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CURVATURE CHARACTERIZATION AND CLASSIFICATION OF RANK-ONE SYMMETRIC SPACES

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We characterize and classify rank-one symmetric spaces by two axioms on $R(\cdot, v)v$, the Jacobi part of the curvature tensor.

1. Introduction. In his book [3], Chavel gave a beautiful account of the rank-one symmetric spaces from a geometric point of view up to the classification of them, which he left for the reader to pursue as a matter in Lie group theory. The purpose of this paper is to extend Chavel's approach to fill in this last step by classifying these spaces, on the Lie algebra level, based on geometric considerations. To be more precise, for each unit vector v, define the Jacobi operator $K_v = R(\cdot, v)v$, where $R(X, Y)Z$ denotes the curvature tensor. Then for a compact rank-one symmetric space one notes that (1) K_v have two distinct constant eigenvalues (1 & 1/4) for all v if the space is not of constant curvature, and (2) $E_1(v)$, the linear space spanned by v and the eigenspace of K_v with eigenvalue 1 is the tangent space of a totally geodesic sphere of curvature 1 (a projective line in fact) through the base point of v, and consequently $E_1(w) = E_1(v)$ whenever w is in $E_1(v)$. These two properties will be adapted in the next section as two axioms, and we will prove then that they turn out to characterize locally rank-one symmetric spaces. Indeed, motivated by [5] and [8], we prove that the curvature tensor, under the two axioms, induces a certain Clifford module, from which the curvature components and the dimension of the space can be read off. It then follows that the space must be locally rank-one symmetric, and the list of such spaces falls out in a natural way.

We would like to mention that there is another interesting geometric classification of the compact symmetric spaces by Karcher [11]. Karcher's construction of the Cayley plane rests on some intriguing properties of isoparametric submanifolds and he has to assume the space is symmetric of positive curvature for the classification, whereas all our results follow from the the two axioms and the technique requires essentially no more than linear algebra. Our analysis reveals by fairly simple constructions why such spaces are closely related to the four standard algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and the Cayley algebra Ca.

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2. A curvature characterization of locally rank-one symmetric spaces. We start out with two axioms.

Axiom 1. Let $K_v(\cdot) = R(\cdot, v)v$ for v in SM, the unit sphere bundle of M. Then K_v has precisely two different constant eigenvalues independent of v (counting multiplicities).

Axiom 2. Let b, c be the two eigenvalues. For $v \in SM$, denote by $E_c(v)$ the span of v and the eigenspace of K_v with eigenvalue c. Then $E_c(w) = E_c(v)$ whenever $w \in E_c(v)$.

REMARK. 1. Axiom 2 is redundant if $\dim E_c(v) = 2$.

2. It's not hard to show (cf. [**3**]) that a two-point homogeneous space of constant curvature must be $\mathbb{R}P^n$ or the space forms.

THEOREM 1. *Locally rank-one symmetric spaces not of constant curvature are characterized by the two axioms.*

We need a few lemmas to complete the proof.

LEMMA 1. *If y is perpendicular to $E_c(v)$, then so is $E_c(y)$. In particular M_p, the tangent space at p, can be decomposed into perpendicular subspaces of the form $E_c(v)$.*

Proof. Since y is perpendicular to $E_c(v)$ and since $E_c(w) = E_c(v)$ for all unit w in $E_c(v)$ by the second axiom above, one sees that $K_w(y) = by$ for all such w, which in turn implies $K_y(w) = bw$; hence $E_c(y)$ is perpendicular to $E_c(v)$. □

LEMMA 2. *Let*

$$M_p = E_c(x) \oplus E_c(y) \oplus E_c(z) \oplus \cdots .$$

Then

(i) $R(y, x_1)x_2 = -R(y, x_2)x_1$ for x_1, x_2 in $E_c(x)$.

(ii) $R(x, y)z = 0$.

(iii) $R(x_1, x_2)x_3 = 0$ if $x_1, x_2, x_3 \in E_c(x)$.

