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BOUNDED HANKEL FORMS WITH WEIGHTED NORMS AND LIFTING THEOREMS

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Bounded Hankel forms with respect to weighted norms are studied. The Nehari's theorem about the norms of the classical Hankel forms is generalized. This is essentially a lifting theorem due to Cotlar and Sadosky. Moreover a theorem about the essential norms of Hankel forms is proved. This relates with a theorem of Adamjan, Arov and Krein in the special case and gives a new lifting theorem which has applications to weighted norm inequalities, and the F. and M. Riesz theorem.

1. Introduction. Let

$$A[a, b] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} A_{ij} a_i b_j$$

where a and b are finite sequences. Then $A[a, b]$ is called a sesquilinear form in the variables a and b.

Let $\mathscr{P}$ be the set of all trigonometric polynomials and m the normalized Lebesgue measure on the unit circle T. If we put $u = \sum_{j=-n}^{n} a_j z^j$ for $a = (\dots, 0, a_{-n}, \dots, a_0, a_1, \dots, a_n, 0, \dots)$ then u belongs to $\mathscr{P}$ and $\int |u|^2\, dm = \sum_{j=-n}^{n} |a_j|^2$. Let

$$A(u, v) = A[a, b]$$

where $u = \sum_{j=-n}^{n} a_j z^j$ and $v = \sum_{j=-m}^{m} \overline{b}_j \overline{z}^j$. Then we say that $A(u, v)$ is a sesquilinear form on $\mathscr{P} \times \mathscr{P}$. It is clear that

$$A(\beta_1 u_1 + \beta_2 u_2, v) = \beta_1 A(u_1, v) + \beta_2 A(u_2, v)$$

and

$$A(u, \alpha_1 v_1 + \alpha_2 v_2) = \overline{\alpha}_1 A(u, v_1) + \overline{\alpha}_2 A(u, v_2).$$

If $A_{ij} = \alpha(i+j)$ then $A(u, v)$ is called a Hankel form on $\mathscr{P} \times \mathscr{P}$ and we will write those forms $\varphi(u, v)$, $\psi(u, v)$ or etc.

Let $\mathscr{P}_+ = \{f \in \mathscr{P} : \hat{f}(j) = 0 \text{ if } j < 0\}$ and $\mathscr{P}_- = \{f \in \mathscr{P} : \hat{f}(j) = 0$ if $j \geq 0\}$. If A is restricted to $\mathscr{P}_+ \times \mathscr{P}_-$ then the restriction of A is called a sesquilinear form on $\mathscr{P}_+ \times \mathscr{P}_-$. If φ is a Hankel form on $\mathscr{P} \times \mathscr{P}$ then we will write

$$H_\varphi = \text{the restriction of } \varphi \text{ to } \mathscr{P}_+ \times \mathscr{P}_-$$

and φ is called a symbol of H_φ.

A sesquilinear form A on $\mathscr{P} \times \mathscr{P}$ is said to be bounded if there exists a positive constant γ such that $|A(u, v)| \leq \gamma$ if $\int |u|^2 \, dm \leq 1$ and $\int |v|^2 \, dm \leq 1$. We will generalize this definition. Let μ and ν be finite positive Borel measures on T. A sesquilinear form A on $\mathscr{P} \times \mathscr{P}$ is said to be bounded w.r.t. (μ, ν) if there exists a positive constant γ such that

$$|A(u, v)|^2 \leq \gamma^2 \int |u|^2 \, d\mu \int |v|^2 \, d\nu \qquad (u, v \in \mathscr{P}).$$

The smallest number γ for which the inequality above is refered to as the norm of the form A and we will write $\gamma = |||A|||$, where the pair of measures is fixed. Similarly for the norm γ of the form A on $\mathscr{P}_+ \times \mathscr{P}_-$ we will write $\gamma = \|A\|$. When the form $A(u, v)$ is bounded on $\mathscr{P} \times \mathscr{P}$ w.r.t. (μ, ν), it can be extended to a form on (the $L^2(\mu)$-closure of $\mathscr{P}$) $\times$ (the $L^2(\nu)$-closure of $\mathscr{P}$). Then we will still write $A(u', v')$ for u' and v' in the closures. It is the same for the case of $\mathscr{P}_+ \times \mathscr{P}_-$.

For $0 < p \leq \infty$ $H^p = H^p(m)$ denotes the usual Hardy space, that is, the $L^p = L^p(m)$-closure of $\mathscr{P}_+$. C denotes the set of all continuous functions on T. Then $H^\infty + C$ is the closure of $\bigcup_{n=1}^{\infty} \bar{z}^n H^\infty$ [**9**, Theorem 2].

Our program is as follows. In §2 we will give representations of bounded Hankel forms on $\mathscr{P} \times \mathscr{P}$. In §3 generalizing Nehari's theorem ([**13**], [**15**, p. 6]) we will calculate the norms of bounded Hankel forms on $\mathscr{P}_+ \times \mathscr{P}_-$. This is, in fact, the lifting theorem of Cotlar and Sadosky [**4**] that appears as a corollary in §6. In §4 we will determine compact bounded Hankel forms on $\mathscr{P}_+ \times \mathscr{P}_-$. This relates with Hartman's theorem [**8**] in a special case. In §5 we will give the distance between a given Hankel form and the set of all compact sesquilinear forms. In §6 as a result of the previous sections we will obtain a new lifting theorem which contains one due to Cotlar and Sadosky [**4**]. In §7 we will apply results in the previous sections to problems in weighted norm inequalities as in [**3**] and to get a quantitative F. and M. Riesz theorem [**16**].

