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NORMING VECTORS OF LINEAR OPERATORS BETWEEN L_p SPACES

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For a bounded linear operator T from an L_p to an L_q space $(1 \leq p, q < \infty)$, we study its *norming vectors*, i.e. those, including the zero vector, on which T attains its norm. The scalar field may be the reals or the complex numbers. Our first two main results are the characterization of the set of norming vectors for a positive T when both $p > 1$ and either (i) $p = q$ or (ii) $p > q$. The descriptions may not hold if T is not positive, but they do in modified forms if $|T|$ exists with norm $\|T\|$. We also prove that if $p > q$ and one of the two underlying measures is purely atomic, then every regular T is norm-attaining. Sufficient conditions for T (of norm 1) to be an extreme contraction in the case $p > q > 1$ are derived from properties of its norming vectors. All results extend to the case of quaternion scalars with little change of the proofs.

1. Introduction. Generic patterns of distribution of norming vectors of elements of the Banach space $\mathscr{L}(\mathbf{E}, \mathbf{F})$ of bounded operators from $\mathbf{E} = L_p$ to $\mathbf{F} = L_q$ reflect the geometric structure of the unit ball of $\mathscr{L}(\mathbf{E}, \mathbf{F})$, including its extremal aspect. (On this aspect, [**10**] contains other results.) Our investigation reveals that these patterns are different for different regions of (p, q), broadly delimited by $p = q$, $p = 2$ and $q = 2$, but are also affected by assumption of positivity on the operator and sometimes the scalars used. The aforementioned result for $p = q > 1$ in the abstract is of particular interest. The characterization therein (Theorem 3.4) is analogous to those of several operator-related subsets $\mathscr{S}$ of Banach or function spaces. These include the two cases $\mathscr{S} =$ the range of a contractive projection (positive if $p = 2$) on $\mathbf{E} = L_p$, $0 < p < \infty$ [**1**, Theorem 2], [**22**, Theorem 6], [**3**, Theorems 3.4-5], and $\mathscr{S} =$ the *convergence set* $\{f\colon T_n f \to f$ in norm$\}$ for a net of contractions $\{T_n\}$ on $\mathbf{E} = L_p$, $1 < p < \infty$, $p \neq 2$ [**2**, Theorem 2.5]. In our result (Theorem 3.4) and these others, $\mathscr{S}$ is a subspace of $\mathbf{E}$ isometrically isomorphic to another L_p space over essentially a measure subspace of the underlying one, with a change of scale. When $p > 1$, Bernau [**2**] characterizes $\mathscr{S}$ also as a subspace $\mathbf{V}$ of $\mathbf{E}$ for which $f, g \in \mathbf{V} \Rightarrow |f| \operatorname{sgn} g \in \mathbf{V}$. Scheffold [**21**] extended this notion to the case $\mathbf{E} =$ a real Banach lattice with order continuous norm. Under some additional assumptions on the norm of $\mathbf{E}$ (which are satisfied if $\mathbf{E} = L_p$, $1 \le p < \infty$) he proved that the following are such subspaces: (a) $\mathrm{Ker}(I - T)$ for a regular operator T on $\mathbf{E}$ for which $|T|$ is contractive; (b) the convergence set for a sequence of regular operators T_n on $\mathbf{E}$, where each $|T_n|$ is a contraction. More recently Hardin [**5**, Theorem 4.2 and Remark (ii)] proved that an isometry on a linear subspace $\mathbf{W}$ of an L_p space to another, when $0 < p < \infty$ and $p \neq 2, 4, 6, \ldots$, extends to one on a subspace of the above type generated by $\mathbf{W}$; see also [**19**, Theorem 1.4] and [**15**, Proposition 1] for the complex case.

We note that for finite dimensional non-negative matrices T, Koskela in different formulations ([**13**], Lemma 1 and Theorem 2 for $p > q$, and Theorems 7–8 for $p = q$) and by proofs different from ours essentially obtained our characterizations (Theorems 3.4 and 4.1(a)) when they are restricted to $\mathscr{N}^+(T)$.

I am very grateful to the referee, whose comments greatly help improve the presentation of the material in this paper.