Proof. For (i), one considers $w = (x_1 + x_2)/\sqrt{2}$. One has $K_w(y) = by$ since w is in $E_c(x)$. This implies (i) by expanding

$$K_w(y) = R\left(y, \frac{x_1 + x_2}{\sqrt{2}}\right)\frac{x_1 + x_2}{\sqrt{2}}.$$

To prove (ii), let $w = (y+z)/\sqrt{2}$. One has $K_x(w) = bw$ since $K_x(y) = by$ and $K_x(z) = bz$; hence $K_w(x) = bx$, which gives, after expanding $K_w(x)$,

$$R(x, y)z = -R(x, z)y.$$

This same relation holds if we cyclically permute x, y, z. Now the first Bianchi identity finishes the proof. The proof of (iii) is similar to that of (ii). □

DEFINITION 1. The type number $\tau = \dim E_c(x) - 1$.

Clearly τ is well-defined since $\dim E_c(x)$ is constant for all unit x.

Given $x_0 \in SM$, choose $x_1, x_2, \ldots, x_\tau$ so that $x_0, x_1, \ldots, x_\tau$ form an orthonormal basis for $E_c(x_0)$. Define, for $1 \leq i \leq \tau$, $J_i\colon E_c(x_0)^\perp \to E_c(x_0)^\perp$ by

$$J_i(y) = R(x_0, x_i)y, \tag{1}$$

where $E_c(x_0)^\perp$ is the subspace perpendicular to $E_c(x_0)$. (That J_i sends $E_c(x)^\perp$ to itself follows from Lemma 2, (iii).)

LEMMA 3. *Given a fixed unit vector $y_0 \perp E_c(x_0)$, the map $R(x_0, \cdot)y_0\colon E_c(x_0) \to E_c(x_0)^\perp$ satisfies $\|R(x_0, x)y_0\| = 2/3|b-c|$ for unit x perpendicular to x_0.*

Proof. Let $w = (x_0 + y_0)/\sqrt{2}$, and let $M_p = E_c(x_0) \oplus E_c(y_0) \oplus E_c(z) \oplus \cdots$. Let (x_i) and (y_i), $i = 0, 1, \ldots, \tau$, be orthonormal bases for $E_c(x_0)$ and $E_c(y_0)$ respectively. It's directly checked, using Lemma

2, that

$$K_w:\begin{cases} x_0 \to \dfrac{b}{2}(x_0 - y_0) \\ x_i \to \dfrac{b+c}{2}x_i + \dfrac{3}{2}\displaystyle\sum_s \langle R(x_i, y_0)x_0, y_s\rangle y_s \\ y_0 \to -\dfrac{b}{2}(x_0 - y_0) \\ y_i \to \dfrac{b+c}{2}y_i + \dfrac{3}{2}\displaystyle\sum_s \langle R(y_i, x_0)y_0, x_s\rangle x_s \\ z \to bz. \end{cases}$$

This precisely says that the restriction of K_w to the subspace V generated by $\{x_i, y_i\}_{i=1}^{\tau}$ has b and c as eigenvalues with equal multiplicity τ by axioms 1 and 2.

When written in matrix form relative to $\{x_i, y_i\}_{i=1}^{\tau}$, K_w restricted to V assumes the form

$$\left(\begin{array}{c|c} \frac{b+c}{2}I & \frac{3}{2}A^t \\ \hline \frac{3}{2}A & \frac{b+c}{2}I \end{array}\right)$$

where $A_{ij} = \langle R(x_i, y_0)x_0, y_j\rangle$. Hence

$$K_w - cI = \left(\begin{array}{c|c} \frac{b-c}{2}I & \frac{3}{2}A^t \\ \hline \frac{3}{2}A & \frac{b-c}{2}I \end{array}\right),$$

so that the null space of $K_w - cI$ is

$$\left\{\begin{pmatrix} X \\ \hline Y \end{pmatrix} : \left(\begin{array}{c|c} \frac{b-c}{2}I & \frac{3}{2}A^t \\ \hline \frac{3}{2}A & \frac{b-c}{2}I \end{array}\right)\begin{pmatrix} X \\ \hline Y \end{pmatrix} = 0\right\}, \tag{2}$$

which has dimension τ since it is the eigenspace of K_w with eigenvalue c.