2. Bounded Hankel forms on $\mathscr{P} \times \mathscr{P}$. For some pair μ and ν of finite positive Borel measures on T, there exist nonzero bounded sesquilinear forms w.r.t. (μ, ν) but in Corollary 1 it is shown that no nonzero Hankel forms can exist.

PROPOSITION 1. *If φ is a bounded Hankel form on $\mathscr{P} \times \mathscr{P}$ w.r.t. (μ, ν) and $|||\varphi||| = \gamma$ then the following are valid.*

(1) *There exists a finite Borel measure* λ *on* T *such that*

$$\varphi(u, v) = \int u\bar{v}\, d\lambda \qquad (u, v \in \mathscr{P})$$

and

$$|\lambda(E)| \leq \gamma |\mu(E)| |\nu(E)|$$

for any Borel set E *in* T.

(2) *If* $\mu = \mu_a + \mu_s$ *and* $\nu = \nu_a + \nu_s$ *are Lebesgue decompositions w.r.t.* λ *then* φ *can be assumed to be a bounded Hankel form on* $\mathscr{P} \times \mathscr{P}$ *with respect to* (μ_a, ν_a).

Proof. There exists a bounded linear operator Φ from $L^2(\mu)$ to $L^2(\nu)$ such that $\varphi(u, v) = \int (\Phi u)\bar{v}\, d\nu$. Since $\varphi(z^i, \bar{z}^j) = \varphi(1, z^{i+j})$,

$$\varphi(u, v) = \int u\bar{v}k\, d\nu \qquad (u, v \in \mathscr{P})$$

where $k = \Phi 1 \in L^2(\nu)$. Set $d\lambda = k\, d\nu$; then

$$\left|\int u\bar{v}\, d\lambda\right|^2 \leq \gamma^2 \int |u|^2\, d\mu \int |v|^2\, d\nu$$

for any $u \in L^2(\mu)$ and $v \in L^2(\nu)$, and hence (1) follows. There is a Borel set E_a in T with $\mu_s(E_a) = \nu_s(E_a) = 0$ on which λ is concentrated. Then $\chi_{E_a} \in L^2(\mu) \cap L^2(\nu)$ and so

$$\left|\int u\bar{v}\, d\lambda\right|^2 \leq \gamma^2 \int |u|^2\, d\mu_a \int |v|^2\, d\nu_a$$

for any $u \in L^2(\mu_a) = \chi_{E_a} L^2(\mu)$ and $v \in L^2(\nu_a) = \chi_{E_a} L^2(\nu)$. This implies (2).

COROLLARY 1. *If* φ *is a bounded Hankel form on* $\mathscr{P} \times \mathscr{P}$ *w.r.t.* (μ, ν), *and* μ *and* ν *are mutually singular, then* $\varphi \equiv 0$.

COROLLARY 2. *If* φ *is a bounded Hankel form on* $\mathscr{P} \times \mathscr{P}$ *w.r.t.* $(w_1 dm, w_2 dm)$, *then for some* k *in* L^∞

$$\varphi(u, v) = \int u\bar{v}k\sqrt{w_1 w_2}\, dm \qquad (u, v \in \mathscr{P}).$$

Conversely such φ is bounded w.r.t. $(w_1\, dm, w_2\, dm)$.

3. Bounded Hankel forms on $\mathscr{P}_+ \times \mathscr{P}_-$. In this section we will give a generalization of Nehari's theorem (see **[13]**, **[15**, p. 6]) which was proved in the case of $\mu = \nu = m$. For any Hankel form φ on $\mathscr{P} \times \mathscr{P}$, if H_φ is bounded on $\mathscr{P}_+ \times \mathscr{P}_-$ w.r.t. (μ, ν) then there exists a finite Borel measure λ on T such that

$$\varphi(u, v) = \int u\bar{v}\, d\lambda \qquad (u \in \mathscr{P}_+, v \in \mathscr{P}_-).$$

The proof is similar to the proof of Proposition 1. Let $\lambda = \lambda_a + \lambda_s$, $\mu = \mu_a + \mu_s$ and $\nu = \nu_a + \nu_s$ be Lebesgue decompositions with respect to m. Put

$$\varphi_a(u, v) = \int u\bar{v}\, d\lambda_a \quad \text{and} \quad \varphi_s(u, v) = \int u\bar{v}\, d\lambda_s$$

for any u, v in $\mathscr{P}$. Then H_{φ_a} and H_{φ_s} are bounded Hankel forms on $\mathscr{P}_+ \times \mathscr{P}_-$ w.r.t. (μ_a, ν_a) and (μ_s, ν_s), respectively. Moreover $\max(\|H_{\varphi_a}\|, \|H_{\varphi_s}\|) = \|H_\varphi\|$.

For set

$$H^2(\mu) = \text{the } L^2(\mu)\text{-closure of } \mathscr{P}_+.$$

Then $\overline{zH}^2(\mu)$ is the $L^2(\mu)$-closure of $\mathscr{P}_-$. Suppose E_s is a Borel set with $m(E_s) = 0$ where μ_s and ν_s are concentrated on E_s, and E_a is a Borel set with $m(E_a) = 1$ where μ_a and ν_a are concentrated on E_a. E_a can be chosen to be the complement of E_s in T. Then both the characteristic functions χ_{E_a} and χ_{E_s} belong to $H^2(\mu) \cap \overline{zH}^2(\nu)$. Moreover $H^2(\mu) = \chi_{E_a} H^2(\mu) \oplus \chi_{E_s} H^2(\mu)$, and $\chi_{E_a} H^2(\mu) = H^2(\mu_a)$ and $\chi_{E_s} H^2(\mu) = H^2(\mu_s) = L^2(\mu_s)$. This implies the above statement about H_{φ_a} and H_{φ_s}.

