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THE GROWTH AND 1/4-THEOREMS FOR STARLIKE MAPPINGS IN $\mathbf{C}^n$

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Certain geometric function theory results are obtained for holomorphic mappings on the unit ball. Specifically, the mappings studied are one-to-one onto domains that are starlike with respect to the origin. For such a mapping $f(z)$, sharp estimates are derived for $|f(z)|$ in terms of $|z|$. Also, a generalization of the Koebe covering theorem is proved. As a corollary of the work, a new proof is given that, in $\mathbf{C}^n$ for $n \geq 2$, a ball and a polydisc are not biholomorphically equivalent.

More than fifty years ago, Henri Cartan [**2**, cf. 1, 3] suggested that geometric function theory of one complex variable should be extended to biholomorphic mappings of n complex variables. In particular, he cited the special classes of starlike functions and of convex functions as appropriate topics for generalization. In noting some of the difficulties of generalization, he pointed out the Growth Theorem for univalent functions as one that would not extend to the polydisc (nor to the ball). Also he observed that for normalized biholomorphic mappings there is no neighborhood about the origin that is always covered by the range of a biholomorphic mapping of the polydisc (or the ball), that is, there is no Koebe 1/4-Theorem for these mappings.

We will show that, for the class of normalized one-to-one starlike mappings on the ball, versions of both the Growth Theorem and the Koebe 1/4-Theorem hold. Specifically, we will find sharp upper and lower bounds on the magnitude of such a mapping in terms of the magnitude of the variable; and, we will show that the ball of radius 1/4 centered at the origin is always covered.

As a corollary of the results we will demonstrate a theorem first shown by Poincaré: there is no biholomorphic mapping of the unit ball in $\mathbf{C}^n$ onto the unit polydisc in case $n \geq 2$.

1. Preliminaries. Let points in $\mathbf{C}^n$ be denoted by column vectors $z = (z_1, z_2, \ldots, z_n)'$, where the prime indicates transpose. The origin $(0, 0, \ldots, 0)'$ is sometimes denoted by 0. The norm of z is $|z| = (\overline{z}'z)^{1/2} = \langle z, z\rangle = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}$, and the ball

$B_r = \{z: z \in \mathbf{C}^n \text{ and } |z| < r\}$ with B denoting the unit ball B_1. Mappings f in $\mathbf{C}^n$ are given as column vectors $f = (f_1, f_2, \dots, f_n)'$ where each function f_k is a function into $\mathbf{C}$. The Jacobian of a holomorphic mapping f is the matrix with ij entry $\partial f_i/\partial z_j$ and it is denoted by J or J_f or $J_f(z)$. A holomorphic mapping f is normalized means that $f(0) = 0$ and $J_f(0) = I$, the identity matrix. A holomorphic mapping f is starlike with respect to the origin means that f is a normalized, one-to-one mapping with an image that is starlike with respect to the origin, that is, each point in the range can be joined to the origin by a straight line interval without leaving the range of f.

The following theorem gives an analytical characterization of starlike mappings of the open ball. Since this theorem will be basic to our work, we indicate a proof of the necessity of the analytical condition in the case that the mapping can be extended holomorphically to the closure of the ball. This argument is intended to facilitate the intuitive understanding of Suffridge's result, Theorem 1.1.

For z_0 on the boundary ∂B of B, the point $f(z_0)$ is a point on the boundary of the range. A vector V starting at $f(z_0)$ and extending in the direction that $f(z_0)$ is from the origin will lie in the complement of the range of f on the open ball. (If V reached a point which came from the interior of the ball, then the starlike property would imply that the point $f(z_0)$ arose from an interior point. But then a neighborhood of $f(z_0)$ would have to be covered by the image of a strictly interior neighborhood of the ball. Since f is one-to-one, the value $f(z_0)$ could not have been reached as a boundary value.)