Now (2) is nothing but

$$\begin{cases} \dfrac{b-c}{2}x + \dfrac{3}{2}A^t y = 0, \\ \dfrac{3}{2}Ax + \dfrac{b-c}{2}y = 0. \end{cases}$$

It follows that $y = \frac{3}{b-c}Ax$ and $A^tAx = (\frac{b-c}{3})^2 x$. Since the null space under consideration has dimension τ and x and y are $\tau \times 1$

matrices, one concludes that $A^tAx = (\frac{b-c}{3})^2 x$ for all $\tau \times 1$ matrices x perpendicular to x_0, meaning $\|R(x, y_0)x_0\| = \frac{|b-c|}{3}$. Now the lemma follows by noting that $R(x_0, x)y_0 = -2R(x, y_0)x_0$ by Lemma 2. $\square$

From now on we normalize the metric so that

$$|b - c| = \frac{3}{2}. \tag{3}$$

PROPOSITION 1. *The operators* $J_i\colon E_c(x_0)^\perp \to E_c(x_0)^\perp$ *defined in* (1) *satisfy*

(i) $J_i^2 = -I$,
(ii) $J_iJ_k = -J_kJ_i$ *for* $i \neq k$,
(iii) $J_iJ_k(y) \in \operatorname{Span}(J_1(y), \dots, J_\tau(y))$.

Proof. Let $M_p = E_c(x_0) \oplus E_c(y) \oplus E_c(z) \oplus \cdots$. By Lemma 2, $J_i\colon E_c(y) \to E_c(y)$, $J_i\colon E_c(z) \to E_c(z)$ etc.; these restrictions of J_i are orthogonal by Lemma 3 and (3). On the other hand J_i is skew-symmetric since the curvature tensor is; hence $(J_i)^2 = J_iJ_i = -J_i^tJ_i = -I$ on $E_c(y)$, and so $(J_i)^2 = -I$ on $E_c(x_0)^\perp$ since $E_c(x_0)^\perp$ is the direct sum of subspaces of the form $E_c(y)$, proving (i). To prove (ii) first note that $\langle J_iJ_k(y), y\rangle = -\langle J_i(y), J_k(y)\rangle = 0$ if $i \neq k$ by Lemma 3. It then follows that $\langle J_iJ_k(v), v\rangle = 0$ for all $v \in E_c^\perp(x_0)$; hence setting $v = y+z$, one has $\langle J_iJ_k(y), z\rangle = -\langle J_iJ_k(z), y\rangle = -\langle z, J_kJ_i(y)\rangle$, which is (ii). (iii) is clear from Lemma 2. $\square$

PROPOSITION 2. *The type number* $\tau = 1, 3$, *or* 7. *If* $\tau = 7$, $\dim M = 16$.

Proof. Notation is as in Proposition 1. Let $M_p = E_c(x_0) \oplus E_c(y) \oplus E_c(z) \oplus \cdots$. Proposition 1 implies $E_c(y)$, which has dimension $\tau + 1$, is a Clifford C_τ module induced by the operators J_i. Hence $\tau = 1, 3$, or 7 (cf. **[10]**).

To prove the second statement, first note that $\dim M = (\tau + 1)s$ for some s since $M_p = E_c(x_0) \oplus E_c(y) \oplus \cdots$.

SUBLEMMA. *If* $s \geq 3$, *then* $\langle J_iJ_kJ_l(y), y\rangle = \langle J_iJ_kJ_l(z), z\rangle$ *for all* y, z *in* $E_c(x_0)^\perp$.

Proof of Sublemma. Since $s \geq 3$, $M_p = E_c(x_0) \oplus E_c(y_0) \oplus E_c(z_0) \oplus \cdots$. Since $K_y(z) = bz$ and $K_z(y) = by$ for $y \in E_c(y_0)$ and

$z \in E_c(z_0)$, we have

$$K_{(y-z)/\sqrt{2}}(y+z) = b(y+z);$$

therefore $\langle J_i J_k J_l(y-z), y+z\rangle = 0$ by (iii) of Proposition 1. Similarly, $0 = \langle J_i J_k J_l(y), z\rangle = \langle J_i J_k J_l(z), y\rangle$ for the same reason. In particular we have

$$0 = \langle J_i J_k J_l(y-z), y+z\rangle = \langle J_i J_k J_l(y), y\rangle - \langle J_i J_k J_l(z), z\rangle,$$

so that $\langle J_i J_k J_l(y), y\rangle = \langle J_i J_k J_l(z), z\rangle$ for all z perpendicular to $E_c(x_0) \oplus E_c(y_0)$, and all y in $E_c(y_0)$ as well, proving the sublemma.