To prove the generalized Nehari's theorem, we need the following lemma which will be used in later sections, too.

Lemma 1. *Let A be a bounded sesquilinear form on $\mathscr{P}_+ \times \mathscr{P}_-$ w.r.t. $(w_1\, dm, w_2\, dm)$ and $w_j = |h_j|^2$ for $j = 1, 2$ where both h_1 and h_2 are outer functions in H^2. If we put*

$$B(f, g) = A(h_1^{-1} f, \bar{h}_2^{-1} g) \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

then B is a bounded sesquilinear form w.r.t. $(m, m,)$ and $\|B\| = \|A\|$.

Proof. Let $\gamma = \|A\|$; then

$$|A(f, g)|^2 \le \gamma^2 \int |f|^2 |h_1|^2\, dm \int |g|^2 |h_2|^2\, dm$$

for any $f \in \mathscr{P}_+$ and $g \in \mathscr{P}_-$. For any $f \in \mathscr{P}_+$ and $g \in \mathscr{P}_-$, set $F = h_1 f$ and $G = \overline{h}_2 g$. Then $F \in H^2$ and $G \in \overline{zH}^2$. Hence

$$|A(h_1^{-1}F, h_2^{-1}G)|^2 \leq \gamma^2 \int |F|^2 \, dm \int |G|^2 \, dm.$$

Since both h_1 and h_2 are outer functions, we get the lemma.

The following theorem is a generalization of Nehari's theorem (cf. [**15**, Theorem 1.3]) but this is the lifting theorem of Cotlar and Sadosky in [**4**], with other notation. A new proof is given here (cf. [**17**]).

THEOREM 2. *Let φ be a Hankel form on $\mathscr{P} \times \mathscr{P}$. If H_φ is bounded w.r.t. (μ, ν) then there exists a Hankel form ψ bounded w.r.t. (μ, ν) on $\mathscr{P} \times \mathscr{P}$ such that*

$$H_\psi = H_\varphi \quad \textit{and} \quad |||\psi||| = \|H_\varphi\|.$$

Proof. Let $\gamma = \|H_\varphi\|$. By the remark above Lemma 1

$$|\varphi_s(f, g)|^2 \leq \gamma^2 \int |f|^2 \, d\mu_s \int |g|^2 \, d\nu_s$$

for all $f \in \mathscr{P}_+$ and $g \in \mathscr{P}_-$. Since $H^2(\mu_s) = L^2(\mu_s)$, this implies that $|||\varphi_s||| \leq \gamma$. Now we will prove that there exists a bounded Hankel form ψ_a with respect to (μ_a, ν_a) such that

$$H_{\psi_a} = H_{\varphi_a} \quad \text{and} \quad |||\psi_a||| = \|H_{\varphi_a}\|.$$

Then setting $\psi = \psi_a + \varphi_s$, the theorem follows because $\varphi = \varphi_a + \varphi_s$ and $\max(\|H_{\varphi_a}\|, \|H_{\varphi_s}\|) = \|H_\varphi\|$. Let $d\mu_a = w_1 \, dm$ and $d\nu_a = w_2 \, dm$.

Case I. $\log w_1 \notin L^1$ or $\log w_2 \notin L^1$. We may assume that $\log w_1 \notin L^1$. By the remark above Lemma 1,

$$|\varphi_a(f, g)|^2 \leq \gamma^2 \int |f|^2 w_1 \, dm \int |g|^2 w_2 \, dm \qquad (f \in \mathscr{P}_+, \, g \in \mathscr{P}_-).$$

Since $\log w_1 \notin L^1$, $H^2(w_1 \, dm) = L^2(w_1 \, dm)$ and hence for any $u \in \mathscr{P}$ and $g \in \mathscr{P}_-$

$$|\varphi_a(u, g)|^2 \leq \gamma^2 \int |u|^2 w_1 \, dm \int |g|^2 w_2 \, dm.$$

Fix any $n \in Z_+$. For any $u_1 \in \mathscr{P}$ and $g_1 \in z^n \mathscr{P}_-$, there exists $u \in \mathscr{P}$ and $g \in \mathscr{P}_-$ such that $u_1 = z^n u$ and $g_1 = z^n g$. Hence

$$\begin{aligned} |\varphi_a(u_1, g_1)|^2 &= |\varphi_a(z^n u, z^n g)|^2 = |\varphi_a(u, g)|^2 \\ &\leq \gamma^2 \int |u_1|^2 w_1 \, dm \int |g_1|^2 w_2 \, dm. \end{aligned}$$

By the same argument for any $u, v \in \mathscr{P}$

$$|\varphi_a(u, v)|^2 \le \gamma^2 \int |u|^2 w_1\, dm \int |v|^2 w_2\, dm.$$

This implies that $|||\varphi_a||| \le \gamma$. Put $\psi_a = \varphi_a$.

