PACIFIC JOURNAL OF MATHEMATICS
Vol. 150, No. 1, 1991

EQUIVARIANT NIELSEN FIXED POINT THEORY FOR G-MAPS

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Let $f: V \to X$ be a G-map defined on an open invariant subset V of a G-ENR X where G is a compact Lie group and the G-action on V is not necessarily free. In this paper, we introduce the notion of an equivariant Nielsen number $N_G^c(f, V)$ which is an ordered k-tuple that depends on the isotropy types $(H_1), \ldots, (H_k)$ of V. When G is finite, $N_G^c(f, V)$ gives a lower bound for the minimal number of fixed points in the (restricted) G-homotopy class of f and this lower bound is sharp when the G-action on V is free. We relate $N_G^c(f, V)$ to a local equivariant obstruction to G-deforming a map to be fixed point free and we discuss the relationship between the equivariant Nielsen number and the ordinary Nielsen number.

1. Preliminaries. Let G be a topological group and X be a (left) G-space. For any subgroup H of G, we denote by NH the normalizer of H in G and by $WH = NH/H$, the Weyl group of H in G. The conjugacy class of H denoted by (H) is called the orbit type of H. If $x \in X$, then G_x denotes the isotropy subgroup of x, i.e., $G_x = \{g \in G | gx = x\}$. For each subgroup H of G, $X^H = \{x \in X | hx = x \text{ for all } h \in H\}$ and $X_H = \{x \in X | G_x = H\}$. An orbit type (H) is called an isotropy type of X if H appears as an isotropy subgroup of some x in X. Suppose X has a finite set of isotropy types denoted by $\{(H_i)\}$. If (H_j) is subconjugate to (H_i), we write $(H_j) \leq (H_i)$. We can choose an *admissible ordering* on $\{(H_j)\}$ so that $(H_j) \leq (H_i)$ implies $i \leq j$. Then we have a filtration of G-subspaces $X_1 \subset \cdots \subset X_k = X$ where $X_i = \{x \in X | (G_x) = (H_j) \text{ for some } j \leq i\}$. Also, $X_{(H)} = GX_H = X_i - X_{i-1}$ with $(H) = (H_i)$. By a free G-subset of X, we mean a G-invariant subset on which the action is free.

Let G be a compact Lie group. A G-space X is a *G-absolute neighborhood retract* (G-ANR), if X is a metric space and for any G-embedding $h: X \to Y$ is a metric G-space Y such that $h(X)$ is closed in Y, the image $h(X)$ is a G-retract of some open invariant neighborhood in Y. If X is a G-ANR then X^H is an ANR for every closed subgroup $H \leq G$. Moreover, if Y is a G-ANR and $f: X \to Y$ is a G-equivalence then $f^H = f|X^H: X^H \to Y^H$ is a homotopy equivalence for every closed $H \leq G$. A G-space X is a *G-euclidean neighborhood retract* (G-ENR) if X can be G-embedded as a G-retract of a G-neighborhood in some Euclidean G-space V.

Let X be a finite dimensional separable metric G-space. Then X is a G-ENR if and only if X is locally compact, has a finite number of isotropy types, and for every isotropy subgroup $H \leq G$, the fixed point set X^H is an ENR. If X is a G-ENR then $X_{(H)}$ is a G-ENR for every closed $H \leq G$ and the orbit space X/G is an ENR (see **[tD]**).

Given any admissible ordering $(H_1), \dots, (H_k)$ of isotropy types of a G-space X, we obtain the associated filtration $X_1 \subset \cdots \subset X_k$. Then any G-map $f\colon X \to X$ preserves the filtration, i.e., $f(X_i) \subset X_i$. Also, the inclusion $X_{i-1} \hookrightarrow X_i$ is a G-cofibration.

Let G be a finite group and K a G-(simplicial) complex (see **[B]**). K/G is a simplicial complex such that the orbit map $p\colon |K| \to |K|/G \approx |K/G|$ is simplicial and p maps each simplex of $|K|$ homeomorphically onto the corresponding image simplex of $|K/G|$, where $|X|$ denotes the underlying space of X. Any G-complex is a G-ENR. We will not distinguish K as a simplicial complex and K as the underlying space.

In §2, we define *G-compactly fixed* maps. We show that every self G-map of a compact G-ENR can be G-deformed to a G-compactly fixed map. In §3, we first define an equivariant Nielsen relation on the fixed point set $\operatorname{Fix} f_H$ of $f_H = f|V_H\colon V_H \to X^H$ for each isotropy type (H) of V. We obtain a WH-Nielsen number $n_{WH}(f_H, V_H)$ which is a lower bound for the number of orbits of fixed points of f_H. Then the *equivariant Nielsen number* $N_G^c(f, V)$ of f is the tuple $\{n_{WH}(f_H, V_H)\}$ and it is invariant under G-compactly fixed homotopy. We give a local version and hence an equivariant analog of the Hopf construction in §4. We prove in §5 a minimality theorem for a certain class of G-spaces satisfying the *equivariant Shi condition* when the action on V is free. The basic technique here is an equivariant version of the "Wecken Trick" (see **[Br]**) of coalescing fixed points of the same class. We relate in §6 the equivariant Nielsen number to a local equivariant obstruction to G-deforming a map to be fixed point free. In §7, we define another equivariant Nielsen type invariant $N_G^*(f)$ which enjoys the usual properties of the ordinary Nielsen number. Finally, in §8, we give an example in which f is G-deformable to be fixed point free but not G-compactly fixed deformable to be fixed point free.

The author would like to thank the referee for a number of helpful suggestions.

2. G-compactly fixed maps. Throughout this section, G will denote a compact Lie group.

2.1. DEFINITION. Let X be a space and $V \subset X$ be an open subset. A map $f: V \to X$ is called *compactly fixed* if the fixed point set $\operatorname{Fix} f = \{x \in V | f(x) = x\}$ is compact in V. If, in addition, X is a G-space, V is invariant and f is a G-map then we call f a *compactly fixed G-map.* A *compactly fixed G-homotopy* is a G-map $F: V \times I \to X$ (the G-action on the unit interval I is trivial) such that $\bigcup_t \operatorname{Fix} F_t$ is compact in V.

2.2. DEFINITION. Let V be an open invariant subset of a G-space X such that V has a finite number of isotropy types. A G-map $f: V \to X$ is called a *G-compactly fixed map* if for each isotropy type (H) of V, $f_H = f|V_H: V_H \to X^H$ is compactly fixed. A *G-compactly fixed homotopy* is a G-map $F: V \times I \to X$ such that $\bigcup_t \operatorname{Fix}(F|V_H \times \{t\})$ is compact in V_H for each H.

2.3. REMARK. Any G-compactly fixed map is a compactly fixed G-map but not the converse. For example, take G to be a finite group and $V = X$, a compact semi-free G-space. Then the identity map 1_X is a compactly fixed G-map but not a G-compactly fixed map.