2. Decomposition system of an operator. In this paper we only consider underlying measure spaces that are direct unions of finite ones. This does not entail any loss of generality [**14**, Corollary to Theorem 15.3], [**10**, p. 615]. An advantage in this case is that the associated measure algebras are complete Boolean algebras, a convenience for formulating concepts and describing properties, e.g. in Theorem 2.1.

In the sequel let $1 \le p$, $q < \infty$, and let $\mathbf{E} = L_p(X, \mathscr{F}, \mu)$ and $\mathbf{F} = L_q(Y, \mathscr{G}, \nu)$ be the usual Lebesgue spaces. Norms in $\mathbf{E}$, $\mathbf{F}$, etc. are all denoted by $\|\cdot\|$ as no confusion seems likely. Let $A \in \mathscr{F}$. Define $\mathbf{E}_A = \{f \in \mathbf{E} \colon \operatorname{supp} f \subset A\}$, where $\operatorname{supp} f = \{f \neq 0\}$, the *support* of f. For any function f on (X, μ) (or even one defined only on (A, μ)), let $f_A = f$ on A, and 0 on A^c, the complement of A. Given $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$, we define the *decomposition system* for T to be

$$\mathscr{F}(T) = \{A \in \mathscr{F} \colon |Tf| \wedge |Tg| = 0 \ \forall f \in \mathbf{E}_A \text{ and } \forall g \in \mathbf{E}_{A^c}\}.$$

When $\mathscr{F}(T) = \mathscr{F}$, T is said to be *disjunctive* (or *Lamperti* in [**8**]). Define $o(T) = \sup\{A \in \mathscr{F} \colon T\mathbf{E}_A = \{0\}\}$, $s(T) = (o(T))^c$ and $\mathscr{F}'(T) = \mathscr{F}(T) \cap s(T)$. Define $\Phi \colon (\mathscr{F}(T), \mu) \to (\mathscr{G}, \nu)$ by

$$\Phi A = \sup\{\operatorname{supp} Tf \colon f \in \mathbf{E}_A\} \qquad (A \in \mathscr{F}(T)).$$

Φ generalizes the natural set mapping $(A \mapsto \operatorname{supp} T1_A$, if $\mu A < \infty)$ for a disjunctive T, by [**8**, Theorem 4.1]. Define $\mathscr{G}(T^*)$, $o(T^*)$, $s(T^*)$, $\mathscr{G}'(T^*)$ and $\Psi \colon (\mathscr{G}(T^*), \nu) \to (\mathscr{F}, \mu)$ similarly.

THEOREM 2.1. *Let* $1 \le p, q < \infty$ *and* $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$. *Then*

(i) $\mathscr{F}(T)$ *is a complete Boolean sub-algebra of* $\mathscr{F}$ *including* $\mathscr{F} \cap o(T)$, *and*

(ii) Φ *is a Boolean ring homomorphism, preserves arbitrary suprema and has* $\operatorname{Ker}\Phi = \mathscr{F} \cap o(T)$.

These also hold with $(T^*, \Psi, \mathscr{G})$ *replacing* $(T, \Phi, \mathscr{F})$, *and* $\Phi|_{\mathscr{F}'(T)}$ *is a* σ*-isomorphism from the measure algebra* $(\mathscr{F}'(T), \mu)$ *to* $(\mathscr{G}'(T^*), \nu)$ *with inverse* $\Psi|_{\mathscr{G}'(T^*)}$.

Proof. First, some ready observations. $\mathscr{F}(T)$ is closed under complementation and finite union. So it is a Boolean sub-algebra. Φ preserves disjointness and finite direct union. So Φ is a Boolean ring homomorphism.

Let $\varnothing \neq \mathscr{K} \subset \mathscr{F}(T)$ and $A = \sup \mathscr{K}$. Let $f \in \mathbf{E}_A$ and $g \in \mathbf{E}_{A^c}$. There are $A^1, A^2, \ldots \in \mathscr{K}$ with $\operatorname{supp} f \subset \bigcup A^n$. Now $B^n \equiv A^1 \cup \cdots \cup A^n \in \mathscr{F}(T)$ and so $|Tf_{B^n}| \wedge |Tg| = 0$. Hence $|Tf| \wedge |Tg| = 0$, as $\|Tf_{B^n} - Tf\| \le \|T\| \cdot \|f_{B^n} - f\| \to 0$. So $A \in \mathscr{F}(T)$. Thus $\mathscr{F}(T)$ is Boolean complete. The same proof shows that $o(T) \in \{A \in \mathscr{F} : T\mathbf{E}_A = \{0\}\}$. (Take the latter as $\mathscr{K}$ and note that $Tf_{B^n} = 0$.) Thus $\mathscr{F} \cap o(T) \subset \mathscr{F}(T)$. This proves (i) and $\operatorname{Ker}\Phi = \mathscr{F} \cap o(T)$. Φ preserves union. So the argument above carries through if we replace $|Tg|$ by any $h \in \mathbf{F}^+_{(\sup \Phi\mathscr{K})^c}$. We get $\Phi \sup \mathscr{K} \subset \sup \Phi\mathscr{K}$. Equality follows. This ends the proof of (ii).

