$. Put differently, the natural projection $\pi_j \colon G/B \to G/P_j$ is followed by the embedding of G/P_j provided by the line bundle $L_j \to G/P_j$. Let X be a torus orbit in G/B. Then $\pi_j(X)$ is a torus orbit in G/P_j, which has a Plücker embedding by the sections of $L_j^{\pi_j(X)}$ as described in §5. The corresponding torus invariant divisor D_{h_j} on $\pi_j(X)$ pulls back to a torus invariant divisor D_j on X. Since these divisors correspond in a natural way to the fundamental weights, we call them (for lack of a better term) the *fundamental torus invariant divisors on X*. We want to study their intersection theory.

To begin, we note that each D_j is defined by the same support function h_j. Indeed, the fan Δ_j of $\pi_j(X)$ has as maximal dimensional cones certain unions of the co-Weyl chamber C^- in $N_\mathbf{R}$, as described by Theorem 1. The support function h_j is defined on the maximal dimensional cones of Δ_j by the formula in Corollary 4. The fan Δ of
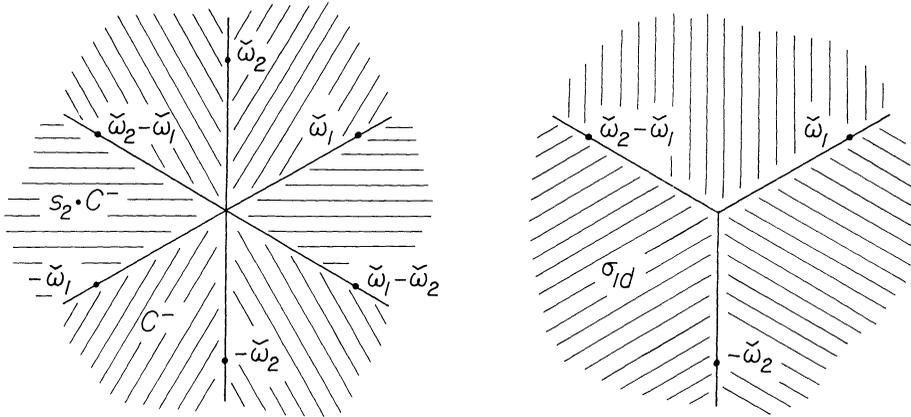
FIGURE 5

X has for its maximal dimensional cones *all* the images wC^- of C^- under the Weyl group (since the parabolic subgroup B corresponds to the choice $S = \{1, \ldots, l\}$, so that $W_S = \{\mathrm{id}\}$). Thus, each h_j is a function on $N_{\mathbf{R}}$ which is not only linear on the cones of Δ_j but also on the cones of Δ. In fact, it is linear across several cones, so it is upper convex but no longer strictly upper convex, which by [8, Corollary 2.14] means that the divisor D_j is not ample. $(\mathscr{O}_X(D_j)$ is, however, generated by global sections [8, Theorem 2.7].) Furthermore, the polytope $\square_{h_j}$ is the one described, for G/P_j, in Theorem 2. Figure 5 has an illustration for the case $G = \mathrm{SL}(3, \mathbf{C})$; the fans Δ and Δ_1 are pictured.

THEOREM 5. *The intersection number* $(D_1 \ldots D_l)$ *is*

$$|W|/\det C,$$

where $|W|$ *is the order of the Weyl group of* G *and* C *is the Cartan matrix.*

REMARK. The intersection number in the theorem is the usual intersection number of cycles whenever X is nonsingular. This happens in our setting: a torus orbit $X \subset G/B$ is always nonsingular. The proof uses the nonsingularity criterion ([8, Theorem 2.22]) stated in Example 3, and Corollary 1 to Lemma 3. $\square$

Proof of Theorem 5. We use [8, Proposition 2.10], according to which the desired intersection number is the *mixed volume* of the polytopes $\square_{h_1}, \ldots, \square_{h_l}$. Normalize volume in $M_{\mathbf{R}}$ so that the basic

parallelopiped

$$\left\{ \sum_{j=1}^{l} x_j \alpha_j \,\middle|\, 0 \le x_j \le 1 \; \forall j \right\}$$

has volume 1. One knows that the volume

(19) $$\mathrm{Vol}(n_1 \square_{h_1} + \cdots + n_l \square_{h_l})$$

is a homogeneous polynomial in the variables n_j. The mixed volume is, by definition, the coefficient of $\prod_1^l n_j$ in this polynomial (we ignore a factor $1/l!$, which cancels out of the final equation).