2.4. PROPOSITION. *Let Y be a G-ENR and X be a G-space. If $h_0, h_1: X \to Y$ are G-maps and $A \subset X$ is a closed invariant subset such that $h_0|A = H_1|A$, then there exist an invariant neighborhood W of A in X and a G-homotopy $\Gamma: W \times I \to Y$ such that $\Gamma_0 = h_0$, $\Gamma_1 = h_1$ and $\Gamma_t|A = h_0|A$ for all $t \in I$.*

Proof. Let $r: U \to i(Y)$ be a G-retraction of an invariant neighborhood of $i(Y)$ in some euclidean G-space, where i is a G-imbedding. Let W be the set of points gx, $g \in G$ such that the line segment from $i(h_0(x))$ to $i(h_1(x))$ lies inside U. Define $\Gamma: W \times I \to Y$ by

$$\Gamma(x, t) = r[(1-t)i(h_0(x)) + ti(h_1(x))]. \qquad \square$$

2.5. PROPOSITION. *Let X and Y be G-ENRs, $A \subset X$ be a closed invariant neighborhood retract, and $f: X \to Y$ be a G-map. Then there is an invariant neighborhood U of A in X and a G-homotopy* *(relative to A) from f to a map f' such that $f'|U = f'|A \circ r$ where $r\colon U \to A$ is a G-retraction.*

Proof. Let $r\colon V \to A$ be a G-retraction of an invariant neighborhood V of A. By 2.4, there exist an invariant neighborhood W of A and a G-homotopy Γ (relative to A) such that $\Gamma_0 = f$ and $\Gamma_1 = f \circ r|W$. Choose invariant neighborhoods $U \subset \widetilde{U} \subset W$ of A where $\widetilde{U}$ is closed. Let $\rho\colon V \to I$ be an invariant function such that $\rho|\widetilde{U} = 1$ and $\rho|V - W = 0$. Define

$$\Gamma'(x, t) = \begin{cases} f(x), & x \notin W, \\ \Gamma(x, \rho(x)t), & x \in W. \end{cases}$$

The map $\Gamma'(x, 1)$ is the desired G-map □

2.6. Proposition. *Let X be a compact G-ENR and $f\colon X \to X$ a G-map. Then f is G-homotopic to a G-compactly fixed map.*

Proof. Choose an admissible ordering $(H_1), \dots, (H_n)$ with the associated filtration of compact G-ENRs $X_1 \subset \cdots \subset X_n$. Since $X_{H_1} = X^{H_1}$, $f_{H_1} = f|X_{H_1}$ is compactly fixed. We assume inductively that f_{H_j} is compactly fixed for $j < k$. Since $X_{k-1}^{H_k}$ is a WH_k-invariant closed neighborhood retract in X^{H_k}, there exist using 2.5 a WH_k-invariant neighborhood U of $X_{k-1}^{H_k}$ and a WH_k-homotopy (relative to $X_{k-1}^{H_k}$) to a map ψ such that $\psi|U = \psi|X_{k-1}^{H_k} \circ r$ where $r\colon U \to X_{k-1}^{H_k}$ is a WH_k-retraction. Since ψ is fixed point free on $U - X_{k-1}^{H_k}$, $\operatorname{Fix} \psi \cap X_{H_k}$ is compact in X_{H_k}. We extend this WH_k-homotopy (relative to $X_{k-1}^{H_k}$) to a G-homotopy (relative to X_{k-1}) from f to a map which is fixed point free in an invariant neighborhood of X_{k-1} (see [F-W, Prop. 2.1]). Induction completes the proof. □

2.7. Remark. When given two G-homotopic maps $f, g\colon X \to X$, each of which is G-compactly fixed, one asks if they are G-compactly fixed homotopic to each other. We will see in the next section that the answer is negative.

3. Equivariant Nielsen numbers. Let W be a compact Lie group, X be a W-ENR, U be an open invariant subset of X such that the W-action of U is free. Suppose $f\colon U \to X$ is a compactly fixed W-map and $\operatorname{Fix} f \neq \varnothing$.

3.1. DEFINITION. Let x and y be fixed points in $\operatorname{Fix} f$. Then x and y are said to be *W-Nielsen equivalent*, denoted by $x \simeq_W y$, if *either* (i) $x = wy$ for some $w \in W$ or (ii) there exists a path $\alpha: I \to U$ such that $\alpha(0) = x$, $\alpha(1) = wy$ for some $w \in W$ and α is homotopic to $f \circ \alpha$ (relative to endpoints) in X.

With this definition, it is easy to show the following

3.2. PROPOSITION. *$\simeq_W$ is an equivalence relation to* $\operatorname{Fix} f$.

3.3. PROPOSITION. *Let $f: U \to X$ be a compactly fixed W-map defined on a free open invariant subset U. The set of W-Nielsen classes is finite.*

Proof. If two fixed points are locally Nielsen equivalent, they are W-Nielsen equivalent. Since there are a finite number of local Nielsen classes, the assertion follows. □

Let $f: V \to X$ be a G-compactly fixed map defined on an open invariant subset V of a G-ENR X, where G is compact Lie and the G-action on V is not necessarily free. Recall that $V_H = V^H_{(H)} = \{x \in V | G_x = H\}$.

3.4. DEFINITION. Let x and y be points in $\operatorname{Fix} f$. Then x and y are said to be *G-Nielsen equivalent*, denoted by $x \approx_G y$ if (i) for some $H \leq G$, x and y lie in X_H, and (ii) $x \simeq_{WG_x} y$ where WG_x is the Weyl group of the isotropy subgroup G_x.

Note that when the action on V is free, 3.4 reduces to 3.1. We also obtain

3.5. PROPOSITION. *$\approx_G$ is an equivalence relation on* $\operatorname{Fix} f$.

3.6. REMARK. The set of equivariant Nielsen classes is *not* finite in general unless $[G : H_i]$ is finite for each i, for example, when G is finite.

In classical Nielsen theory, the Nielsen number of f is defined as the number of essential Nielsen classes. We will also define essential classes on the set of WH_i-Nielsen classes (which is finite by 3.3) instead of the set of G-Nielsen classes which may be infinite.

3.7. DEFINITION. Let $f: U \to X$ be a compactly fixed W-map defined on a free open invariant subset of a W-ENR X. Then the set of W-Nielsen classes is finite. A W-Nielsen class N is *essential* if $I(f, N) \neq 0$ where I is the local fixed point index. Define the *W-Nielsen number* denoted by $n_W(f, U)$ to be the number of essential W-Nielsen classes.

3.8. DEFINITION. Let $f: V \to X$ be a G-compactly fixed map defined on an open invariant (not necessarily free) subset of a G-ENR X. Define the *equivariant Nielsen numbers*, denoted by $N_G^c(f, V)$ to be the k-tuple

$$(n_{W_1}(f_1, V_{H_1}), \dots, n_{W_k}(f_k, V_{H_k}))$$

where $\{(H_1), \dots, (H_k)\}$ are the isotropy types of V and $n_{W_i}(f_i, V_{H_i})$ is the $W_i = WH_i$-Nielsen number of $f_i = f|V_{H_i}$.

3.9. REMARK. For every isotropy type (H), the WH-Nielsen number $n_{WH}(f|V_H, V_H)$ is finite and hence $N_G^c(f, V)$ is well defined. Since V_H (resp. X^H) is homeomorphic to V_K (resp. X^K) if K is conjugate to H in G, $n_{WH}(f|V_H, V_H)$ is independent of the choice of the representative of (H) and hence so is $N_G^c(f, V)$.

Of any Nielsen type invariant, the most important property is the invariance under homotopy. Our next objective here is to verify this property for $N_G^c(f, V)$.