Case II. $\log w_1 \in L^1$ and $\log w_2 \in L^1$. There exist outer functions h_1 and h_2 in H^2 such that $w_1 = |h_1|^2$ and $w_2 = |h_2|^2$ (cf. [**6**, p. 53]). Let $d\lambda_a = w_3\, dm$. By Lemma 1

$$\left|\int f\bar{g}(h_1h_2)^{-1} w_3\, dm\right|^2 \le \gamma^2 \int |f|^2\, dm \int |g|^2\, dm \qquad (f \in \mathscr{P}_+,\ g \in \mathscr{P}_-).$$

Let $s = w_3(h_1h_2)^{-1}$; then by a duality argument there exists $l \in H^\infty$ such that $\|s + l\|_\infty \le \gamma$. By Schwarz's lemma, this implies that

$$\left|\int (s + l)u_1\bar{u}_2\, dm\right|^2 \le \gamma^2 \int |u_1|^2\, dm \int |u_2|^2\, dm \qquad (u_1, u_2 \in \mathscr{P}).$$

Let $v_1 = h_1^{-1} u_1$ and $v_2 = \overline{h_2}^{-1} u_2$ for any $u_1, u_2 \in \mathscr{P}$. Then $v_1 \in L^2(w_1\, dm)$ and $v_2 \in L^2(w_2\, dm)$. Hence

$$\left|\int v_1 v_2 w_3\, dm + \int v_1\bar{v}_2(lh_1h_2)\, dm\right|^2 \le \gamma^2 \int |v_1|^2 w_1\, dm \int |v_2|^2 w_2\, dm.$$

Since $h_1^{-1}\mathscr{P}$ and $h_2^{-1}\mathscr{P}$ are dense in $L^2(w_1\, dm)$ and $L^2(w_2\, dm)$, respectively, if we put

$$\varphi_0(u, v) = \int (lh_1h_2)u\bar{v}\, dm \qquad (u, v \in \mathscr{P})$$

then φ_0 is a bounded Hankel form on $\mathscr{P} \times \mathscr{P}$ w.r.t. $(w_1\, dm, w_2\, dm)$, $H_{\varphi_0} \equiv 0$ and $|||\varphi_a + \varphi_0||| \le \gamma$. Put $\psi_a = \varphi_a + \varphi_0$.

Theorem 2 implies that $\|H_\varphi\| = \inf\{|||\varphi + \varphi_0|||: H_{\varphi_0} \equiv 0\}$.

In Theorem 2 if $d\mu = d\nu = dm$ then Nehari's theorem follows and if $d\mu = d\nu = w\, dm$ then the scalar version of a theorem of Page [**9**] follows.

4. Compact bounded Hankel forms on $\mathscr{P}_+ \times \mathscr{P}_-$. The ideas of this section are closely related to those of [**2**]. In particular, the concept of compact form and Theorem 3 are in Theorem 1a in [**2**]. Let A be a bounded sesquilinear form on $\mathscr{P}_+ \times \mathscr{P}_-$ w.r.t. (μ, ν). We say that A is compact if there exists a null decreasing sequence $\{\gamma_n\}$ such that

$$|A(z^n f, g)|^2 \le \gamma_n^2 \int |f|^2 \, d\mu \int |g|^2 \, d\nu \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

and

$$|A(f, \bar{z}^n g)|^2 \le \gamma_n^2 \int |f|^2 \, d\mu \int |g|^2 \, d\nu \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

for $n = 1, 2, \ldots$. When $\gamma_n = 0$ and $\gamma_{n-1} \ne 0$ for some n, A is called finite n. In this section we will give a generalization of Hartman's theorem **[8]** which was proved in the case of $\mu = \nu = m$ and describes compact Hankel forms. However Theorem 4 does not show Hartman's theorem (see Remark).

LEMMA 2. *If A is a nonzero compact (finite $n \ne 0$, resp.) sesquilinear form w.r.t. (μ, ν) associated with $\{\gamma_n\}$, then it is a nonzero compact (finite $n \ne 0$, resp.) sesquilinear form w.r.t. $(w_1\, dm, w_2\, dm)$ associated with $\{\gamma_n\}$ where $d\mu/dm = w_1$ and $d\nu/dm = w_2$. Moreover both $\log w_1$ and $\log w_2$ are integrable.*

Proof. Let E_a and E_s be Borel sets as in the remark before Lemma 1. Then χ_{E_a} and χ_{E_s} belong to $H^2(\mu) \cap \bar{z}\overline{H}^2(\nu)$. Hence for $n = 1, 2, \ldots$

$$|A(\chi_{E_s} z^n f, g)|^2 \le \gamma_n^2 \int |f|^2 \, d\mu_s \int |g|^2 \, d\nu \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

and

$$|A(f, \chi_{E_s} \bar{z}^n g)|^2 \le \gamma_n^2 \int |f|^2 \, d\mu \int |g|^2 \, d\nu_s \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Since $H^2(\mu_s) = L^2(\mu_s)$ and $H^2(\nu_s) = L^2(\nu_s)$, for $n = 1, 2, \ldots$

$$|A(\chi_{E_s} u, g)|^2 \le \gamma_n^2 \int |u|^2 \, d\mu_s \int |g|^2 \, d\nu \qquad (u \in \mathscr{P}, g \in \mathscr{P}_-)$$

and

$$|A(f, \chi_{E_s} v)|^2 \le \gamma_n^2 \int |f|^2 \, d\mu \int |v|^2 \, d\nu_s \qquad (f \in \mathscr{P}_+, v \in \mathscr{P}_-).$$

As $n \to \infty$, it follows that $A(\chi_{E_s} f, g) = A(f, \chi_{E_s} g) = 0$ for all $f \in \mathscr{P}_+$ and $g \in \mathscr{P}_-$. Hence $A(z^n f, g) = A(\chi_{E_a} z^n f, \chi_{E_a} g)$ and $A(f, \bar{z}^n g) = A(\chi_{E_a} f, \chi_{E_a} \bar{z}^n g)$. This implies that A is a nonzero compact (finite $n \neq 0$, resp.) sesquilinear form w.r.t. $(w_1\, dm, w_2\, dm)$ associated with $\{\gamma_n\}$. If $\log w_1 \notin L^1$ or $\log w_2 \notin L^1$ then $H^2(w_1\, dm) = L^2(w_1\, dm)$ or $H^2(w_2\, dm) = L^2(w_2\, dm)$. By the same argument to the above, we can show that A is a zero form. Thus the lemma follows.