Now fix $y_0 \in E_c(x_0)^{\perp}$ and let $y_i = J_i(y_0)$ so that $y_0, y_1, \ldots, y_\tau$ form an orthonormal basis for $E_c(y_0)$. Define a product "$\cdot$" on $E_c(y_0)$ so that $E_c(y_0)$ under this product is isomorphic to one of the three algebras $\mathbb{C}$, $\mathbb{H}$, and the nonassociative Cayley algebra Ca corresponding to the type number $\tau = 1, 3, 7$ respectively (cf. **[13]**), namely, $y_0 \cdot y_i = y_i$, and $y_i \cdot y_k = J_i(y_k)$ for $i \neq k$. Now let $J_i J_k(y_0) = \sum_s a_{ik}^s J_s(y_0)$. Then $a_{ik}^s = -\langle J_s J_i J_k(y_0), y_0\rangle$, which is a constant by the above sublemma. It follows that $J_i J_k(y) = \sum_s a_{ik}^s J_s(y)$ for all y in $E_c(y_0)$, and hence $(y_i \cdot y_k) \cdot y_l = y_i \cdot (y_k \cdot y_l)$, so that $E_c(y_0)$ under the product is associative, and therefore $\tau \neq 7$. In other words, if $\tau = 7$ then $s = 2$, and so $\dim M = (\tau+1)s = 16$. □

We now make a convention that if $0 \leq \alpha$, $\beta \leq \tau$, we denote by $\alpha\beta$ the number $\pm\gamma$ such that $\pm e_\gamma = e_\alpha e_\beta$, where e_i's are the basis elements in the standard multiplication tables for the three algebras above (cf. **[13]**). Also denote $X_{-\alpha} = -X_\alpha$ for vectors with subscripts.

LEMMA 4. *In a neighborhood U of each point p, given a unit vector field x_0, one can pick an orthonormal frame $x_0, x_1, \ldots, x_\tau$; $y_0, y_1, \ldots, y_\tau$; $z_0, \ldots$ such that for $q \in U$, one has $M_q = E_c(x_0) \oplus E_c(y_0) \oplus E_c(z_0) \cdots$ with $x_i \in E_c(x_0)$, $y_i \in E_c(y_0)$, etc., such that $R(x_0, x_\alpha)y_\beta = -y_{\beta\alpha}$, $R(x_0, x_\alpha)z_\beta = -z_{\beta\alpha}$, etc.*

Proof. Recall that τ is the type number of the space. That $M_q = E_c(x_0) \oplus (y_0) \oplus \cdots$ follows from Lemma 1. Pick smooth fields $x_1, \ldots, x_\tau$ such that $x_0, x_1, \ldots, x_\tau$ form an orthonormal basis for $E_c(x_0)$. Define for $1 \leq i \leq \tau$, $-y_j = R(x_0, x_j)y_0$, $-z_j = R(x_0, x_j)z_0$, etc. For $\tau = 1$, it is readily checked that $R(x_0, x_\alpha)y_\beta = -y_{\beta\alpha}$, etc. For $\tau = 3$, let $\dim M = (\tau+1)s = 4s$. If $s \geq 3$, then by the sublemma in Proposition 2 $\mathrm{span}(\mathrm{Id}, J_1, J_2, J_3) = \mathbb{H}$, where $J_i = R(x_0, x_i)$. It follows easily that $J_1 \cdot J_2 = \pm J_3$. Changing x_0 into $-x_0$ if necessary, one may always assume $J_1 \cdot J_2 = J_3$; in other words, $R(x_0, x_\alpha) \cdot R(x_0, x_\beta) = R(x_0, x_{\alpha\beta})$ if $1 \le \alpha, \beta \le 3$, i.e., $R(x_0, x_\alpha)y_\beta = y_{\alpha\beta} = -y_{\beta\alpha}$ etc., which is the conclusion. On the other hand if $\tau = 3$ and $\dim M = 8$, then $M_q = E_c(x_0) \oplus E_c(y_0)$. Now $E_c(y_0)$ is isomorphic to $\mathbb{H}$ under the product $y_i \cdot y_j = J_i(y_j) = J_iJ_j(y_0)$. Therefore again it is easy to see that $y_1 \cdot y_2 = \pm y_3$, i.e., $J_1 \cdot J_2(y_0) = \pm J_3(y_0)$. We may assume $J_1J_2(y_0) = J_3(y_0)$ by changing x_0 into $-x_0$, so that once more $R(x_0, x_\alpha)y_\beta = -y_{\beta\alpha}$.