3.10. PROPOSITION. (*G-Compactly Fixed Homotopy Invariance.*) *Let $f: V \to X$ be a G-compactly fixed map. The G-Nielsen number $N_G^c(f, V)$ is invariant under G-compactly fixed homotopy.*

Proof. Since V is a disjoint union of $V_{(H)}$ where H appears as an isotropy subgroup, it suffices to show that given a G-compactly fixed homotopy $F: V \to I \to X$, $n_{WH}(F_0|V_H, V_H) = n_{WH}(F_0|V_H, V_H) = n_{WH}(F_1|V_H, V_H)$ for every isotropy type (H). Let $\widehat{F} = F|V_H \times I$.

Let $\mathcal{N}(\widehat{F}_t)$ denote the set of WH-Nielsen classes of $\widehat{F}_t$, $t \in I$. Suppose $N_0 \in \mathcal{N}(\widehat{F}_0)$, $N_1 \in \mathcal{N}(\widehat{F}_1)$ then we say N_0 and N_1 are *$\widehat{F}$-related* if there exists $x_0 \in N_0$, $x_1 \in N_1$, and a path $C = \{x_t\}_{t\in I}$ in V_H such that $\{\widehat{F}_t(x_t)\} \sim \{x_t\}$ (rel. endpoints) in X^H. Let $\widetilde{F}: V_H \times I \to X^H \times I$ be the *fat* homotopy defined by

$$\widetilde{F}(x, t) = (\widehat{F}(x, t), t).$$

If N_0 and N_1 are $\widehat{F}$-related then they belong to the same WH-Nielsen class of $\widetilde{F}$ because $\{\widehat{F}_t(x_t)\} \sim \{x_t\}$ is equivalent to $\{\widetilde{F}(x_t, s_t)\} \sim \{(x_t, s_t)\}$. Also, each WH-Nielsen class of $\widetilde{F}$ is open (and closed) in $\operatorname{Fix}\widetilde{F}$ and hence an isolated fixed point set of $\widetilde{F}$. Following **[J1,** I.3.10] we conclude that

(1) N_0 and N_1 are $\widehat{F}$-related $\Rightarrow I(\widehat{F}_0, N_0) = I(\widehat{F}_1, N_1)$.

(2) $N \in \mathscr{N}(\widehat{F}_0)$ is not $\widehat{F}$-related to any $N' \in \mathscr{N}(\widehat{F}_1) \Rightarrow I(\widehat{F}_0, N) = 0$.

Hence there is a one to one correspondence between the essential WH-Nielsen classes of $\widehat{F}_0$ and those of $\widehat{F}_1$. Thus $n_{WH}(\widehat{F}_0, V_H) = n_{WH}(\widehat{F}_1, V_H)$ for every type (H) of V. □

When the action on V is free, $N_G^c(f, V) = n_G(f, V)$. When $G = \{1\}$, $N_G^c(f, V) = n(f, V)$ the local Nielsen number of f on V. When $V = X$, we write $N_G^c(f) = N_G^c(f, X)$.

3.11. PROPOSITION (*Lower Bound*). *Let $f: V \to X$ be a G-compactly fixed map. If f' is G-compactly fixed homotopic to f then for every isotropy type (H) of V with WH finite, we have*

$$n_{WH}(f_H, V_H) \leq \frac{1}{|WH|} \cdot |\operatorname{Fix} f'_H|.$$

Proof. $n_{WH}(f_H, V_H)$ is the number of essential WH-Nielsen classes each of which contains at least one *fixed orbit* (orbit of a fixed point), i.e., at least $|WH|$ many fixed points. □

We will now illustrate by an example that two G-homotopic maps each of which is G-compactly fixed, need not be related by a G-compactly fixed homotopy.

3.12. EXAMPLE. Consider $X = S^1$ the circle of radius $1/2$ centered at $(1/2, 0)$ in $\mathbf{R}^2$. Let $G = \mathbf{Z}_2 = \{\pm 1\}$ and the action be given by

$$\xi \cdot (x, y) = (x, \xi y)$$

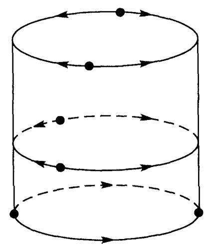
where $\xi \in G$. Thus $X^G = \{(0, 0), (1, 0)\}$. Define a homotopy

$$H: X \times I \to X$$

by

$$H_t(x, y) = \begin{cases} (2x, \frac{1}{2}\delta(y)(1 - (4x - 1)^2)^{1/2}), & 0 \leq x \leq \frac{t}{4}, \\ (\frac{t}{2}, \frac{1}{2}\delta(y)(1 - (t - 1)^2)^{1/2}), & \frac{t}{4} \leq x \leq \frac{2+t}{4}, \\ (2x - 1, \frac{1}{2}\delta(y)(1 - (4x - 3)^2)^{1/2}), & \frac{2+t}{4} \leq x \leq 1, \end{cases}$$

where $\delta(y) = 0$ if $y = 0$ and $\delta(y) = y/|y|$ otherwise.



It is easy to see that H is a G-homotopy and

$$\operatorname{Fix} H_0 = X^G,$$
$$\operatorname{Fix} H_1 = X^G \cup \{(1/2, 1/2), (1/2, -1/2)\}.$$

Both H_0 and H_1 are G-compactly fixed but H_0 is fixed point free on $X - X^G$ whereas H_1 has fixed points $(1/2, 1/2)$ and $(1/2, -1/2)$ each of which has index 1. Thus $n_G(H_0, X - X^G) = 0$ and $n_G(H_1, X - X^G) = 1$. By 3.10, we conclude that H_0 and H_1 cannot be G-compactly fixed homotopic.

4. Equivariant Hopf's construction. In this section, we modify the proof of the classical Hopf construction given in **[Br]** and obtain a local version of it. Then we apply the Covering Homotopy Theorem to obtain the equivariant Hopf's construction.

4.1. PROPOSITION. *Let U be a connected open subset of a connected locally finite simplicial complex X. Let $f: U \to X$ be a compactly fixed map. Then given $\varepsilon > 0$, there exists a finite polyhedron $L \subset U$ with $\operatorname{Fix} f \subset \operatorname{int} L$ and a compactly fixed ε-homotopy H_t (relative to $U - \operatorname{int} L$) so that $H_0 = f$ and H_1 has finitely many fixed points, each lying in the interior of some maximal simplex.*

Proof. Since f is compactly fixed and X is locally compact, we can find a compact set $C \subset U$ such that $\operatorname{Fix} f \subset \operatorname{int} C$. Let N be a finite polyhedron so that N is a neighborhood of C inside U. Let K be a regular neighborhood of N and L be a regular neighborhood of K. We may choose K and L so that both of them lie inside U and their boundaries ∂L, ∂K in X are disjoint (∂L and ∂K are finite polyhedra). Therefore, f is fixed point free on $L - \operatorname{int} K$. Let $\min\{d(x, f(x)) | x \in L - \operatorname{int} K\} = 2\delta > 0$, where d is the metric on X.