THEOREM 3. *Let n be a nonnegative integer.*

(1) *H_φ is finite $n = 0$ if and only if there exists a function h in H^1 such that $\varphi(f, g) = \int f\bar{g}h\, dm$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$.*

(2) *When $n \neq 0$, H_φ is finite n if and only if there exists a function h in $\bar{z}^n H^1$ and out of H^1 such that $\varphi(f, g) = \int f\bar{g}h\, dm$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$.*

Proof. (1) There exists a finite Borel measure λ such that $\varphi(f, g) = \int f\bar{g}\, d\lambda$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$. If H_φ is zero, by the proof of Lemma 2 $\varphi(f, g) = \varphi(\chi_{E_a} f, \chi_{E_a} g)$ and hence λ is absolutely continuous w.r.t. dm. Let $d\lambda = h\, dm$; then $h\, dm$ annihilates $z\mathscr{P}_+$ and so $h \in H^1$. The converse is clear.

(2) Let H_φ be finite, $n \neq 0$. By Corollary 2, Theorem 2 and Lemma 2, there exists a nonzero function h in L^1 such that

$$\varphi(f, g) = \int f\bar{g}h\, dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Since H_φ is finite, $n \neq 0$, by Lemma 2 there exist $\gamma_1, \gamma_2, \dots, \gamma_n$ with $\gamma_n = 0$ such that for $1 \leq j \leq n$

$$\left|\int z^j f\bar{g}h\, dm\right|^2 \leq \gamma_j^2 \int |f|^2 w_1\, dm \int |g|^2 w_2\, dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-),$$

where $w_1 = d\mu/dm$ and $w_2 = d\nu/dm$. Moreover there exist outer functions h_1 and h_2 such that $|h_j|^2 = w_j$ for $j = 1, 2$. By Lemma 1, for $1 \leq j \leq n$

$$\left|\int z^j f\bar{g}(h_1h_2)^{-1}h\, dm\right|^2 \leq \gamma_j^2 \int |f|^2\, dm \int |g|^2\, dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

and hence $\|z^j(h_1h_2)^{-1}h + H^\infty\| \leq \gamma_j$. Since $\gamma_n = 0$, $(h_1h_2)^{-1}h \in \bar{z}^n H^\infty$ and hence $h \in \bar{z}^n H^1$ and $h \notin H^1$ because H_φ is rank $n \neq 0$. The converse is clear because for such h, $\int z^n f\bar{g}h\, dm = 0$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$.

In the proof of Theorem 3, $h_1h_2 \in H^1$ and $h = (h_1h_2)u$ where $u \in \bar{z}^n H^\infty$. The following theorem is the generalization of this result.

THEOREM 4. *H_φ is nonzero and compact w.r.t. (μ, ν) if and only if there exists a function $h = h_0 \times u$ in $H^1 \times (H^\infty + C)$ and out of H^1 such that*

$$\varphi(f, g) = \int f\bar{g}h\,dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

and $h_0 = h_1h_2$ where h_j is an outer function in H^2, $w_j = |h_j|$, $d\mu/dm = w_1$ and $d\nu/dm = w_2$.

Proof. Let H_φ be nonzero and compact. By Lemma 2, we may assume that $d\mu = w_1\,dm$ and $d\nu = w_2\,dm$, and there exists an outer function h_j in H^2 with $w_j = |h_j|^2$. By the proof of Theorem 3, $\|z^j(h_1h_2)^{-1}h + H^\infty\| \leq \gamma_j$ and $\gamma_j \to 0$ as $j \to \infty$. Thus $(h_1h_2)^{-1}h \in H^\infty + C$ and hence $h = (h_1h_2)u \in H^1 \times (H^\infty + C)$ and out of H^1. For the converse, put $\|z^j u + H^\infty\| = \gamma_j$; then $\gamma_j \to 0$ as $j \to \infty$ and for each j there exists $g_j \in H^\infty$ such that

$$|z^j h + h_1h_2g_j| \leq \gamma_j|h_1h_2|.$$

Hence for each j

$$\begin{aligned}|\varphi(z^jf, g)|^2 = \left|\int z^j f\bar{g}h\,dm\right|^2 &\leq \gamma_j^2 \int |f\bar{g}||h_1h_2|\,dm \\ &\leq \gamma_j^2 \int |f|^2 w_1\,dm \int |g|^2 w_2\,dm\end{aligned}$$

for all $f \in \mathscr{P}_+$ and $g \in \mathscr{P}_-$. This implies that H_φ is nonzero and compact w.r.t. (μ, ν).

If $h = h_0 \times u$ is in $H^1 \times (H^\infty + C)$ and $\varphi_1(f, g) = \int f\bar{g}h\,dm$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$ then H_{φ_1} is compact w.r.t. (μ_1, ν_1) where $d\mu_1 = d\nu_1 = |h_0|^2\,dm$.