Now $\int f T^* g \, d\mu = \int Tf \cdot g \, d\nu$ $(f \in \mathbf{E},\ g \in \mathbf{F}')$. So for all $C \in \mathscr{F}$ and $D \in \mathscr{G}$, $T\mathbf{E}_C \subset \mathbf{F}_D \Leftrightarrow T^*\mathbf{F}'_{D^c} \subset \mathbf{E}'_{C^c}$. Let $B \in \mathscr{G}(T^*)$. The latter inclusion holds for $(C, D) = ((\Psi B)^c, B^c)$ or $(\Psi B, B)$. Thus so does the former. It follows that

$$\Psi B \in \mathscr{F}(T) \quad \text{and} \quad \Phi\Psi B = B \cap \Phi X.$$

Like Φ, Ψ is a Boolean ring homomorphism. So dually for all $A \in \mathscr{F}(T)$ we have

$$\Phi A \in \mathscr{G}(T^*) \quad \text{and} \quad \Psi\Phi A = A \cap \Psi Y.$$

From these two results it follows that $(\Psi Y)^c \in \operatorname{Ker}\Phi$, $(\Phi X)^c \in \operatorname{Ker}\Psi$,

$$\operatorname{Range}\Psi = \mathscr{F}(T) \cap \Psi Y, \qquad \operatorname{Range}\Phi = \mathscr{G}(T^*) \cap \Phi X,$$

and Φ is a bijection from the former range onto the latter, with inverse Ψ. Hence $o(T) = (\Psi Y)^c$ and $o(T^*) = (\Phi X)^c$. The rest follows.

For $A \in \mathscr{F}$ and $B \in \mathscr{G}$, define $T_{BA} \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ by $T_{BA} f = (Tf_A)_B$ $(f \in \mathbf{E})$.

THEOREM 2.2. *Let* $1 \le p < q < \infty$ *and* $O \ne T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$. *Then* $(\mathscr{F}'(T), \mu)$ *is purely atomic and*

$$\|T\| = \sup\{\|T_{YA}\| : A \text{ is an atom of } \mathscr{F}'(T)\}.$$

Proof. Assume that $\mathscr{F}'(T)$ has a diffuse part $D \ne \varnothing$. Fix $0 \ne f \in \mathbf{E}_D$ with $Tf \ne 0$. Then $D \supset A^0 \equiv$ the $\mathscr{F}'(T)$-measurable cover of $\operatorname{supp} f$. Now $\|Tf_A\|^q$ and $\|f_A\|^p$ are additive on $A \in \mathscr{F}'(T)$. So if $\varnothing \ne A \in \mathscr{F}'(T) \cap A^0$ is partitioned into non-null $B, C \in \mathscr{F}'(T)$, then $\rho(A) \equiv \|Tf_A\|^q/\|f_A\|^p \le \max\{\rho(B), \rho(C)\}$. Hence there exist $A^1, A^2, \ldots \in \mathscr{F}'(T)$ with $A^{n+1} \subset A^n$, $\|f_{A^{n+1}}\|^p = \|f_{A^n}\|^p/2$ and $\rho(A^{n+1}) \ge \rho(A^n)$ $(n \ge 0)$. So $\|T\|^q \ge (\|Tf_{A^n}\|/\|f_{A^n}\|)^q = \rho(A^n)/\|f_{A^n}\|^{q-p} \uparrow \infty$, impossible. So $D = \varnothing$.