By the simplicial approximation theorem (with a subdivision of L), there exists a small homotopy $F: L \times I \to X$ given by

$$F(x, t) = (1 - t)f(x) + t\varphi(x)$$

for some simplicial map φ. Also, $F_t(x)$ lies in the same simplex as $f(x)$ for every $t \in I$. Choose a continuous function $\varrho: L \to I$ such that $\varrho|\partial L = 0$ and $\varrho|K = 1$. Define

$$\begin{aligned} G(x, t) &= F(x, \varrho(x)t) \\ &= (1 - \varrho(x)t)f(x) + \varrho(x)t\varphi(x). \end{aligned}$$

Then we have

$$\begin{aligned} G(x, 0) &\equiv f(x), && x \in L, \\ G(x, t) &= f(x), && x \in \partial L, \\ G(x, t) &= F(x, t), && x \in K. \end{aligned}$$

The map G_t is fixed point free on $L - \operatorname{int} K$ for all t. To see that, we first subdivide L such that $\operatorname{mesh}(L) < \delta$. If $G(x, t) = x$ for some x in $L - \operatorname{int} K$ then $f(x)$ and x would have to lie in the same simplex since $G(x, t) = F_{\varrho(x)t}(x)$. However, $d(x, f(x)) > 2\delta$ and thus a contradiction. Thus, F_t is fixed point free on ∂K. We now apply the Hopf's construction (see **[Br]**) to F_1 on K and obtain an ε-homotopy $H'_t: K \to X$ such that $H'_t|\partial K \equiv F_1|\partial K$ and H'_1 has finitely many fixed points, each lying inside some maximal simplex in K. Define $H: U \times I \to X$ by

$$H(x, t) = \begin{cases} f(x), & x \in U - L, \\ G_t(x), & x \in L - K, \\ H'_t(x), & x \in K. \end{cases}$$

By making $\operatorname{mesh}(L)$ sufficiently small, the homotopy $\{G_t\}$ can be made to be an ε-homotopy and hence $\{H_t\}$ is the required compactly fixed ε-homotopy. □

Next we prove an equivariant analog of 4.1.

4.2. PROPOSITION. *Let G be a finite group and X be a locally finite G-simplicial complex. Suppose U is a free open invariant subset of X and $f: U \to X$ is a compactly fixed G-map. Given any $\varepsilon > 0$, there exists a finite G-complex $L \subset U$ with* $\operatorname{Fix} f \subset \operatorname{int} L$ *and a compactly fixed G ε-homotopy $H: U \times I \to X$ (relative to $U - \operatorname{int} L$) such that $H_0 = f$ and* $\operatorname{Fix} H_1$ *is finite. Furthermore, each fixed point of H_1 lies in the interior of some maximal simplex.*

Proof. Let $\mathscr{F} = p(\operatorname{Fix} f)$ where $p: X \to X/G$ is the orbit map. Since $\operatorname{Fix} f$ is compact in U, $\mathscr{F}$ is a compact subset of U/G. Denote by $\overline{f}$ the induced map of f on U/G. Let $\{(H_1), \ldots, (H_k)\}$ be an admissible ordering on the isotropy types of X with the associated filtration $X_1 \subset \cdots \subset X_k = X$. Since $\operatorname{Fix} f \subset X - f^{-1}(X_{k-1})$, $\mathscr{F} \subset (X - f^{-1}(X_{k-1}))/G$. Without loss of generality, we may assume that $(X - f^{-1}(X_{k-1}))/G$ is connected. Applying 4.1 to $\overline{f}$ restricted to $(X - f^{-1}(X_{k-1}))/G$, we obtain an ε-homotopy which can be lifted by the Covering Homotopy Theorem. Since $U \to U/G$ is a finite cover, the lifted homotopy can be made to be an ε-homotopy. □

4.3. THEOREM (*Equivariant Hopf's Construction*). *Let G be a finite group, X be a finite G-simplicial complex and $f: X \to X$ be a G-map. Given $\varepsilon > 0$, there exists an equivariant ε-homotopy $\mathscr{H}: X \times I \to X$ such that $\mathscr{H}_0 = f$ and $\mathscr{H}_1|X_H$ has finitely many fixed points each lying in the interior of some maximal simplex in X^H for each isotropy type (H) of X.*

Proof. By 2.6, f is G-homotopic to a G-compactly fixed map and the homotopy can be made arbitrarily small. Without loss of generality, we may assume that f is G-compactly fixed. Choose an admissible ordering $(H_1), \ldots, (H_k)$ on the isotropy types of X with the associated filtration $X_1 \subset \cdots \subset X_k \subseteq X$. We assume inductively that $f|X_i: X_i \to X_i$ has finitely many fixed points each lying in the interior of some maximal simplex in the corresponding subcomplex, for $i < j$. Since X_{H_j} is a free open WH_j-invariant subset of X^{H_j}, we apply 4.2 to $f|X_{H_j}: X_{H_j} \to X^{H_j}$ to obtain WH_j-homotopy Γ relative to $X_{H_j} - \operatorname{int} K$ for some WH_i invariant compact polyhedron K containing $\operatorname{Fix} f|V_{H_j}$. Extend Γ to a G-homotopy on $X_{(H_j)}$ relative to $X_{(H_j)} - \operatorname{int} GK$. Since $X_j = X_{j-1} \sqcup X_{(H_j)}$, we extend the homotopy to a G-homotopy Γ' on X_j. Finally, we can extend Γ' to a G-homotopy on X because $X_j \hookrightarrow X$ is a closed G-cofibration. The inductive step is complete. □

4.4. REMARK. Let G be a compact Lie group and $f: M \to M$ be a G-map on a compact smooth G-manifold M. Since M/G is also a triangulable manifold, we can use the techniques in the proofs of 4.2 and 4.3 to show that f can be G-deformed to a map with finitely many fixed orbits. This is also proved in [Wi] using a different approach.

5. Minimal number of fixed points. In this section, we study the minimal number of fixed points in the G-compactly fixed homotopy class of a G-compactly fixed map. When X satisfies a certain connectedness condition and the G-action on U is free, any compactly fixed G-map $f: U \to X$ can be equivariantly deformable to a G-map with the minimal number of fixed points in its compactly fixed G-homotopy class.

5.1. DEFINITION. A locally finite simplicial complex K is of *type* S if (i) there is a 3-simplex, and (ii) for every 0- or 1-simplex σ, the link $\mathrm{lk}(\sigma, K)$ is path connected.

5.2. DEFINITION. Let G be a finite group and X a G-complex. Then X is said to satisfy the *equivariant Shi condition* or is a *G-complex of type S* if every connected component of X^H is of type S for every isotropy type (H) of X. In the case when G is compact Lie, acting smoothly on a smooth G-manifold M, we call M a *G-manifold of type S* if every connected component of M^H is of type S for every isotropy type (H) of M with WH finite.

Note that if X is a G-complex of type S, then X/G is a complex of type S. In [**Br**, VIII.D], a space of type S is defined to be a simplicial complex satisfying the conditions of Definition 5.1 except that the link of a 1-simplex is not required to be path connected. The following is easy to verify

5.3. PROPOSITION. *Every maximal simplex of a complex of type S is at least three dimensional.*

Next we show how to coalesce fixed points of the same class using an equivariant analog of the Wecken trick.