Now consider the vector V under the inverse of the Jacobian: $J_f(z_0)^{-1}V$. If this vector is placed so that its base is at z_0, then it must be in the complement of the ball. (Note that the linear mapping $J_f(z_0)^{-1}$ may fail to be conformal.) Since $J_f(z_0)^{-1}V$ is in the complement of the ball, the real part of its inner product with z_0 must be nonnegative. That is,

$$\operatorname{Re}\langle J_f(z_0)^{-1}V, z_0\rangle = |J_f(z_0)^{-1}V| \cdot |z_0| \cos\theta,$$

where θ is the angle between the two entries of the inner product, is nonnegative. Furthermore, since V is a real, positive scalar times $f(z_0)$, the necessity of the analytical condition in the following theorem is obtained for points on ∂B. The condition (1.1) then follows for the points of B by the maximum principle.

THEOREM 1.1 (*Suffridge* [5]). *Suppose that* $f(z)$ *is a normalized holomorphic mapping on the unit ball* B *and that* $J_f(z)$ *is nonzero in* B. *Then* f *is a one-to-one mapping that takes the ball onto a domain that is starlike with respect to the origin if and only if the normalized mapping*

$$w(z) = J_f(z)^{-1} f(z) \tag{1.1}$$

has the property that

$$\operatorname{Re} \bar{z}' w(z) \geq 0 \quad \textit{for every point } z \textit{ in the ball } B. \tag{1.2}$$

An easy corollary will be of use later in our work: If f is starlike with respect to the origin, then, when f is restricted to a smaller ball centered at the origin, it is still a starlike mapping.

The following lemma will be important in working with condition (1.2). A proof is included in order that this work be as self-contained as possible.

LEMMA 1.2 (*Pfaltzgraff* [4]). *Let* $w(z)$ *be a normalized mapping such that*

$$\operatorname{Re} \sum_{j=1}^{n} \bar{z}_j w_j \geq 0 \quad \textit{on } B. \tag{1.3}$$

Then

$$\frac{1-|z|}{|z|(1+|z|)} \cos \Theta \leq \frac{1}{|w(z)|} \leq \frac{1+|z|}{|z|(1-|z|)} \cos \Theta \tag{1.4}$$

where Θ *is the angle between* z *and* $w(z)$.

Proof. Fix $\zeta \in \partial B$. Let ξ be a scalar complex number with $0 < |\xi| < 1$. By hypothesis,

$$\operatorname{Re} \sum_{j=1}^{n} \bar{\xi}_j \bar{\xi} w_j(\zeta\xi) \geq 0.$$

Rewriting the sum, we obtain

$$\operatorname{Re} \sum_{j=1}^{n} \bar{\zeta}_j \zeta_j \bar{\xi} \xi \frac{w_j(\zeta\xi)}{\zeta_j \xi} \geq 0.$$

Since $\bar{\xi}\xi > 0$, we have

$$\operatorname{Re} \sum_{j=1}^{n} \bar{\zeta}_j \frac{w_j(\zeta\xi)}{\xi} \geq 0. \tag{1.5}$$

We define $P(\xi)$ to be $\sum_{j=1}^{n} \overline{\zeta}_j (w_j(\zeta\xi)/\xi)$. Then $P(\xi)$ is an analytic function of a one dimensional complex variable. From the normalization of w, it is clear that we can extend P to be defined at zero and then $P(0) = 1$. From inequality (1.5), $P(\xi)$ has positive real part on the unit disc $\{\xi : |\xi| < 1\}$; and the estimates

$$\frac{1-|\xi|}{1+|\xi|} \leq |\operatorname{Re} P(\xi)| \leq \frac{1+|\xi|}{1-|\xi|}$$

hold on the unit disc.

If the expression $\operatorname{Re}\sum_{j=1}^{n} \overline{\zeta}_j (w_j(\zeta\xi)/\xi)$ is considered as an inner product, we see this real expression equals $(|w(\zeta\xi)|/|\xi|)|\zeta| \cos\theta$ where θ is the angle between $w(\zeta\xi)/\xi$ and ζ. This angle θ is easily seen to be the same as the angle Θ in the conclusion of the theorem; just note that the angle θ is unchanged if each vector is rotated by multiplying by the complex scalar ξ and then think of z as being represented by $\zeta\xi$. Also we use that $|\zeta| = 1$ so that $|z| = |\xi|$. The following inequalities result:

$$\frac{1-|z|}{1+|z|} \leq \frac{|w(z)|}{|z|} \cos\Theta \leq \frac{1+|z|}{1-|z|}.$$

The conclusion of the lemma follows by using these inequalities to estimate $1/|w(z)|$.