Let $A \in \mathscr{F}(T)$ and $B = A^c$. For all $f \in \mathbf{E}$, we have

$$\|Tf\|^p = (\|Tf_A\|^q + \|Tf_B\|^q)^{p/q}$$
$$\le \|Tf_A\|^p + \|Tf_B\|^p \le \|T_{YA}\|^p\|f_A\|^p + \|T_{YB}\|^p\|f_B\|^p.$$

It follows that $\|T\| = \max\{\|T_{YA}\|, \|T_{YB}\|\}$. By this formula,

$$\|T_{Y(A^1\cup\cdots\cup A^n)}\| = \max\{\|T_{YA^1}\|, \ldots, \|T_{YA^n}\|\}$$

for any atoms $A^1, A^2, \ldots, A^n$ of $\mathscr{F}'(T)$. If $f \in \mathbf{E}_{s(T)}$, then $f_{A^1\cup\cdots\cup A^n} \to f$ for some such atoms $A^1, A^2, \ldots$. By these and a continuity argument, $\|T\| \le$ stated supremum. Equality follows.

Suppose $(X, \mathscr{F}, \mu)$ is purely atomic. Then $s(T) = \{x \in X : T1_x \ne 0\}$. Call $x, y \in s(T)$ *T-linked* if $|T1_{x^m}| \wedge |T1_{x^{m+1}}| \ne 0$ $(1 \le m \le n-1)$ for some $x^1 = x, x^2, \ldots, x^n = y$ in $s(T)$. This is an equivalence relation. It is easy to prove:

PROPOSITION 2.3. *If* $(X, \mathscr{F}, \mu)$ *is purely atomic, then the equivalence classes of T-linked points are precisely the atoms of* $\mathscr{F}'(T)$, *and for each such atom* A, $\Phi A = \sup\{\operatorname{supp} T1_x : x \in A\}$. *If further* $(Y, \mathscr{G}, \nu)$ *is purely atomic, then* ΦA *is an equivalence class of* T^*-*linked points.*

Φ induces a unique positive linear operator $\Phi^\#$ from $\mathscr{F}(T)$- to $\mathscr{G}$-measurable functions, satisfying $\Phi^\# 1_A = 1_{\Phi A}$ $(A \in \mathscr{F}(T))$ and behaving like a composition operator **[8**, §4] (see also **[16]**, p. 159 or **[4]**, pp. 453–454).

Let

$$\mathscr{L}(\mathbf{E}^+, \mathbf{F}^+) = \{\text{positive operators in } \mathscr{L}(\mathbf{E}, \mathbf{F})\}$$
$$= \{T \in \mathscr{L}(\mathbf{E}, \mathbf{F}) : T\mathbf{E}^+ \subset \mathbf{F}^+\}.$$

For a scalar $a \neq 0$, let $\operatorname{sgn} a = a/|a|$; let $\operatorname{sgn} 0 = 0$. This defines the *signum* of a.

LEMMA 2.4. *Let* $1 \leq p, q < \infty$, $T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$ *and* $f \in \mathbf{E}$.
(a) *If* θ *is* $\mathscr{F}(T)$*-measurable and* $\theta f \in \mathbf{E}$, *then* $T(\theta f) = \Phi^{\#}\theta \cdot Tf$.
(b) *If* $f \geq 0$, *then* $T\mathbf{E}_{\operatorname{supp} f} \subset \mathbf{F}_{\operatorname{supp} Tf}$.
(c) *If* $f \geq 0$, $\operatorname{supp} f \in \mathscr{F}(T)$, $B \in \mathscr{F} \cap \operatorname{supp} f$ *and* $Tf_B \wedge Tf_{B^c} = 0$, *then* $B \in \mathscr{F}(T)$.
(d) *If* $\operatorname{supp} f \in \mathscr{F}(T)$ *and* $|Tf| = T|f|$, *then* $\operatorname{sgn} f$ *is* $\mathscr{F}(T)$*-measurable.*

Proof. (a) This is easy for θ simple. The general case follows.
(b) T preserves monotone limits. So $g \in \mathbf{E}_{\operatorname{supp} f}$ implies

$$\operatorname{supp} Tg \subset \operatorname{supp} T|g| = \bigcup \operatorname{supp} T(|g| \wedge (nf)) \subset \operatorname{supp} Tf.$$