5.4. LEMMA. *Let $f: U \to X$ be a compactly fixed G-map where X is a G-complex of type S and U is a free invariant subset of X. Suppose $\mathscr{O}_1$ and $\mathscr{O}_2$ are isolated fixed orbits belonging to the same G-Nielsen class such that each fixed point in $\mathscr{O}_1 \cup \mathscr{O}_2$ lies in the interior of some maximal simplex of X. Then f is G-homotopic via compactly fixed G-homotopy to a map φ with one less fixed orbit.*

Proof. Since $\mathscr{O}_1$ and $\mathscr{O}_2$ belong to the same G-Nielsen class, there exist $x_1 \in \mathscr{O}_1$, $x_2 \in \mathscr{O}_2$ and a path α in U from x_1 to x_2 so that $\alpha \sim f \circ \alpha$ in X (rel. endpoints). Let $\overline{\alpha}$ denote the image of α in U/G. We first cover the path $\overline{\alpha}$ by a finite number of open vertex-stars. By taking the closure of these open stars, $\overline{\alpha}$ lies inside a closed simplicial neighborhood. Following [**Br**, VIII.D.1], α is homotopic (rel. endpoints) to a polygonal path $\overline{\beta}$ so that the interior of each segment lies inside some maximal simplex and each endpoint lies in some simplex of dimension at least one. Moreover this homotopy can be made arbitrarily small so that the track of the homotopy stays inside U/G. Since X/G is also of type S, each maximal simplex is at least three dimensional. If s and s' are maximal simplices intersecting at a one dimensional face, by the connectivity of $\mathrm{lk}(s \cap s', X/G)$ there is a finite chain of maximal simplices $s = s_1, \ldots, s_k = s'$ such that $s \cap s' \subset s_i$ and $s_i \cap s_{i+1}$ is at least two dimensional for $i = 1, \ldots, k-1$. So $\overline{\beta}$ lies inside the union of the closed simplices $\mathrm{cl}(s_i)$. By taking a fine equivariant subdivision of X and hence a fine subdivision of X/G we may assume that $s_i \subset U/G$ for all i.

We deform the path $\overline{\beta}$ slightly to a polygonal path $\overline{\gamma}$ (rel. endpoints) so that the interior of each segment lies inside some maximal simplex of dimension at least three and each endpoint lies in the interior of some simplex of dimension at least two. By general position, we may assume that $\overline{\gamma}$ is simple. Thus $\overline{\alpha} \sim \overline{\gamma}$ (rel. endpoints) in U/G. Lifting this homotopy to a G-homotopy $\alpha \sim \gamma$ (rel. endpoints) in U, we have $\gamma \sim f \circ \gamma$ (rel. endpoints) in X. We then coalesce the fixed points x_1 and x_2 along γ by the Wecken method ([**Br**, VIII]). The path $\overline{\gamma}$ being simple implies that γ is a cross section. By taking all the G-translates of γ we unite the fixed orbits $\mathscr{O}_1$ and $\mathscr{O}_2$ along $g\gamma$, for each $g \in G$ (also see [**F-W**]). Hence f is G-homotopic to a map φ with one less fixed orbit. Since φ coincides with f outside a small contractible neighborhood of $G\gamma$, this G-homotopy is indeed compactly fixed. □

5.5. Definition. Let $f: V \to X$ be a G-compactly fixed map where V is an open invariant subset of a G-ENR X and G is compact Lie. We define the *minimal number of fixed points in the G-compactly fixed homotopy class of f* to be

$$m_G^c(f, V) = \min\{|\operatorname{Fix} h| \mid h \text{ is } G\text{-compactly fixed homotopic to } f\}.$$

5.6. Theorem (*Minimality*). *Let $f: U \to X$ be a compactly fixed G-map where X is a G-complex of type S and U is a free invariant open subset of of X. Then f is G-homotopic via a compactly fixed G-homotopy to a G-map φ such that*

$$|\operatorname{Fix}\varphi| = |G| \cdot n_G(f, U) = m_G^c(f, U).$$

Proof. By 4.2, f is G-homotopic via a compactly fixed G-homotopy to a map f' with finitely many fixed points each lying in the interior of some maximal simplex. Applying 5.4 finitely many times, each

G-Nielsen class contains only one fixed orbit which can be removed if the index is zero (see **[F-W]**). Thus we arrive at a G-map φ with $n_G(f, U)$ many fixed orbits and hence $|G| \cdot n_G(f, U)$ many fixed points. The minimality follows from 3.11. □

5.7. COROLLARY. *Let X be a G-complex of type S and $A \subset X$ be a closed invariant subset such that the G-action on $X - A$ is free. Suppose that $f: X \to X$ is a G-map and $f|X - A$ is compactly fixed. Then $n_G(f, X - A) = 0$ if, and only if, f is G-homotopic (relative to A) to a G-map which is fixed point free on $X - A$.*

5.8. REMARK. Note that for any finite group G, we have from 3.11 the following:

$$\sum_i [G : H_i] \cdot n_{WH_i}(f, V_{H_i}) \leq m_G^c(f, V).$$

We have equality when G acts freely on V. It would be interesting to know when equality can be achieved in general.

6. G-deformation via obstruction theory. A local obstruction to deforming a map to be fixed point free has been defined and calculated in terms of the local Nielsen number in **[F-H]**. Moreover, equivariant obstructions have been used to prove an equivariant analog of the converse to the Lefschetz fixed point theorem **[V]**. In this section, we define a local obstruction in terms of $n_{WH}(f_H, V_H)$ to deforming f_H to be fixed point free equivariantly.

6.1. LEMMA. *Let M be a compact smooth manifold of dimension ≥ 3 and $f: U \to M$ be a compactly fixed map on an open set $U \subset M$. Suppose that L is a connected compact codimension 0 submanifold with boundary ∂L such that $L \subset U$ and $\mathrm{Fix}\, f \cap \partial L = \varnothing$. Then there exists a local (primary) obstruction $o(f, L) \in H^m(L, \partial L; \pi_m)$ of f on L in U such that $f|L$ is deformable in M (relative to ∂L) to a fixed point free map if, and only if, $o(f, L) = 0$, where $m = \dim L = \dim M$, $\pi_m = \pi_m(L \times M, L \times M - \Delta)$.*

Proof. This follows from 2.6 and 5.4 of **[F-H]**. □

We now give an obstruction theoretic proof of 5.7.

6.2. THEOREM. *Let G be a finite group acting smoothly on a connected compact smooth manifold of dimension ≥ 3. Suppose that $A \subset M$ is a closed invariant subset of M so that the G-action on*

$M - A$ is free. Suppose that $f\colon (M, A) \to (M, A)$ is a G-map so that $f|M - A$ is compactly fixed. Then f is G-homotopic (relative to A) to a G-map f' such that f' is fixed point free on $M - A$ if, and only if, $n_G(f, M - A) = 0$.