2. The Growth Theorem for starlike functions.

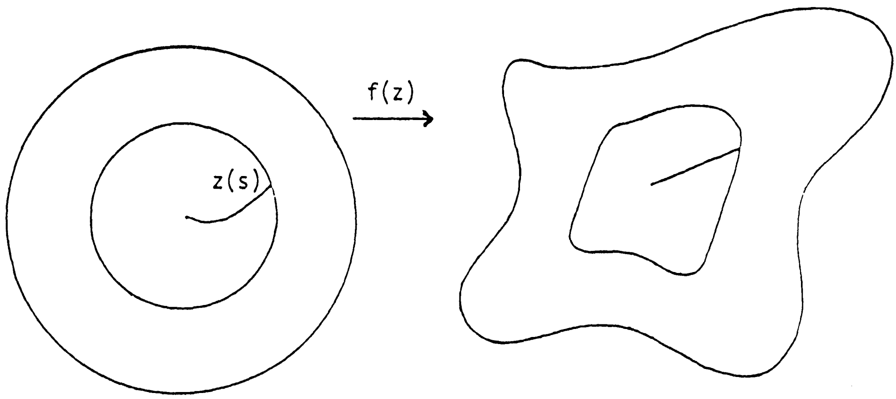
We can now state the Growth Theorem for starlike mappings.

THEOREM 2.1. *Let $f = (f_1, f_2, \ldots, f_n)'$ be a starlike mapping of the unit ball B in $\mathbf{C}^n$. For any point $z \in B$,*

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Furthermore the estimates are sharp.

Proof. Consider a number $0 < r < 1$. Let z_1 be a point such that $|z_1| = r$ and $|f(z_1)| =$ maximal magnitude of f on $\{z : |z| = r\}$. Since f is starlike, there is a ray from the origin to $f(z_1)$ which is contained in the image of $\{z : |z| \leq r\}$. Hence the preimage of this ray is in the closure of the ball B_r. Indeed, as one moves along the preimage starting at the origin, the magnitude of z is monotonically



increasing. Since this preimage curve is analytic, we can certainly think of it as $z(s)$ parameterized with arc length.

$$\begin{aligned} \frac{df(z(s))}{ds} &= \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \frac{dz_j}{ds} = J\frac{dz}{ds}, \\ \frac{df(z(s))}{ds} &= \lambda(z(s)) f(z(s)), \end{aligned} \tag{2.1}$$

where λ is a positive, scalar-valued function, and the existence of λ follows from the fact that the image of the arc $z(s)$ is a straight line extending in the direction of $f(z(s))$.

Since f is starlike, we can apply equation (1.1) of Suffridge's theorem to obtain

$$\frac{df(z(s))}{ds} = \lambda J w.$$

We can solve for dz/ds by applying J^{-1}. Since λ is a scalar, we obtain

$$\frac{dz}{ds} = \lambda w.$$

Taking the magnitude of both sides of this equation, and recalling that s is the arc length parameter, we note that

$$1 = \lambda |w|, \qquad \text{that is}, \quad \lambda = \frac{1}{|w|}. \tag{2.2}$$

With this expression for λ, the equation (2.1) can be expressed as a vector equation as follows:

$$\frac{df(z(s))}{ds} = \frac{1}{|w(s)|} f(z(s)).$$

Our estimation of the magnitude of f will be facilitated by introducing a scalar function: $g(s) = |f(z(s))|2 = \overline{f(z(s))'}f(z(s))$. Then we have

$$\begin{aligned}\frac{dg}{ds} &= 2\,\mathrm{Re}\left\{\overline{f(z(s))'}\frac{df(z(s))}{ds}\right\}\\ &= 2\frac{1}{|w|}\mathrm{Re}\{\overline{f(z(s))'}f(z(s))\}\\ &= 2\frac{1}{|w|}g(z(s)).\end{aligned}$$