If $\tau = 7$, then $\dim M = 16$, and $M_q = E_c(x_0) \oplus E_c(y_0)$. Although $E_c(y_0)$ is isomorphic to the Cayley algebra Ca under the product $y_i \cdot y_j = J_iJ_j(y_0)$, it might happen that y_i's are not the standard basis elements for Ca in general. However one observes that $y_0, y_1, y_2, y_1 \cdot y_2$ form a standard basis for $\mathbb{H}$, and by picking a smooth unit vector field w perpendicular to them one verifies that $y_0, y_1, y_2, y_1 \cdot y_2, w \cdot y_0, w \cdot y_1, w \cdot y_2, w \cdot (y_1 \cdot y_2)$ form a standard basis; let's call this new basis $(v_0 = y_0, v_1, \dots, v_7)$ so that $v_\alpha \cdot v_\beta = v_{\alpha\beta}$. Let $v_i = \sum_{j=0}^{7} a_{ji}y_j$. Define $\overline{x}_i = \sum_{j=0}^{7} A_{ji}x_j$. A straightforward computation gives $R(x_0, \overline{x}_\alpha)R(x_0, \overline{x}_\beta)y_0 = R(x_0, \overline{x}_{\alpha\beta})y_0$ for $\alpha, \beta \ne 0$. Now let $\overline{y}_\alpha = R(x_0, \overline{x}_\alpha)y_0$, one has $R(x_0, \overline{x}_\alpha)\overline{y}_\beta = \overline{y}_{\alpha\beta} = -\overline{y}_{\beta\alpha}$, proving the lemma. □

LEMMA 5. *Let M be a Riemannian manifold. If for every geodesic $r(t)$ the operator K_v with $v = \dot{r}(t)$ is parallel, then the entire curvature tensor is parallel along $r(t)$ and M is locally symmetric.*

Proof. See [2]. □

LEMMA 6. *Assume the same conditions as in Lemma 4. Then*

$$\begin{aligned}(\nabla_{x_0}R)(x_i, x_0, x_0, x_j) &= (\nabla_{x_0}R)(y_i, x_0, x_0, y_j)\\ &= (\nabla_{x_0}R)(z_i, x_0, x_0, z_j)\\ &= (\nabla_{x_0}R)(y_i, x_0, x_0, z_j) = 0,\end{aligned}$$

etc., where $R(x, y, z, w) = \langle R(x, y)z, w\rangle$, provided x_0 is tangent to the geodesics emanating from p.

Proof. We'll prove $(\nabla_{x_0}R)(y_i, x_0, x_0, z_j) = 0$. The proof of the others are similar. Since $R(y_i, x_0, x_0, z_j) = 0$ by Lemma 2 and since

$\nabla_{x_0} x_0 = 0$ by the way x_0 is chosen, one sees that

$$\begin{aligned}
&(\nabla_{x_0} R)(y_i, x_0, x_0, z_j) \\
&\quad = -R(\nabla_{x_0} y_i, x_0, x_0, z_j) - R(y_i, x_0, x_0, \nabla_{x_0} z_j) \\
&\quad = -\langle \nabla_{x_0} y_i, z_j \rangle R(z_j, x_0, x_0, z_j) \\
&\quad\quad - \langle y_i, \nabla_{x_0} z_j \rangle R(y_i, x_0, x_0, y_i) = 0,
\end{aligned}$$

in view of Lemma 2 and $K_{x_0}(x_i) = cx_i$, $K_{x_0}(y_i) = by_i$, $K_{x_0}(z_i) = bz_i$, etc. □

Now we prove the promised characterization theorem.