If μ is a complex finite Borel measure on T and $\hat{\mu}(n) = \int e^{-in\theta}\,d\mu = 0$ for any negative integer n, then $d\mu = h\,dm$ for some h in H^1. This is the famous F. and M. Riesz theorem (cf. [**11**, p. 47]) and a corollary of the following corollary which follows from Theorem 3 and 4. That is, it is just the case of $\varepsilon_0 = 0$.

COROLLARY 4. *Let μ be a complex finite Borel measure on T and*

$$\varepsilon_n = \sup\left\{\left|\int z^n F\,d\mu\right|; F \in \mathscr{P}_+, \int |F|\,d|\mu| \leq 1\right\}.$$

If $\varepsilon_n \to 0$ *as* $n \to \infty$ *then* $\mu = h\,dm$ *and* h *is in* $H^1 \times (H^\infty + C)$. *If* $\varepsilon_n = 0$ *for some* $n \geq 0$ *then* h *belongs to* $\bar{z}^n H^1$.

Proof. By Schwarz's lemma,

$$\sup\left\{\left|\int z^n f\bar{g}\,d\mu\right| ; f \in \mathscr{P}_+, g \in \mathscr{P}_-, \int |f|^2\,d|\mu| \leq 1 \right.$$
$$\left. \text{and} \int |g|^2\,d|\mu| \leq 1\right\} \leq \varepsilon_n.$$

Now apply Theorems 3 and 4 for $\varphi(z^n f, g) = \int z^n f\bar{g}\,d\mu$.

5. Distance between H_φ and the set of all compact sesquilinear forms.

THEOREM 5. *Let* H_φ *be a bounded Hankel form and* A *a compact (finite* n, *resp.) sesquilinear form on* $\mathscr{P}_+ \times \mathscr{P}_-$ *w.r.t.* (μ, ν). *If* $\|H_\varphi + A\| \leq \gamma$ *then there exists a symbol* ψ *such that* H_ψ *is a compact (finite* n, *resp.) Hankel form w.r.t.* (μ, ν) *and* $|||\varphi + \psi||| \leq \gamma$.

Proof. By the remark preceding Lemma 1, we can decompose $\varphi = \varphi_a + \varphi_s$ where H_{φ_a} is bounded w.r.t. (μ_a, ν_a) and H_{φ_s} is bounded w.r.t. (μ_s, ν_s). If $\|H_\varphi + A\| \leq \gamma$ then by Lemma 2 and the proof of Theorem 2 $|||\varphi_s||| \leq \gamma$ and $\|H_{\varphi_a} + A\| \leq \gamma$. Hence we may assume that $\varphi = \varphi_a$, $\mu = \mu_a = w_1\,dm$ and $\nu = \nu_a = w_2\,dm$. If $\log w_1 \notin L^2(m)$ or $\log w_2 \notin L^1(m)$, by Lemma 2 $A(f, g) = 0$ $(f \in \mathscr{P}_+, g \in \mathscr{P}_-)$ and hence Theorem 2 implies the theorem. By Lemma 1

$$|\varphi(h_1^{-1} f, \bar{h}_2^{-1} g) + A(h_1^{-1} f, h_2^{-1} g)|^2$$
$$\leq \gamma^2 \int |f|^2\,dm \int |g|^2\,dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-)$$

and there exists a null decreasing sequence $\{\gamma_n\}$ such that

$$|A(h_1^{-1} z^n f, h_2^{-1} g)|^2$$
$$\leq \gamma_n^2 \int |z^n f|^2\,dm \int |g|^2\,dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Hence there exist bounded linear operators H_l and $\mathscr{A}$ from $H^2(m)$ to $\bar{z}\overline{H}^2(m)$ such that

$$(H_l f, g) = (lf, g) = \varphi(h_1^{-1} f, h_2^{-1} g)$$

and

$$(\mathscr{A} f, g) = A(h_1^{-1} f, h_2^{-1} g)$$

where $l \in L^\infty(m)$ and $(\ ,\ )$ denotes the usual inner product with respect to m. Let U be a unilateral shift on H^2; then $\|\mathscr{A} U^n\| \to 0$ because $\gamma_n \to 0$. By the same argument as in **[10**, p. 6], there exists a function $k \in H^\infty + C$ such that $\|l + k\|_\infty < 1$. Similarly to the proof of Theorem 2 put

$$\psi(u, v) = \int (kh_1 h_2) u\bar{v}\, dm \qquad (u, v \in \mathscr{P}).$$

Then ψ is a bounded Hankel form w.r.t. $(w_1\, dm, w_2\, dm)$ and by Theorem 4 H_ψ is compact. Thus $|||\varphi + \psi||| \leq \gamma$.

Theorem 5 implies that $\inf\{\|H_\varphi + A\| : A$ ranges over all compact sesquilinear forms$\}$ $=$ $\inf\{|||\varphi + \psi||| : H_\psi$ ranges over all compact Hankel forms$\}$. When $d\mu = d\nu = dm$, this relates a theorem of Adamjan, Arov and Krein (cf. **[1]**, **[15**, p. 6]). However the former does not imply the latter (see Remark).