(c) By (b) with f replaced by both f_B and f_{B^c}, we have $B \in \mathscr{F}(T_{YA})$, where $A = \operatorname{supp} f$. Since $A \in \mathscr{F}(T)$, this implies $B \in \mathscr{F}(T)$.
(d) We need only prove $B \equiv \{\operatorname{Re}(\operatorname{sgn} f/s) > 0\} \in \mathscr{F}(T)$ for any unimodular scalar s. Let $f' = f/s$ and $g = \operatorname{Re} f'$. Then $B = \operatorname{supp} g^+$. Let $C = B^c$. As

$$\begin{aligned} |Tf'_B + Tf'_C| &= |Tf'| = |Tf| = T|f| = T|f'| \\ &= T|f'_B| + T|f'_C| \geq |Tf'_B| + |Tf'_C|, \end{aligned}$$

so we have equality. Hence $\operatorname{sgn} Tf'_B = \operatorname{sgn} Tf'_C$ on $D \equiv \{|Tf'_B| \wedge |Tf'_C| \neq 0\}$ and $|Tf'_Z| = T|f'_Z|$ $(Z = B, C)$. On $\operatorname{supp} Tf'_B$ ($= \operatorname{supp} Tg^+$ by (b)) $\operatorname{Re} Tf'_B = Tg^+ > 0$. But $\operatorname{Re} Tf'_C = -Tg^- \leq 0$. So $D = \varnothing$, or $T|f'_B| \wedge T|f'_C| = 0$. By (c) applied to $|f'|$, $B \in \mathscr{F}(T)$.

3. Norming vectors: $\infty > p = q > 1$. The set of *norming vectors* of $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is defined to be

$$\mathscr{N}(T) = \{f \in \mathbf{E}\colon \|Tf\| = \|T\| \cdot \|f\|\}.$$

T is *norm-attaining* if $\mathscr{N}(T) \neq \{0\}$. Let $\mathscr{N}^+(T) = \mathscr{N}(T) \cap \mathbf{E}^+$. For a scalar $a \neq 0$, let $a^{p-1} = |a|^{p-2}a$; let $O^{p-1} = 0$. This is applied on L_p vectors. For a sub-σ-ring $\mathscr{R}$ of $\mathscr{F}$ with largest element A, a function $f = f_A$ on $(X, \mathscr{F}, \mu)$ is *$\mathscr{R}$-measurable* if $f|_A$ is.

Lemma 3.1 (from [**10**, Lemma 2.10]) dates back to M. Riesz [**17**, §6] (see also [**6**, §8.14]) in the case of finite complex sequence spaces.

LEMMA 3.1. *Let* $1 < p, q < \infty$, $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ *and* $0 \neq f \in \mathbf{E}$. *Then* $f \in \mathscr{N}(T)$ *if and only if*

$$T^*((\overline{Tf})^{q-1}) = \|T\|^q \|f\|^{q-p} \overline{f}^{p-1}, \tag{3.1}$$

in which case $(\overline{Tf})^{q-1} \in \mathscr{N}(T^*)$.

LEMMA 3.2. *Suppose* $1 \leq p, q < \infty$, $O \neq T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$ *and* $0 \neq f \in \mathscr{N}(T)$. *Then* $|f| \in \mathscr{N}^+(T)$. *When* $p > 1$, $\operatorname{sgn} f$ *is* $\mathscr{F}'(T)$-*measurable and* $\operatorname{supp} f \in \mathscr{F}'(T)$. *Further,* $\operatorname{supp} f$ *is* (i) $s(T)$ *if* $p > q$ *or* (ii) *an atom of* $\mathscr{F}'(T)$ *if* $1 < p < q$.

Proof. We have $|Tf| \leq T|f|$. So equality holds and $|f| \in \mathscr{N}^+(T)$. Let $p > 1$. The assertion on $\operatorname{sgn} f$ follows from Lemma 2.4(d) and $A \equiv \operatorname{supp} f \in \mathscr{F}'(T)$. To prove $A \in \mathscr{F}'(T)$, we may assume $f \geq 0$ (or replace f by $|f|$). Clearly $A \subset s(T)$. Let $g \in \mathbf{E}^+$ with $g \wedge f = 0$. When $1 < p \leq q$, $\langle Tg, (Tf)^{q-1}\rangle = 0$ by (3.1). So $Tg \wedge Tf = 0$. By Lemma 2.4(b), $A \in \mathscr{F}'(T)$. Assume further, as we may, that $\|f\| = \|T\| = 1 = \|Tf\| = \|g\|$. When $p > q$, with $t = \|Tg\|^{q/(p-q)}$ and $r = pq/(p-q)$ we have $t^p = \|Tg\|^r = \|tTg\|^q$. So