Proof of Theorem 1. Here in this proof x_0 as in Lemma 6 is tangent to the geodesics emanating from $p \in U$. Let $x_0, x_1, \dots, x_\tau$; $y_0, y_1, \dots, y_\tau$; $z_0, \dots$ be as in the previous lemma. We only have to show that

$$(\nabla_{x_0} R)(x_i, x_0, x_0, y_j) = 0,$$

in view of Lemmas 5 and 6. In fact it suffices to check

$$(\nabla_{x_0} R)(x_i, x_0, x_0, y_0) = 0,$$

since $R(x_0, x_\alpha) y_\beta = -y_{\beta\alpha}$ says that one may rename the y_i's so that y_β becomes y_0. Now a direct computation using $R(x_0, x_\alpha) y_\beta = -y_{\beta\alpha}$, (i) of Lemma 2 and the first Bianchi identity, and (3) gives

$$(\nabla R)(x_i, x_0, x_0, y_0) = \pm(\nabla R)(y_i, y_0, y_0, x_0),$$

depending on whether $c > b$ or $c < b$. In particular

$$(\nabla_{x_0} R)(x_i, x_0, x_0, y_0) = \pm(\nabla_{x_0} R)(y_i, y_0, y_0, x_0).$$

Now the second Bianchi identity says

$$\begin{aligned}
(\nabla_{x_0} R)(y_i, y_0, y_0, x_0) &= (\nabla_{x_0} R)(y_0, x_0, y_i, y_0) \\
&= -(\nabla_{y_i} R)(y_0, x_0, y_0, x_0) - (\nabla_{y_0} R)(y_0, x_0, x_0, y_i) = 0,
\end{aligned}$$

in view of the proof of Lemma 6. Hence $(\nabla_{x_0} R)(x_i, x_0, x_0, y_0) = 0$, and the space is locally symmetric; therefore it must be of rank one by the constancy of eigenvalues of the Jacobi operator K_v. □

3. Lie algebra classification of rank-one symmetric spaces. Our analysis has been based primarily on Lemma 4, where the curvature structure of the space under consideration was partially displayed. Using Lemma 2, Lemma 7 and Corollary 1 in this section, which give explicitly all the curvature components in their full generality, it will then be a straightforward matter, with the aid of the fact that the Lie algebra of the isotropy group of a symmetric space is the linear span of all $R(x, y)$ at the origin (cf. **[9]**), to write down the Lie algebra structure of the space to see that a compact symmetric space of rank one must necessarily be either a sphere or one of the projective spaces. We shall thus be brief, and leave the details to the reader.

Let $M_p = E_c(x_0) \oplus E_c(y_0) \oplus E_c(z_0) \oplus \cdots$ as in Lemma 4 so that

$$R(x_0, x_\alpha)y_\beta = -y_{\beta\alpha}. \tag{4}$$

By the symmetry of the curvature tensor one has

$$R(y_\alpha, y_\beta)x_0 = -x_{\beta\alpha}. \tag{5}$$

LEMMA 7. *Assume* $c > b$. *Then* $R(x_\alpha, x_\beta)y_\gamma = -y_{(\gamma\beta)\alpha}$, *where* $\alpha \neq \beta$ *and* $\beta \neq 0$.