6. Lifting theorem. In this section we obtain a new lifting theorem which contains one due to Cotlar and Sadosky **[2]**. Let A_{ij} $(i, j = 1, 2)$ be bilinear forms on $\mathscr{P} \times \mathscr{P}$ and suppose

$$A_{11}(u, u) \geq 0, \qquad A_{22}(u, u) \geq 0 \quad \text{and} \quad A_{12}(u, v) = \overline{A_{21}(u, v)}.$$

Set

$$\mathbf{A}(\mathbf{u}, \mathbf{u}) = \sum_{i,j=1}^{2} A_{ij}(u_i, u_j)$$

where $\mathbf{u} = (u_1, u_2)$ and $u_i \in \mathscr{P}$ for $i = 1, 2$. We write $\mathbf{A} = [A_{ij}]$. If ρ_{ij} $(i, j = 1, 2)$ are finite Borel measures on T and

$$A_{ij}(u, v) = \int u\bar{v}\, d\rho_{ij} \qquad (u \in \mathscr{P}_+, v \in \mathscr{P}_-),$$

then A_{ij} $(i, j = 1, 2)$ are bounded Hankel forms on $\mathscr{P} \times \mathscr{P}$ w.r.t. $(|\rho_{ij}|, |\rho_{ij}|)$. By the hypothesis on $[A_{ij}]$

$$\rho_{11} \geq 0, \quad \rho_{22} \geq 0 \quad \text{and} \quad \rho_{12} = \bar{\rho}_{21}.$$

We write $\mathbf{A} = [A_{ij}] = [\rho_{ij}] = \rho$ and we call $\boldsymbol{\rho}$ a matrix of measures. $\mathbf{A} \succ 0$ w.r.t. Γ means that $\mathbf{A}$ is positive w.r.t. Γ:

$$\mathbf{A}(\mathbf{u}, \mathbf{u}) = \sum_{i,j=1}^{2} A_{ij}(u_i, u_j) \geq 0 \qquad (\mathbf{u} \in \Gamma)$$

where Γ denotes $\mathscr{P} \times \mathscr{P}$ or $\mathscr{P}_+ \times \mathscr{P}_-$.

We say that $\mathbf{A}$ is compact (finite n, resp.) w.r.t. ρ if $A_{11} = A_{22} = 0$ and A_{12} is compact (finite n) w.r.t. (ρ_{11}, ρ_{22}).

THEOREM 6. *Let* $\boldsymbol{\rho}$ *be a matrix of measures. If*

$$\boldsymbol{\rho} + \mathbf{A} \succ 0 \quad w.r.t.\ \mathscr{P}_+ \times \mathscr{P}_-$$

where $\mathbf{A}$ *is compact (finite* n, *resp.) w.r.t.* $\boldsymbol{\rho}$, *then there exists a compact (finite* n, *resp.) matrix* $\boldsymbol{\tau}$ *of measures w.r.t.* $\boldsymbol{\rho}$ *such that*

$$\boldsymbol{\rho} + \boldsymbol{\tau} \succ 0 \quad w.r.t.\ \mathscr{P} \times \mathscr{P}.$$

Proof. Let

$$\varphi_{12}(f, g) = \int f\overline{g}\, d\rho_{12} \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Then $\varphi_{12} + A_{12}$ is a bounded bilinear form on $\mathscr{P}_+ \times \mathscr{P}_-$ w.r.t. (ρ_{11}, ρ_{22}) because $\boldsymbol{\rho} + \mathbf{A} \succ 0$. Let $\|\varphi_{12} + A_{12}\| \leq \gamma$. By Theorem 5, there exists a symbol ψ such that H_ψ is a compact (finite n, resp.) w.r.t. (ρ_{11}, ρ_{22}) and $|||\varphi_{12} + \psi||| \leq \gamma$. By Theorems 3 and 4, there exists a function h in L^1 such that

$$\psi(f, g) = \int f\overline{g}h\, dm \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Then $d\tau_{12} = h\, dm$ is the desired measure.

COROLLARY 3 (Cotlar and Sadosky). *Let* $\boldsymbol{\rho}$ *be a matrix of measures. If*

$$\boldsymbol{\rho} \succ 0 \quad w.r.t.\ \mathscr{P}_+ \times \mathscr{P}_-$$

then there exists a finite $n = 0$ *matrix* τ *of measures such that*

$$\boldsymbol{\rho} + \boldsymbol{\tau} \succ 0 \quad w.r.t.\ \mathscr{P} \times \mathscr{P}.$$

By Theorems 3 *and* 4, *we can describe compact (finite* n, *resp.) matrices of measures w.r.t.* ρ.

7. Weighted norm inequalities. In this section we show known results in the L^2 weighted problem, using the theorems of §§3, 4 and 5. For any fixed nonnegative integer n, we want to find the positive measure μ for which there is a nonzero positive measure ν_n such that

$$\int |z^n f|^2\, d\nu_n \leq \int |z^n f + g|^2\, d\mu \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

The inequality above is equivalent to the following one:

$$\left|\int z^n f\overline{g}\, d\mu\right|^2 \leq \int |f|^2 d(\mu - \nu_n) \int |g|^2\, d\mu \qquad (f \in \mathscr{P}_+, g \in \mathscr{P}_-).$$

Hence the problem is related with prediction problems when such a measure μ arises as the spectral density of a discrete weakly stationary Gaussian stochastic process. The following proposition is due to Arocena, Cotlar and Sadosky [**3**]. The Helson-Szegö theorem [**10**] and the Koosis theorem [**12**] follow from the first part in it.