Proof. By 4.2, we may assume without loss of generality that f has only a finite number of fixed points. Let $\mathcal{N}$ be a G-Nielsen class. Choose a representative fixed orbit $\mathcal{O} \subset \mathcal{N}$. For any fixed orbit $\mathcal{O}' \neq \mathcal{O}$ in $\mathcal{N}$, there exist $x \in \mathcal{O}$, $x' \in \mathcal{O}'$ and a path $\alpha(x, x')$ in $M - A$ such that $\alpha(x, x')(0) = x$, $\alpha(x, x')(1) = x'$ and $\alpha \sim f \circ \alpha$ relative to the ends in M. As in 5.4, we may choose $\alpha(x, x')$ to be a cross section and $\alpha(x, x') \cap (\operatorname{Fix} f|M - A) = \{x, x'\}$. Now fix this point $x \in \mathcal{O}$. For all the other fixed orbits $\mathcal{O}'$ in $\mathcal{N}$, we form a wedge of paths

$$\mathcal{P}(x) = \bigvee_{\substack{x' \in \mathcal{O}' \\ \mathcal{O}' \subset \mathcal{N}}} \alpha(x, x')$$

with wedge point x.

We can take a small closed invariant tubular neighborhood L of $G\mathcal{P}(x)$ of the form $G\overline{L}$ where $\overline{L}$ is a connected compact submanifold of codimension 0 with boundary $\partial\overline{L}$ such that $\mathcal{P}(x) \subset \operatorname{int}\overline{L}$ and $\operatorname{Fix} f \cap L = \mathcal{N}$. Now consider the fiber bundle

$$L \times_G M \xrightarrow{p} L/G.$$

Since L is a product, p is in fact the product bundle

$$\overline{L} \times M \to \overline{L}.$$

There is also a one-to-one correspondence (see **[tD**, I.7]) between

$$\{G\text{-maps: } L \to M\} \quad \text{and} \quad \{\text{cross sections of } p\}.$$

Thus the fixed point free G-maps correspond to the sections lying in

$$L \times_G M \to d(L/G),$$

where $d\colon L/G \to L \times M$ is the section corresponding to the inclusion $i\colon L \hookrightarrow M$ and hence to those sections lying in

$$\overline{L} \times M \to \overline{L}.$$

If $\hat{s}$ is the section corresponding to $f|L$, then there exists a primary obstruction $o(f, \overline{L})$ for $\hat{s}$ to be deformable (relative to $\partial\overline{L}$) into $\overline{L} \times M - \Delta$ and

$$o(f, \overline{L}) \in H^m(\overline{L}, \partial\overline{L}; \pi_m(\overline{L} \times M, \overline{L} \times M - \Delta)).$$

By 6.1, $o(f, \overline{L})$ has a cochain representation

$$c(f, \overline{L}) = (-1)^m \sum_i I(f, N_i)[h_{\sigma_i}]\sigma_i$$

where the sum is over the local Nielsen classes $\{N_i\}$ of $f|\overline{L}$.

Since $L = G\overline{L}$ is a disjoint union of the translates of $\overline{L}$ and there is only one local Nielsen class in $\overline{L}$, then $|G| \cdot I(f, N_1) = I(f, \mathscr{N})$ and from 6.1,

$$c(f, \overline{L}) = \frac{(-1)^m}{|G|} \cdot I(f, \mathscr{N})[h_\sigma]\sigma.$$

We then apply the above argument to every G-Nielsen class $\mathscr{N}$. □

6.3. Remark. We can now define a sequence of local obstructions $\{o(f_H, \overline{L}_{(H)})\}$ associated with $\{n_{WH}(f_H, V_H)\}$. Thus a necessary condition for f to be deformable to a fixed point free G-map via a G-compactly fixed homotopy is the vanishing of these obstructions.

6.4. Remark. In the case where M is simply connected and $M - A$ is connected, there is exactly one G-Nielsen class $\mathscr{N}$ of $f|M - A$. If the relative Lefschetz number $L(f|_{(M,A)}) = L(f) - L(f|A) = 0$, then $\mathscr{N}$ must have index 0 and hence $n_G(f, M - A) = 0$. Thus 6.2 reduces to the main result in **[V]**.

6.5. Remark. Suppose that $M - A$ and M are connected and

$$\pi_1(M - A) \xrightarrow{\iota_\#} \pi_1(M)$$

is surjective. Let $x, y \in \operatorname{Fix} f \cap (M - A)$. If x and y belong to a Nielsen class of f then they belong to a local Nielsen class of $f|M - A$. Thus, as in the main theorem of **[F-W]**, if the codimension of $M_{i-1}^{H_i}$ in M^{H_i} is at least 2, then the Nielsen equivalence of f^{H_i} restricted to M_{H_i} coincides with the local Nielsen equivalence of f_{H_i} on M_{H_i}. Therefore if f is fixed point free in M_{i-1} then $n(f^{H_i}) = 0 \Leftrightarrow n_{WH_i}(f_{H_i}, M_{H_i}) = 0$. Hence 6.2 gives an alternative proof of **[F-W, 2.2]**.

7. An equivariant Nielsen type invariant. There is another natural equivariant Nielsen type invariant $N_G^*(f)$ which enjoys the usual properties of the ordinary Nielsen number. We may also extend to a local definition $N_G^*(f, V)$. Throughout this section, G will denote a compact Lie group unless further restricted.

7.1. Definition. Let M be a compact G-ANR and $\mathscr{H} = \{H | H$ closed subgroup of G with $M^H \neq \varnothing\}$. For any G-map $f: M \to M$, the *G-Nielsen invariant of f on M*, denoted by $N_G^*(f)$, is the function given by $N_G^*(f)(H) = n(f^H)$ for all $H \in \mathscr{H}$ where $n(f^H)$ is the ordinary Nielsen number of $f^H = f|M^H: M^H \to M^H$.

7.2. REMARK. For any orbit type (H), if $K \in (H)$ then M^H is homeomorphic to M^K by the assignment $x \mapsto g^{-1}x$ where $x \in M^H$ and $K = g^{-1}Hg$, $g \in G$. Hence $N_G^*(f)(K) = N_G^*(f)(H)$. So $N_G^*(f)$ does not depend on the choice of representatives of orbit types.

We now give an equivariant analog of the homotopy invariant property for $N_G^*(f)$.

7.3. THEOREM (*G-homotopy invariance*). *$N_G^*(f)$ is invariant under G-homotopy, i.e., if $\hat{f}$ is G-homotopic to f then $N_G^*(\hat{f}) = N_G^*(f)$.*

Proof. If $\hat{f}$ is G-homotopic to f then for every closed subgroup $H \leq G$ with $M^H \neq \varnothing$, the map $\hat{f}^H$ is homotopic to f^H. Thus, $N_G^*(\hat{f})(H) = n(\hat{f}^H) = n(f^H) = N_G^*(f)(H)$ by the homotopy invariance of the ordinary Nielsen number $n(f^H)$. □

7.4. THEOREM (*Commutativity*). *Let X and Y be compact G-ANRs and $f: X \to Y$, $g: Y \to X$ be G-maps. Then,*

$$N_G^*(g \circ f) = N_G^*(f \circ g).$$

Proof. This follows from the commutativity of the ordinary Nielsen number. □

7.5. THEOREM (*G-homotopy type invariance*). *Let X and Y be a compact G-ANRs. Given the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\big\downarrow & & \big\uparrow{\scriptstyle k} \\ Y & \xrightarrow{g} & Y \end{array}$$

where all maps are G-maps and h is a G-homotopy equivalence with inverse k, then $N_G^*(f) = N_G^*(g)$.