$$\|T(f+tg)\| \geq (\|Tf\|^q + \|tTg\|^q)^{1/q} = (1 + \|Tg\|^r)^{1/r}(1+t^p)^{1/p}$$

and $\|f+tg\| = (1+t^p)^{1/p}$. Hence $\|T\|^r \geq 1 + \|Tg\|^r$. Thus $Tg = 0$. So $A = s(T)$. This gives result (i) and ends the proof that $A \in \mathscr{F}'(T)$ if $p > 1$.

When $1 < p < q$, let A be decomposed into $B, C \in \mathscr{F}'(T)$. Then

$$\begin{aligned}\|Tf\|^p &= (\|Tf_B\|^q + \|Tf_C\|^q)^{p/q} \\ &\leq \|Tf_B\|^p + \|Tf_C\|^p \leq \|f_B\|^p + \|f_C\|^p = 1.\end{aligned}$$

Thus equalities hold. Hence $f_B, f_C \in \mathscr{N}(T)$ and as $p/q < 1$, one of Tf_B and Tf_C, and so one of f_B and f_C, is 0. This proves result (ii).

Lemma 3.3(b) is a crucial step towards proving Theorem 3.4. It is based on the condition of equality for an integral inequality [**9**, p. 324].

LEMMA 3.3. *Let* $\infty > p = q > 1$, $O \neq T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$ *and* $0 \neq f \in \mathscr{N}^+(T)$. *Then*:

(a) $\theta f \in \mathscr{N}(T)$ *for all* $\mathscr{F}(T)$-*measurable functions* θ *such that* $\theta f \in \mathbf{E}$.

(b) *If* $g \in \mathcal{N}^+(T) \cap \mathbf{E}_{\operatorname{supp} f}$, *then* $(g/f)_{\operatorname{supp} f}$ *is* $\mathcal{F}'(T)$*-measurable.*

Proof. (a) The norms in $\mathbf{E}$ and $\mathbf{F}$ being p-additive this holds for simple whence general θ.

(b) We may assume $\|T\| = \|g\| = 1$. For any scalar $t > 0$,

$$(3.2)\quad \int (Tg - tTf)^+ (Tf)^{p-1}\, d\nu \le \int T(g - tf)^+ \cdot (Tf)^{p-1}\, d\nu = \int (g - tf)^+ f^{p-1}\, d\mu$$

by Lemma 3.1. Integrate both ends of (3.2) with respect to $t^{p-2}\,dt$ over $(0, \infty)$ and interchange the order of integration. We get $c\|Tg\|^p \le c \equiv 1/(p-1) - 1/p$. As $g \in \mathcal{N}(T)$, equality holds here, whence also in (3.2) for all $t > 0$, as the integrals shown are continuous in t. Further as $Tg - tTf = T(g - tf)^+ - T(g - tf)^-$, where all the terms have supports $\subset \operatorname{supp} Tf$ (Lemma 2.4(b)), this implies

$$(3.3)\qquad T(g - tf)^+ \wedge T(g - tf)^- = 0 \qquad (t > 0).$$

For those $t > 0$ with $\{0 < g = tf\} = \emptyset$, $\operatorname{supp}|g - tf| = \operatorname{supp} f \in \mathcal{F}'(T)$ by Lemma 3.2. So by (3.3) and Lemma 2.4(c) applied to $|g - tf|$,

$$\{(g/f)_{\operatorname{supp} f} > t\} = \operatorname{supp}(g - tf)^+ \in \mathcal{F}'(T).$$

As such scalars $t > 0$ are co-countable and so dense in $(0, \infty)$, the conclusion follows.