Some simplifications are in order. It suffices, in the definition of mixed volume, to let the n_j be positive integers. We may translate each polytope in (19) so that its vertices lie in the weight lattice. The *volume* of the sum in (19) will change, but the *mixed volume* remains the same. This follows from the linearity of mixed volume in each of the arguments, see [8, Appendix]. The sum

$$n_1(\square_{h_1} + \omega_1) + \cdots + n_l(\square_{h_l} + \omega_l)$$

then has a simple description: it is the convex hull of the Weyl group orbit through

$$\lambda = n_1 \omega_1 + \cdots + n_l \omega_l.$$

Call this polytope $\square_n$. We need its volume, i.e. the number of root lattice points in $\square_n$, as function of the n_j. It is easier to count *weight lattice* points, so in the end we will divide by $\det C$, which is the index of the root lattice in the weight lattice. Furthermore, it is enough to count the number of points in the positive Weyl chamber C^+, and to multiply the result by $|W|$. The structure of the formula in the Theorem should now be clear. (Note also that we write C^+ rather than $\check{C}^+$; the notation is simpler if we forget for the moment that the root lattice is dual to our basic lattice N.)

The intersection $\square_n \cap C^+$ is bounded by the walls of C^+

$$\{x \,|\, (x, \check{\alpha}_j) = 0\}, \qquad j = 1, \ldots, l,$$

(the $\check{\alpha}_j$ are the simple coroots) and by the hyperplanes through λ and orthogonal to the fundamental weights ω_j,

$$\{x \,|\, (x, \omega_j) = (\lambda, \omega_j)\}, \qquad j = 1, \ldots, l.$$

Introduce the parallelopiped Π bounded by the walls of C^+ and by the hyperplanes through λ and parallel to the walls:

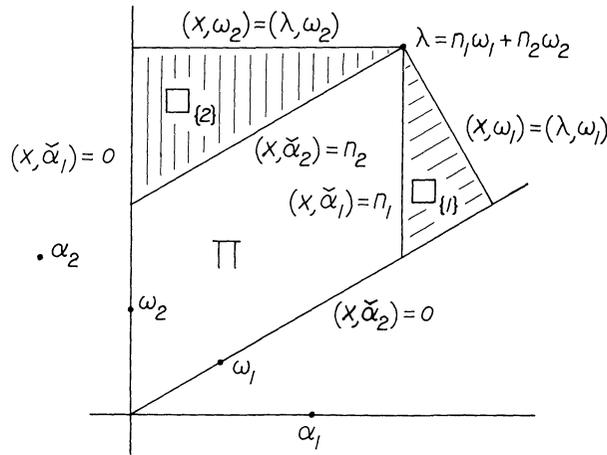$$\{x \,|\, (x, \check{\alpha}_j) = n_j\}, \qquad j = 1, \ldots, l.$$

FIGURE 6

(See Figure 6 for an $SL(3, \mathbf{C})$ example.) The number of weight lattice points on and inside Π is $\prod_1^l (n_j + 1)$, and so the volume of Π is

$$(\det C)^{-1} n_1 \cdots n_l.$$

We show that the difference

$$(\square_n \cap C^+) \backslash \Pi$$

consists of polyhedra whose volumes, as function of $n_1, \ldots, n_l$, are independent of at least one n_j. Suppose this has been proved. Then the volume of $\square_n \cap C^+$ is

$$(\det C)^{-1} n_1 \cdots n_l + \text{ other monomials},$$

whence the mixed volume of all of $\square_n$ is $|W|/\det C$, as desired.

First we prove that

$$\Pi \subset \square_n \cap C^+.$$

Let $x \in \Pi$, $x = x_1 \omega_1 + \cdots + x_l \omega_l$. Of course, $x \in C^+$. Since $0 \le (x, \check{\alpha}_j) \le n_j$ for all j, we have $0 \le x_j \le n_j$. Then

$$\sum_{k=1}^l x_k (\omega_k, \omega_j) \le \sum_{k=1}^l n_k (\omega_k, \omega_j) \quad \forall j$$

since $(\omega_k, \omega_j) \ge 0$, and so

$$(x, \omega_j) \le (\lambda, \omega_j) \quad \forall j.$$

This shows that $x \in \square_n$.