Proof. Since f is G-homotopic to $k \circ h \circ f$ we have $N_G^*(f) = N_G^*(k \circ h \circ f)$. Similarly, $N_G^*(g) = N_G^*(h \circ k \circ g)$. By 7.4, we have

$$\begin{aligned} N_G^*(k \circ h \circ f) &= N_G^*(k \circ (h \circ f)) = N_G^*((h \circ f) \circ k) \\ &= N_G^*((g \circ h) \circ k) = N_G^*(g \circ (h \circ k)) = N_G^*(h \circ k \circ g). \ \square \end{aligned}$$

7.6. REMARK. Theorem 7.5 can also be proven directly from the homotopy type invariance of $n(f^H)$ for each $H \leq G$. Also in 7.5, suppose X and Y have isotropy types $\mathrm{Iso}(X)$ and $\mathrm{Iso}(Y)$ respectively, and $\mathrm{Iso}(X) \cup \mathrm{Iso}(Y) = \{(H_1), \ldots, (H_k)\}$; then $N_G^*(f)(H_i) = N_G^*(g)(H_i)$ for $i = 1, \ldots, k$.

7.7. DEFINITION. Let V be an open invariant subset of a G-ENR X and $f: V \to X$ be a compactly fixed G-map. Then for every closed subgroup $H \leq G$ with $V^H \neq \varnothing$, we define the *local G-Nielsen type invariant*, denoted by $N_G^*(f, V)$, to be the function given by $N_G^*(f, V)(H) = n(f^H, V^H)$ the local Nielsen number as defined in **[F-H]**.

Note that if f is compactly fixed then $f^H: V^H \to X^H$ is compactly fixed. It is clear that when $V = X$, $N_G^*(f, X) = N_G^*(f)$ and when $G = \{1\}$, $N_G^*(f, V) = n(f, V)$. If $\Gamma: V \times I \to X$ is a compactly fixed G-homotopy, then $\Gamma^H: V^H \times I \to X^H$ is compactly fixed and so by the homotopy invariance of the local Nielsen number, we obtain the following

7.8. THEOREM (*G-homotopy invariance*). *$N_G^*(f, V)$ is invariant under compactly fixed G-homotopy.*

Next we illustrate the relationship between $N_G^c(f)$ and $N_G^*(f)$ which was first explored implicitly in **[F-W]**.

7.9. LEMMA. *Let G be a finite group and X be a G-complex. Let $f: V \to X$ be a G-map on an open invariant subset V with finite number of fixed points. Suppose that (H) is an isotropy type of V and $\mathcal{N}$ is a WH-Nielsen class of $f_H = f|V_H$. If N is a local class of f_H, then either $N \subset \mathcal{N}$ or $N \cap \mathcal{N} = \varnothing$. Furthermore, in the case $N \subset \mathcal{N}$, $I(f_H, N) = 0$, if, and only if, $I(f_H, \mathcal{N}) = 0$.*

Proof. From the definition of WH-Nielsen relation and that of ordinary local Nielsen relation on $\mathrm{Fix}\, f_H$, if two fixed points are locally Nielsen equivalent then they are WH-Nielsen equivalent. It follows that $\mathcal{N}$ is a disjoint union of local Nielsen classes. Therefore, if $N \cap \mathcal{N} \neq \varnothing$ then $N \subset \mathcal{N}$. Otherwise $N \cap \mathcal{N} = \varnothing$.

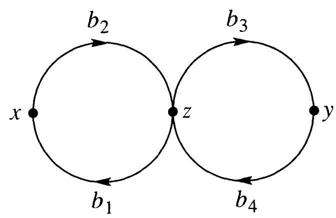
If $N = \mathcal{N}$ then the last assertion is trivial. Suppose that N is a proper subset of $\mathcal{N}$. Let $p_H: V_H \to V_H/WH$ be the orbit map. Since

V_H is a free WH-space, p_H is a finite covering map. We also have $p_H(N) \subset p_H(\mathcal{N})$. If $x, y \in \mathcal{N}$ belong to distinct orbits then $p_H(x)$ and $p_H(y)$ are locally Nielsen equivalent fixed points of the induced map $\overline{f}_H$ on V_H/WH. Thus any two points in $p_H(\mathcal{N})$ are in the same local Nielsen class of $\overline{f}_H$, so $p_H(N) = p_H(\mathcal{N})$. Since we may assume that $\overline{f}_H$ has isolated fixed points, it follows that $I(f_H, N)$ is an integer (nonzero) multiple of $I(\overline{f}_H, p_H(N))$ while $I(f_H, \mathcal{N}) = |WH| \cdot I(\overline{f}_H, p_H(N)) = |WH| \cdot I(\overline{f}_H, p_H(\mathcal{N}))$. Hence $I(f_H, N) = 0$ if and only if $I(f_H, \mathcal{N}) = 0$. □

7.10. THEOREM. *Let G be a finite group and X be a G-complex. Let $f: V \to X$ be a G-map on an open invariant subset V with finite number of fixed points. Let $(H_1), \ldots, (H_k)$ be an admissible ordering on the isotropy types of V. Suppose that* $\operatorname{Fix} f \subset V_{(H_k)}$. *If $n_{WH_k}(f_{H_k}, V_{H_k}) = 0$ then $N_G^*(f, V)(H_i) = 0$ for $i = 1, \ldots, k$.*

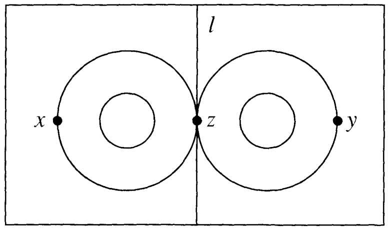
Proof. Since $\operatorname{Fix} f \subset V_{(H_k)}$, f is fixed point free on V_{k-1}. Thus $N_G^*(f, V)(H_i) = 0$ for $i = 1, \ldots, k-1$. Let F be a Nielsen class of f^{H_k}. Since $F \subset V_{H_k}$ and if two points are Nielsen equivalent in V_{H_k} they are Nielsen equivalent in V^{H_i}, F is a disjoint union of Nielsen classes of f_{H_k}. By 7.9, $n_{WH_k}(f_{H_k}, V_{H_k}) = 0$ implies F is of index zero. □

8. An example with $N_G^*(f) = 0$ but $N_G^c(f) \neq 0$. Consider the figure-eight P_0:

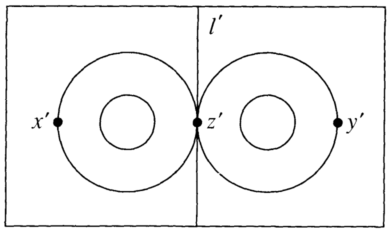


Define a self-map $f_0: P_0 \to P_0$ so that $f_0(b_1) = b_2^{-1}$, $f_0(b_2) = b_1^{-1}$, $f_0(b_3) = b_2^{-1}b_1^{-1}b_3$, $f_0(b_4) = b_4b_3b_4$. Put $\alpha = b_1b_2$, $\beta = b_3b_4$. Then $f_0(\alpha) = \alpha^{-1}$, $f_0(\beta) = \alpha^{-1}\beta^2$ and $\operatorname{Fix} f_0 = \{x, y, z\}$. Since $f_0(b_1^{-1}b_3) = b_2b_2^{-1}b_3 = b_1^{-1}b_3$, x and y are Nielsen equivalent.