Hence we can write the equation of differentials:

$$\frac{dg}{g} = 2\frac{1}{|w|}ds. \tag{2.3}$$

Thus, from the upper estimate of $|w|$ given by Lemma 1.2, we have

$$\frac{d\log g}{ds} \le 2\frac{1+|z(s)|}{|z(s)|(1-|z(s)|)}\cos\theta(s), \tag{2.4}$$

where $\theta(s)$ is the angle between $z(s)$ and $dz(s)/ds$. From geometrical considerations, it is easy to obtain the general formula $d|z(s)|/ds = \cos\theta(s)$ provided $z(s)$ is not at the origin. Then from formula (2.4) we find that

$$\begin{aligned}\log g(s_1) - \log g(s_0) &\le \int_{s_0}^{s_1} 2\frac{(1+|z(s)|)}{|z(s)|(1-|z(s)|)}\cos\theta(s)\,ds\\ &= \int_{|z(s_0)|}^{|z(s_1)|} 2\frac{1+|z|}{|z|(1-|z|)}\,d|z|\\ &= \int_{|z(s_0)|}^{|z(s_1)|}\left(\frac{2}{|z|}+\frac{4}{1-|z|}\right)d|z|\\ &= 2\log|z(s_1)| - 4\log(1-|z(s_1)|)\\ &\quad - [2\log|z(s_0)| - 4\log(1-|z(s_0)|)].\end{aligned}$$

We recall that f is normalized so that its Jacobian at the origin is equal to the identity matrix. Hence, if we let s_0 be a small positive number ε, then $|z(s_0)| = \varepsilon + o(\varepsilon)$ and $g(s_0) = \varepsilon^2 + o(\varepsilon^2)$. Thus

$$g(s_1) \le \left\{\frac{|z(s_1)|^2}{(1-|z(s_1)|)^4}\right\}\frac{\varepsilon^2+o(\varepsilon^2)}{\varepsilon^2+o(\varepsilon^2)}.$$

If we consider the limit as ε tends to zero, and pick s_1 so that $z(s_1)$ is $z_1 = z$, and use the definition of g, we obtain

$$|f(z)| \le \frac{|z|}{(1-|z|)^2} \quad \text{as claimed in the theorem.}$$

The other inequality of the theorem is easily proved. Again we use equality (2.3), but we use the lower bound on $1/|w(z)|$ given by Lemma 1.2. The rest of the argument follows as before.

The inequalities in Theorem 2.1 are best possible. To demonstrate this fact we define

$$f(z) = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_2)^2}\right)'.$$

If the variable z is restricted to be of the form $(z_1, 0)'$, then equality can be obtained in both inequalities of Theorem 2.1. That f satisfies the hypothesis that it be normalized at the origin is clear from power series expansions. That it is one-to-one in the ball is obvious since each coordinate function of f is a univalent function of one variable: thus given $(w_1, w_2)'$ the value of z_1 can be obtained by applying the inverse of the function in the first coordinate to w_1 and the value of z_2 is obtained by a similar process from w_2. Finally, that f takes the ball onto a starlike domain follows from the fact that each coordinate function of f is a starlike function of one variable. For any point in the range of f, consider the straight line from that point to the origin. The projection of that line onto the plane of points $(*, 0)'$ is also a straight line to the origin. Its preimage under the first coordinate function is a curve that moves to the origin in a monotonic fashion, that is, the straight line distance to the origin decreases monotonically. Hence if we start at a point in the range of f and move along the straight line to the origin, the points considered are always in the range of f on the unit ball.

Now, it is clear that there are many examples showing that the estimates of Theorem 2.1 are sharp. For example, the second coordinate function of f can be replaced by any one-to-one analytic function f_2 of the single variable z_2 onto a domain that is starlike with respect to the origin provided f_2 takes the origin to the origin and has derivative at the origin equal to one.

3. The covering theorem for starlike mappings. From Theorem 2.1 it is easy to prove the following analogue of Koebe's 1/4-Theorem:

COROLLARY 3.1. *If $f(z)$ is a holomorphic mapping on the unit ball in $\mathbf{C}^n$ and is starlike with respect to the origin, then the image of f contains a ball of radius $1/4$ centered at the origin. The value $1/4$ is best possible.*

Proof. Consider the image of a ball B_r of radius $0 < r < 1$. The image is an open set with boundary being the image of the boundary of B_r. By Theorem 2.1, the image of the ∂B_r is at least $r/(1+r)^2$ away from the origin. By letting r tend to 1 from below, we complete the proof of the corollary.