A similar argument also shows that if $(x, \check{\alpha}_j) > n_j$ for all j, then $(x, \omega_j) > (\lambda, \omega_j)$ for all j, which contradicts $x \in \square_n$. Therefore, for each $x \in (\square_n \cap C^+) \backslash \Pi$ there is a subset $\Theta \subset \{1, \ldots, l\}$ such that

$$0 \le (x, \check{\alpha}_j) \le n_j \quad \text{for } j \notin \Theta$$

and

$$n_k \le (x, \check{\alpha}_k), \quad (x, \omega_k) \le (\lambda, \omega_k) \quad \text{for } k \in \Theta.$$

Let $\square_\Theta$ be the set of such x; the $\square_\Theta$ are disjoint up to sets of measure zero. Now if $x \in \square_\Theta$, then

$$y = x - \sum_{i \in \Theta} n_i \omega_i$$

satisfies:

$$0 \le (y, \check{\alpha}_j) \le n_j \quad \text{for } j \notin \Theta,$$

and

$$0 \le (y, \check{\alpha}_k), \quad (y, \omega_k) \le \left(\sum_{m \notin \Theta} n_m \omega_m, \omega_k \right) \quad \text{for } k \in \Theta.$$

Let $\square^\Theta$ be the set of such y. Clearly, $\square_\Theta$ and $\square^\Theta$ differ only by a translation. Hence, they have the same volume. Since $\mathrm{Vol}(\square^\Theta)$ is independent of n_k, $k \in \Theta$, this volume as function of the n_j cannot involve the factor $\prod_1^l n_j$. The theorem is proved. $\square$

Finally, we generalize Theorem 5 to find the intersection multiplicity of arbitrary intersections of the D_j. Choose

$$\Theta = \{i_1, \ldots, i_s\} \subset \{1, \ldots, l\},$$

and let V be an S-dimensional closed subvariety of X. The intersection number

$$(D_{i_1} \cdots D_{i_s}; V)$$

is defined to be the coefficient of $\nu_1 \cdots \nu_s$ in the polynomial

$$\chi(X, \mathscr{O}_V \otimes_{\mathscr{O}_X} \mathscr{O}_X(\nu_1 D_{i_1} + \cdots + \nu_s D_{i_s})),$$

where χ is the Euler characteristic.

Let $\mathscr{G}^\Theta$ be the Lie algebra generated by α_j, $j \in \Theta$, let G^Θ and B^Θ be the corresponding Lie group and its Borel subgroup, and let W^Θ and C^Θ be the Weyl group and Cartan matrix of $\mathscr{G}^\Theta$.

THEOREM 6. *Let $\tau \in \Delta$ be the cone spanned by $\check{\omega}_k$, $k \notin \Theta$. Then*

$$(D_{i_1} \cdots D_{i_s} ; V(\tau)) = |W^{\Theta}| / \det C^{\Theta}.$$

Proof. We need some notation. As in [8, Corollary 1.7], introduce

$$\overline{N}(\tau) = N / \mathbf{Z}(\tau \cap N), \quad \overline{N}(\tau)_{\mathbf{R}} = N_{\mathbf{R}} / \mathbf{R}\tau,$$

and

$$\overline{\Delta}(\tau) = \{\overline{\sigma} = \text{ image of } \sigma \text{ in } \overline{N}(\tau)_{\mathbf{R}} | \tau < \sigma \in \Delta\}.$$

As abstract toric variety,

$$V(\tau) = T_{\overline{N}(\tau)} \text{emb}(\overline{\Delta}(\tau)).$$

Let $h \in \text{SF}(N, \Delta)$ be the support function defining the divisor $\sum_{j \in \Theta} \nu_j D_j$ on X. By [8, Lemma 2.11], there is an $\overline{h} \in \text{SF}(\overline{N}(\tau), \overline{\Delta}(\tau))$ such that

(20) $$\mathscr{O}_X(D_h) \otimes_{\mathscr{O}_X} \mathscr{O}_{V(\tau)} = \mathscr{O}_{V(\tau)}(D_{\overline{h}}).$$

The argument now consists of two steps.

Step 1. $V(\tau)$ can be identified with a generic torus orbit in G^{Θ} / B^{Θ}.

Step 2. Since by Step 1, $V(\tau)$ is a torus orbit, we can speak about the fundamental torus invariant divisors. Call them $D_1^{\Theta}, \ldots, D_s^{\Theta}$. Then: $\overline{h}$ defines the divisor $\nu_1 D_1^{\Theta} + \cdots + \nu_s D_s^{\Theta}$ on $V(\tau)$.