PROPOSITION 7. *Let μ be a positive measure. There is a nonzero positive measure ν such that*

$$\int |f|^2\, d\nu \le \int |f+g|^2\, d\mu \qquad (f \in \mathscr{P}_+,\ g \in \mathscr{P}_-)$$

if and only if $d\nu = u\, dm$ and there is a nonzero k in H^1 such that

$$|w+k|^2 \le (w-u)w$$

where $d\mu = w\, dm + d\mu_s$. Then if $\log(w-u)$ is in L^1 then $u \le (1-\gamma^{-1})w$ and $\gamma > 1$.

We can prove Proposition 7 using the lifting theorem of Cotlar and Sadosky (Theorem 2 or Corollary 3) as that in [**3**]. The following theorem is closely related to results in [**3**]. We will give a proof using Theorems 3 and 4.

THEOREM 8. *Let μ be a positive measure. For any fixed nonnegative integer n, let ν_n be a nonzero positive measure such that*

$$\int |z^n f|^2\, d\nu_n \le \int |z^n f + g|^2\, d\mu \qquad (f \in \mathscr{P}_+,\ g \in \mathscr{P}_-).$$

Suppose that there exists a positive measure λ and a decreasing sequence $\{\varepsilon_n\}$ such that $\nu_n = \mu - \varepsilon_n \lambda$ and $0 \le \varepsilon_n \le 1$.

(1) *$\varepsilon_n = 0$ for some n if and only if $d\nu_n = d\mu = w\, dm$ and $w = sh$ where h is an outer function with $w = |h|$ and s is in $\overline{z}^n H^\infty$.*

(2) *$\varepsilon_n \to 0$ as $n \to \infty$ if and only if $d\nu_n = (w_1 - \varepsilon_n w_2)\, dm$, $d\mu = w_1\, dm$, $d\lambda = w_2\, dm + d\lambda_s$ and $w_1 = s h_1 h_2$ where h_j is an outer function with $w_j = |h_j|^2$ for $j = 1, 2$ and s is in $H^\infty + C$.*

Proof. Set

$$\varphi(u, v) = \int u \overline{v}\, d\mu \qquad (u, v \in \mathscr{P});$$

then by the remark before Theorem 7 H_φ is finite n and compact w.r.t. (λ, μ) for (1) and (2), respectively. (1) follows from (2) of Theorem 3. For if $\varepsilon_n = 0$ for some n then $w \in \overline{z}^n H^1$ and hence $w = |h| = \overline{z}^n q h$ where q is in H^∞. (2) follows from Theorem 4.

In Theorem 8, if $\lambda = \mu$ this was proved by Helson and Sarason **[10]**. Theorem 8 is also a corollary of Theorem 6 which is a new lifting theorem.

REMARK. Hankel operators from $H^2(\mu)$ to $\overline{zH}^2(\nu)$. Let μ and ν be finite positive Borel measures on T. M_z^μ and M_z^ν are multiplication operators by the coordinate function z on $L^2(\mu)$ and $L^2(\nu)$, respectively. Let $\mathbf{\Phi}$ be a bounded linear operator from $L^2(\mu)$ to $L^2(\nu)$ and $(\mathbf{\Phi}u, v) = \int(\mathbf{\Phi}u)\overline{v}\,d\nu$ for u, v in $\mathscr{P}$. Then $\mathbf{\Phi}M_z^\mu = M_z^\nu\mathbf{\Phi}$ if and only if $\varphi(u, v) = (\mathbf{\Phi}u, v)$ is a bounded Hankel form on $\mathscr{P} \times \mathscr{P}$ w.r.t. (μ, ν). Let P and Q be the orthogonal projections from $L^2(\mu)$ to $H^2(\mu)$ and from $L^2(\nu)$ to $\overline{zH}^2(\nu)$, respectively. Put $H = Q\mathbf{\Phi}P$; then $(Hf, g) = H_\varphi(f, g)$ for f in $\mathscr{P}_+$ and g in $\mathscr{P}_-$. Put $S_z^\mu = PM_z^\mu|H^2(\mu)$ and $S_{\overline{z}}^\nu = QM_{\overline{z}}^\nu|\overline{zH}^2(\nu)$; then $HS_z^\mu = (S_{\overline{z}}^\nu)^*H$. Theorem 2 calculates the norm of H. In general, even if H is a compact linear operator, H_φ may not be a compact sesquilinear form.

When $\mu = \nu = m$, $\mathbf{\Phi}$ is a multiplication operator M_Φ by a function Φ in $L^\infty(m)$ and $\|\mathbf{\Phi}\| = \|\Phi\|_\infty = |||\varphi|||$. H is called a Hankel operator and $\|H\| = \|H_\varphi\|$. H_φ is a compact Hankel form if and only if H is a compact Hankel operator. For by Theorem 4 H_φ is compact if and only if $\varphi(f, g) = \int f\overline{g}h\,dm$ $(f \in \mathscr{P}_+,\ f \in \mathscr{P}_-)$ and $h \in H^\infty + C$. By Hartman's theorem (cf. **[15**, Theorem 1.4**]**) H is compact if and only if $\Phi \in H^\infty + C$. Moreover the essential norm $\|H\|_e$ of H coincides with $\inf\{\|H_\varphi + A\|$: A ranges over all compact sesquilinear forms$\}$. For by a theorem of Adamjan, Arov and Krein **[1]**, $\|H\|_e = \|\Phi + H^\infty + C\|$. While by Theorems 4 and 5 $\inf\|H_\varphi + A\| = \inf\{|||\varphi + \psi|||$: H_ψ ranges over all compact Hankel forms$\} = \|h + H^\infty + C\|$ where $\varphi(f, g) = \int f\overline{g}h\,dm$.

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