Proof. We may assume $\alpha \neq 0$ by (4). Fix δ and let $w = (x_0 + y_\delta)/\sqrt{2}$. Then

$$\begin{aligned} K_w(x_\beta \pm y_{\delta\beta}) &= R(x_\beta \pm y_{\delta\beta}, w)w \\ &= \tfrac{1}{2}[cx_\beta \pm by_{\delta\beta} + bx_\beta \pm cy_{\delta\beta} + R(x_\beta, x_0)y_\delta \\ &\qquad \pm R(y_{\delta\beta}, x_0)y_\delta + R(x_\beta, y_\delta)x_0 \pm R(y_{\delta\beta}, y_\delta)x_0]. \end{aligned} \tag{6}$$

Now $R(x_\beta, x_0)y_\delta = y_{\delta\beta}$ by (4). And by the first Bianchi identity

$$\begin{aligned} y_{\delta\beta} &= R(x_\beta, x_0)y_\delta \\ &= R(x_\beta, y_\delta)x_0 - R(x_0, y_\delta)x_\beta \\ &= 2R(x_\beta, y_\delta)x_0, \end{aligned}$$

by (i) of Lemma 2. Similarly $R(y_{\delta\beta}, y_\delta)x_0 = x_\beta$ by (5) and $R(y_{\delta\beta}, x_0)y_\delta = x_\beta/2$ by (i) of Lemma 2 and the Bianchi identity. Hence the right-hand side of (6) is

$$\begin{aligned} &\frac{(b+c)}{2}x_\beta \pm \frac{3}{4}x_\beta \pm \frac{(b+c)}{2}y_{\delta\beta} + \frac{3}{4}y_{\delta\beta} \\ &\qquad = \frac{(b+c) \pm |b-c|}{2}x_\beta + \frac{|b-c| \pm (b+c)}{2}y_{\delta\beta}, \end{aligned}$$

by (3). By assumption $c > b$, therefore

$$K_w(x_\beta + y_{\delta\beta}) = c(x_\beta + y_{\delta\beta}), \tag{7}$$

$$K_w(x_\beta - y_{\delta\beta}) = b(x_\beta - y_{\delta\beta}). \tag{8}$$

Replacing β by α in (8) one gets

$$K_w(x_\alpha - y_{\delta\alpha}) = b(x_\alpha - y_{\delta\alpha}). \tag{9}$$

Now Axiom 1, (7), and (9) imply

$$K_{(x_\beta+y_{\delta\beta})/\sqrt{2}}(x_\alpha - y_{\delta\alpha}) = b(x_\alpha - y_{\delta\alpha}),$$

or equivalently,

$$(10)\qquad b(x_\alpha - y_{\delta\alpha}) = \frac{b+c}{2}x_\alpha - \frac{b+c}{2}y_{\delta\alpha} + \frac{3}{4}R(x_\alpha, x_\beta)y_{\delta\beta} - \frac{3}{4}R(y_{\delta\alpha}, y_{\delta\beta})x_\beta.$$

On the other hand, $R(x_\alpha, x_\beta)y_{\delta\beta} \in E_c(y_0)$ and $R(y_{\delta\alpha}, y_{\delta\beta})x_\beta \in E_c(x_0)$ by (ii) of Lemma 2. Hence comparing both sides of (10) one concludes

$$\frac{3}{4}R(x_\alpha, x_\beta)y_{\delta\beta} = \frac{b+c}{2}y_{\delta\alpha} - by_{\delta\alpha} = \frac{c-b}{2}y_{\delta\alpha},$$

or

$$R(x_\alpha, x_\beta)y_{\delta\beta} = \frac{2}{3}(c-b)y_{\delta\alpha} = y_{\delta\alpha}$$

by (3). Let $\gamma = \delta\beta$. Then $\gamma\beta = (\delta\beta)\beta = -\delta$, so that $\delta\alpha = -(\gamma\beta)\alpha$, i.e.,

$$R(x_\alpha, x_\beta)y_\gamma = -y_{(\gamma\beta)\alpha}. \qquad \square$$

COROLLARY 1. *Assume $c > b$. Let $M_p = E_c(x_0) \oplus E_c(y_0) \oplus E_c(z_0) \oplus \cdots$, and $x_i, y_i, z_i \cdots$ be as in Lemma 4. Then $R(y_\alpha, y_\beta)z_\gamma = -z_{(\gamma\beta)\alpha}$, etc.*