Let P be the disk with two holes and the embedded figure-eight:

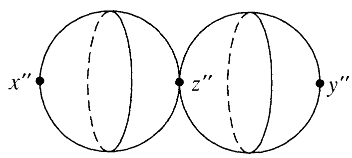


Let $f' = \imath \circ f_0 \circ r \colon P \to P$ where r is a retraction of P onto P_0 and $\imath \colon P_0 \to P$ is the inclusion. (The map f' is the same map as in Jiang's example in [J2].) Furthermore we require that r contracts the line l to the point z.



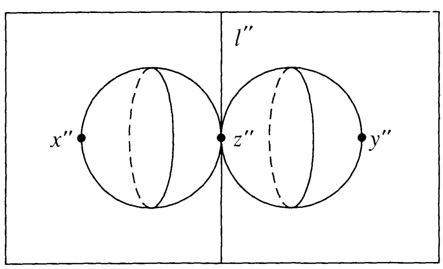
Let $X(1)$ be P, $f_{(1)} = f'$ and let $X(2)$ be another copy of P and $f_{(2)} = f'$ with $\operatorname{Fix} f_{(2)} = \{x', y', z'\}$.

Let W be the wedge of two 2-spheres:



and the map $g \colon W \to W$ which is a $180°$ rotation about the axis through x'', z'' and y'' with $\operatorname{Fix} g = \{x'', y'', z''\}$.

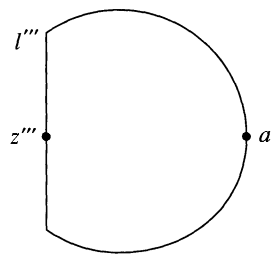
Let $X(3)$ be the following space:



which is the union of W and a 2-disc with a subset removed whose boundary is the figure-eight P_0.

Take $f_{(3)}\colon X(3) \to X(3)$ to be $\hat{\imath} \circ g \circ \hat{r}$ where $\hat{r}\colon X(3) \to W$ is the retraction which sends the line l'' to the point z'' and $\hat{\imath}\colon W \to X(3)$ is the inclusion.

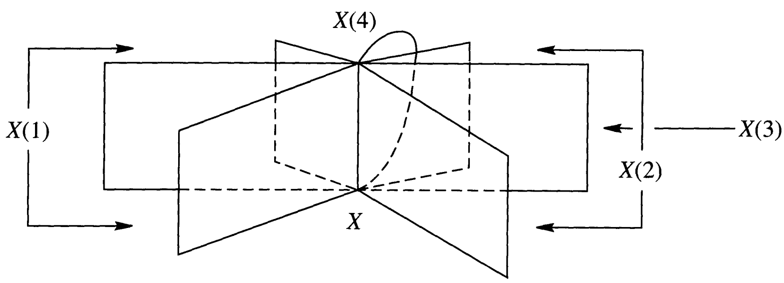
Let $X(4)$ be the unit circle



and $f_{(4)}\colon X(4) \to X(4)$ to be the composition $\lambda \circ \kappa$ where κ is a flow on $X(4)$ so that a is a "source" and z''' is a 'sink' and λ is a retraction which takes the line segment l''' to the point z'''. Therefore, $\operatorname{Fix} f_{(4)} = \{z''', a\}$.

We now let X be the union $\bigcup_{i=1}^{4} X(i)$ with the lines l, l', l'', l''' all identified so that $z = z' = z'' = z'''$. Embed X in $\mathbf{R}^3$ and let $G = \mathbf{Z}_2$ act on X by reflection in the plane containing $X(4) = S^1$.

Let $f\colon X \to X$ be $f_{(1)} \cup F_{(2)} \cup f_{(3)} \cup f_{(4)}$ so that $\operatorname{Fix} f = \{x, y, x', y', x'', y'', z, a\}$.



Let Y be a symmetric regular neighborhood of X in $\mathbf{R}^4$ and extend f to $\varphi = \imath_X \circ f \circ r_Y$ where $\imath_X\colon X \hookrightarrow Y$ is the inclusion and $r_Y\colon Y \to X$ is a G-invariant retraction. Note that Y is a finite G-complex of type S and $\operatorname{Fix} \varphi = \operatorname{Fix} f$.

We are now to calculate $N_G^*(\varphi)$ and $N_G^c(\varphi)$. Since $\varphi = \iota_X \circ f \circ r_Y$ and $f^G = f|X(4) = f_{(4)}$, the Lefschetz number

$$L(\varphi^G) = L(f^G) = i(f_{(4)}, a) + i(f_{(4)}, z)$$
$$= (-1) + (+1) = 0.$$

Thus the Nielsen number $n(\varphi^G) = 0$ since $Y^G \approx D^2 \times S^1$ is a solid torus which is a Jiang space (see **[Br]** or **[J1]**). Recall on $X(1)$, $f_{(1)}(\alpha) = \alpha^{-1}$; $f_{(1)}(\beta) = \alpha^{-1}\beta^2$ and $f_{(4)}$ is homotopic to the identity on $X(4)$. Hence

$$f_{*1}: H_1(X; \mathbf{Q}) \to H_1(X; \mathbf{Q})$$

is given by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the trace $\operatorname{tr}(f_{*1}) = 3$. We also have

$$f_{*2} = (f_{(3)})_{*2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $f_{(3)}$ is homotopic to the identity on $X(3)$. Thus, $\operatorname{tr}(f_{*2}) = 2$. Hence

$$L(f) = 1 - \operatorname{tr}(f_{*1}) + \operatorname{tr}(f_{*2}) = 1 - 3 + 2 = 0.$$

Thus, $L(\varphi) = L(f) = 0$.

An easy computation shows that x and x' are fixed points of f of index 1 while y and y' are of index -1. Also, x and x' are Nielsen equivalent to y and y' respectively. Since W is simply connected the fixed points x'', y'' and z are Nielsen equivalent to each other. Since $L(f) = 0$ then $\{x'', y'', z, a\}$ has index zero. Therefore f has at most three Nielsen classes each of which is of index 0. It follows that $n(\varphi) = n(f) = 0$ and hence $N_G^*(\varphi) = 0$.

Since φ has only a finite number of fixed points, it is G-compactly fixed. The subspace $Y - Y^G$ consists of two disjoint components each of which contains three fixed points whose index sum is nonzero. In fact, the fixed orbits of φ in $Y - Y^G$ are $\{x, x'\}$, $\{y, y'\}$ and $\{x'', y''\}$. So $n_G(\varphi, Y - Y^G) = 1 \neq 0$ and thus $N_G^c(\varphi) \neq 0$.

Since z and a are Nielsen equivalent in Y^G, the fixed point a can be coalesced with z. We can also move x'' to z along a path γ so that $\gamma - \{z\}$ is connected in $Y - Y^G$. Then we can move x'' and

y'' to z equivariantly. The fixed point z has index zero. Therefore it can be removed locally. We are now left with fixed points x, y, x', y'. Since each component of $Y - Y^G$ has no local cut points, x and y can cancel and hence the fixed orbits $\{x, y\}$ and $\{x', y'\}$ can be removed equivariantly. We then conclude that the map φ is G-deformable but not G-compactly fixed deformable to be fixed point free.

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Received December 1, 1989. This paper is based on part of the author's Ph.D. thesis written at the University of Wisconsin under the supervision of Professor Edward R. Fadell whose guidance, patience and constant encouragement are deeply appreciated.

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