Also it is easy to see that the estimate is sharp. The examples given at the end of the previous section show that the point $w_1 = -1/4$, $w_2 = 0$ can fail to be in the range.

If k-fold symmetry of f is assumed, that is, the image of f is unchanged when multiplied by the scalar complex number $\exp(2\pi i/k)$ for some positive integer k, then Theorem 2.1 and Corollary 3.1 can be strengthened:

COROLLARY 3.2. *If f is holomorphic in the unit ball and is starlike with a k-fold symmetric image for $k \geq 1$, then*

$$\frac{|z|}{(1+|z|^k)^{2/k}} \leq |f(z)| \leq \frac{|z|}{(1-|z|^k)^{2/k}}.$$

Then the image of the ball under f contains a ball of radius $2^{-2/k}$ and these estimates are best possible.

Proof. The geometric hypothesis that the image of the normalized holomorphic mapping is k-fold symmetric has a related analytical property of $f(z)$. Specifically, $\exp(-2\pi i/k) f(\exp(2\pi i/k) z) = f(z)$ for all z in the ball. This latter expression of symmetry is obviously satisfied by the mapping $w(z)$ of Theorem 1.1. Consequently, the positive real part function $P(\xi)$ in Lemma 1.1 has the symmetry property that $P(\exp(2\pi i/k) z) = P(z)$. Then the estimate for this function is that

$$\frac{1-|\xi|^{1/k}}{1+|\xi|^{1/k}} \leq \operatorname{Re} P(\xi) \leq \frac{1+|\xi|^{1/k}}{1-|\xi|^{1/k}}.$$

Following the rest of the proof of Theorem 2.1 gives the result claimed. Many mappings show that the result is sharp, for example,

$$f(z) = \left(\frac{z_1}{(1-z_1^k)^{2/k}}, \frac{z_2}{(1-z_2^k)^{2/k}}\right)'.$$

Finally, the estimate for the Koebe ball is found the same way as before. The example just described shows that the result obtained is sharp.

This corollary implies some restrictions on which symmetrical domains can be mapped onto each other by biholomorphic mappings. We recall that a domain is balanced means it is starlike with respect to the origin and is circular; a domain is circular means that if w belongs to the domain, then ξw also belongs to the domain for every complex scalar ξ of magnitude 1.

COROLLARY 3.3. *The only balanced domain which is the image of the unit ball under a normalized biholomorphic mapping is the unit ball.*

Proof. Consider a normalized, biholomorphic mapping of the ball onto a balanced domain D. Since the image domain is k-fold symmetric for all positive integers k, the estimates obtained in the previous corollary imply that $|f(z)| = |z|$ for all z in B. By the normalization of f, it follows that $f(z) = z$ for all z in B; hence the domain D is the unit ball B.

As an easy consequence of this corollary, we deduce a well-known, classic observation.

COROLLARY 3.4 (*Poincaré* 1907, *Cartan* 1932). *For $n \geq 2$, the unit ball in $\mathbf{C}^n$ is not biholomorphically equivalent to a polydisc in $\mathbf{C}^n$.*

Proof. Suppose there were such a mapping f from B onto a polydisc D. Using a holomorphic affine transformation in the range space, we can require D to be of a certain form: $D = \{w: |w_k| \leq r_k$ for $k = 1, 2, \ldots, n\}$ for some positive numbers $r_1, r_2, \ldots, r_n$. By an application of a Möbius transformation in the domain, we can assume that f takes the origin into the origin. Also, by using a unitary transformation of the ball, we can assume that the Jacobian of f at the origin is diagonal. By rescaling the complex coordinates in the image space, it can be assumed that f is normalized. Since the polydisc, positioned as described, is balanced, a contradiction to the previous corollary has been reached for $n \geq 2$. Hence there cannot be a biholomorphic mapping of B onto a polydisc.

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