Proof. It suffices to show that $R(y_0, y_\alpha)z_\beta = -z_{\beta\alpha}$ in view of Lemma 7. A direct computation shows

$$K_{(x_0+y_0)/\sqrt{2}}(x_\alpha - y_\alpha) = b(x_\alpha - y_\alpha),$$

if $c > b$. Hence by (ii) of Lemma 2

$$\begin{aligned} 0 &= R(x_0 + y_0, x_\alpha - y_\alpha)z_\beta \\ &= R(x_0, x_\alpha)z_\beta - R(y_0, y_\alpha)z_\beta, \end{aligned}$$

i.e.,

$$\begin{aligned} R(y_0, y_\alpha)z_\beta &= R(x_0, x_\alpha)z_\beta \\ &= -z_{\beta\alpha} \end{aligned}$$

by Lemma 4. $\square$

Now it is well known that the symmetric Lie algebra decomposition of the underlying space is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the tangent space at the origin, and $\mathfrak{k}$, the isotropy algebra, is the linear span of $R(X, Y)$

for all X, Y in $\mathscr{m}$ with the natural Lie algebra structure. It follows easily from Lemma 2, Lemma 7, and Corollary 1 that if the type number $\tau = 1$, or 3, then $R(u, v)$ corresponds to the Lie bracket of the matrices

$$\left(\begin{array}{c|c} 0 & u \\ \hline -u^* & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|c} 0 & v \\ \hline -v^* & 0 \end{array}\right),$$

where u and v are regarded as column vectors over $\mathbb{C}$ and $\mathbb{H}$ respectively, and u^* and v^* their conjugates. Thus the symmetric Lie pair must be either $(u(n+1), u(n)\times u(1))$ or $(\mathrm{sp}(n+1), \mathrm{sp}(n)\times\mathrm{sp}(1))$. On the other hand let the type number $\tau = 7$ so that $M_p = E_c(x_0)\oplus E_c(y_0)$ with $\dim M_p = 16$ corresponding to the Cayley algebra Ca. Recall that if we denote by $e_0, e_1, \ldots, e_8$ the generator of C_9, the Clifford algebra of rank 9, then the Lie algebra of Spin(9) is linearly spanned by e_0e_i and e_ie_j for all $1 \le i < j \le 8$. In view of this it is also easy to see that e_0e_i and e_ie_j correspond to $R(x_0, y_i)$ and $R(x_0, y_i)R(x_0, y_j)$ respectively, so that the isotropy algebra of the space is $o(9)$ and Lemma 7 gives explicitly the irreducible representation of $o(9)$ on $\mathbb{R}^{16}$. Let

$$\begin{aligned}
\mathscr{D}_0 &= \mathrm{Span}(R(x_0, y_i)R(x_0, y_j) | 1 \le i < j \le 8), \\
\mathscr{D}_1 &= \mathrm{Span}(R(x_0, y_i) | 1 \le i \le 8), \\
\mathscr{D}_2 &= E_c(x_0), \\
\mathscr{D}_3 &= E_c(y_0).
\end{aligned}$$

Then $\mathscr{g} = \mathscr{D}_0 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3$ has the property that $[\mathscr{D}_0, \mathscr{D}_i] \subset \mathscr{D}_i$, and $[\mathscr{D}_i, \mathscr{D}_j] \subset \mathscr{D}_k$, where $1 \le i, j, k \le 3$, and i, j, k are mutually distinct. In other words $\mathscr{g}$ is f_4 **[13]**, and the symmetric pair is $(f_4, o(9))$. Conversely it is well known that such symmetric pairs give rise to the symmetric spaces of rank one.

Lastly, we would like to mention a conjecture of Bob Osserman, which states that nonzero $R(\cdot, v)v$ having constant eigenvalues with fixed multiplicities for all unit v characterizes locally rank-one symmetric spaces **[5]**. It follows from Theorem 1 that this conjecture would be true if the curvature condition in the conjecture would imply the two axioms in §2.

Added in proof. Recently the author received a preprint by Z. I. Szabó and P. B. Gilkey, entitled "A simple topological proof that two-point homogeneous spaces are symmetric", in which an elegant proof of what the title addresses is given. Szabó and Gilkey's result together with the characterization and classification of rank-one symmetric spaces in our paper furnish a geometric-topological understanding of two-point homogeneous spaces.

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