Let us show how the theorem follows from Steps 1 and 2.

$$
\begin{aligned}
(D_{i_1} \cdots D_{i_s} ; V(\tau)) & \overset{\text{def}}{=} \text{ coeff of } \nu_1 \cdots \nu_s \\
& \quad \text{in } \chi(X, \mathscr{O}_X(\nu_1 D_{i_1} + \cdots + \nu_s D_{i_s}) \otimes_{\mathscr{O}_X} \mathscr{O}_{V(\tau)}) \\
& \overset{\text{by}(20)}{=} \text{ coeff of } \nu_1 \cdots \nu_s \text{ in } \chi(V(\tau), \mathscr{O}_{V(\tau)}(D_{\overline{h}})) \\
& \overset{\text{step 2}}{=} \text{ coeff of } \nu_1 \cdots \nu_s \\
& \quad \text{in } \chi(V(\tau), \mathscr{O}_{V(\tau)}(\nu_1 D_1^{\Theta} + \cdots + \nu_s D_s^{\Theta})) \\
& \overset{\text{def}}{=} (D_1^{\Theta} \ldots D_s^{\Theta}) \\
& = |W^{\Theta}| / \det C^{\Theta},
\end{aligned}
$$

the last equality following from Step 1, which allows Theorem 5 to be applied to the torus orbit $V(\tau)$ in G^{Θ} / B^{Θ}.

Proof of Step 1. We need to show that the fan $\overline{\Delta}(\tau)$ defining

$$V(\tau) = T_{\overline{N}(\tau)} \text{emb}(\overline{\Delta}(\tau))$$

can be identified with the fan of a torus orbit in G^Θ/B^Θ as described in Theorem 1.

Since the sets $\{\breve{\omega}_j,\, j \in \Theta\}$ and $\{\breve{\omega}_k,\, k \notin \Theta\}$ span complementary sublattices of N and complementary subspaces of $N_{\mathbf{R}}$, it is clear that we can identify $\overline{N}(\tau)$ with the coweight lattice N^Θ of $\mathscr{G}^\Theta$ and $\overline{N}(\tau)_{\mathbf{R}}$ with $N_{\mathbf{R}}^\Theta$. The cones $\sigma \in \Delta$ for which $\tau < \sigma$ are of the form

$$\sigma = \tau + w \cdot \left(\sum_{i \in \Theta'} \mathbf{R}_{\geq 0} \breve{\omega}_i \right),$$

where $\Theta' \subseteq \Theta$, and $w \in W^\Theta =$ subgroup of W generated by s_{α_j}, $j \in \Theta$. The projections onto $\overline{N}(\tau)_{\mathbf{R}}$ of these cones are identified with

$$\overline{\sigma} = (\sigma + \mathbf{R}\tau)/\mathbf{R}\tau = w \cdot \left(\sum_{i \in \Theta'} \mathbf{R}_{\geq 0} \breve{\omega}_i^\Theta \right), \qquad w \in W^\Theta,$$

which by Theorem 1 are the cones of the fan Δ^Θ of a torus orbit in G^Θ/B^Θ.

Proof of Step 2. We know that the divisor $\nu_1 D_{i_1} + \cdots + \nu_s D_{i_s}$ is defined by the function

$$h = \langle w \cdot \lambda - \lambda, \cdot \rangle \quad \text{on } wC^-,$$

with $\lambda = \nu_1 \omega_{i_1} + \cdots + \nu_s \omega_{i_s}$. Since $h|_\tau = 0$, [8, Lemma 2.11] says that we may take $\overline{h} = h$.

We compute the divisor determined by $\overline{h}$. Let M^Θ be the root lattice of $\mathscr{G}^\Theta$ (spanned by $\alpha_j,\, j \in \Theta$); it is a sublattice of M and $M_{\mathbf{R}}^\Theta$ is a subspace of $M_{\mathbf{R}}$. The fundamental weights ω_j^Θ of $\mathscr{G}^\Theta$ are not always identified with fundamental weights of $\mathscr{G}$, since the latter may not lie in $M_{\mathbf{R}}^\Theta$. Rather, we have (for $i_p \in \Theta$)

$$\omega_p^\Theta = \omega_{i_p} \mod \omega_k,\, k \notin \Theta.$$

Hence

$$\lambda = \sum_{p=1}^s \nu_p \omega_p^\Theta + \sum_{k \notin \Theta} \mu_k \omega_k \stackrel{\text{def}}{=} \lambda^\Theta + \sum_{k \notin \Theta} \mu_k \omega_k.$$

Now, $h = \langle w\lambda - \lambda, \cdot \rangle$ on $w \cdot C^-$ for $w \in W^\Theta$, but since w stabilizes $\omega_k,\, k \notin \Theta$, this is the same as $\langle w\lambda^\Theta - \lambda^\Theta, \cdot \rangle$, which in turn gives the values of $\overline{h}$ on the cones of Δ^Θ. It then follows from Theorem 2 that $\overline{h}$ determines the divisor $\nu_1 D_1^\Theta + \cdots + \nu_s D_s^\Theta$ on $V(\tau)$. Step 2 is proved. $\square$

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