THEOREM 3.4. *Let* $\infty > p = q > 1$ *and let* $O \neq T \in \mathcal{L}(\mathbf{E}^+, \mathbf{F}^+)$ *be norm-attaining. Then when* μ *is* σ*-finite there exists* $0 \neq f \in \mathcal{N}^+(T)$ *with* $\operatorname{supp} f \in \mathcal{F}'(T)$ *such that* $\mathcal{N}(T)$ *is given by the closed linear subspace*

$$\{\theta f \in \mathbf{E}\colon \theta \text{ is } \mathcal{F}'(T) \cap \operatorname{supp} f\text{-measurable}\},$$

and in the general case $\mathcal{N}(T)$ *is a direct* l_p*-sum of such subspaces. In any case* $\mathcal{N}(T)$ *is a closed linear subspace isometrically isomorphic to an* L_p *space and is also a Banach sub-lattice of* $\mathbf{E}$.

Proof. Let μ be σ-finite. Let $A = \sup\{\operatorname{supp} g\colon g \in \mathcal{N}(T)\}$. For some $f_1, f_2, \ldots \in \mathcal{N}(T)$, $\bigcup \operatorname{supp} f_n = A$. Let $A^1 = \operatorname{supp} f_1$ and $A^n = \operatorname{supp} f_n \backslash (A^1 \cup \cdots \cup A^{n-1})$ $(n \ge 2)$. Let $g_n = |f_n|_{A^n}$ $(n \ge 1)$ and choose scalars $a_1, a_2, \ldots > 0$ with $\sum a_n^p \|g_n\|^p < \infty$. By Lemmas 3.2, 3.3(a) and Theorem 2.1(i), A^n and $A = \bigcup A^n$ are in

$\mathscr{F}'(T)$, and g_n and $f \equiv \sum a_n g_n$ are in $\mathscr{N}^+(T)$. For any $h \in \mathscr{N}(T)$, we have $\operatorname{supp} h \subset \operatorname{supp} f = A$. So $\theta_1 \equiv \operatorname{sgn} h$ and $\theta_2 \equiv (|h|/f)_A$ are $\mathscr{F}'(T)$-measurable, by Lemmas 3.2 and 3.3(b), and $h = \theta_1\theta_2 f$. This and Lemma 3.3(a) prove the first case. In general by the same principles and transfinite induction, we can find a maximal family of elements of $\mathscr{N}^+(T)\backslash\{0\}$ with disjoint supports $\in \mathscr{F}'(T)$. The general description follows. The last statement is an easy consequence. (The displayed subspace in the theorem is isometrically isomorphic to $L_p(\operatorname{supp} f, \mathscr{F}'(T) \cap \operatorname{supp} f, f^p\, d\mu)$.)

Let $\infty > p > 1$ and let P be a norm-one projection (positive if $p = 2$) on $\mathbf{E}$. $\operatorname{Ker}(I-P)$ has a structure [**1**, **3**] similar to that of $\mathscr{N}(T)$ given in Theorem 3.4, with $\mathscr{F}'(T)$ replaced by a differently defined sub-σ-ring $\mathscr{F}_1$ of $\mathscr{F}$; $\mathscr{F}_1$ consists of supports of functions invariant under P. The following implies that for positive operators, Theorem 3.4 generalizes this. See also Theorem 5.1(a); note that $|P|$ has norm 1.

PROPOSITION 3.5. *$\mathscr{N}(P) = \operatorname{Ker}(I-P)$ and $\mathscr{F}'(P) = \mathscr{F}'(P^*)$, which as a complete Boolean sub-ring is generated by $\mathscr{F}_1$.*

Proof. If $f \in \mathscr{N}(P)$, then applying Lemma 3.1 with $T = P$ to f and to Pf, we get $f = Pf$. So $\mathscr{N}(P) \subset \operatorname{Ker}(I-P)$. Equality follows. The rest follows from properties of conditional expectation operators and from P being, essentially, unitarily equivalent with one of them through a multiplication operator [**1**, **3**], in our setting with the underlying measure space being a direct sum of finite ones.

4. Norming vectors: $\infty > p > q \geq 1$.

THEOREM 4.1. *Let $\infty > p > q \geq 1$.*

(a) *Let $O \neq T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$ attain its norm. For an $f \in \mathscr{N}^+(T)$ with support $s(T)$,*

$$(4.1)\quad \mathscr{N}(T) = \{c\theta f \in \mathbf{E} : c \geq 0 \text{ is a scalar, } \theta \text{ is } \mathscr{F}'(T)\text{-measurable and } |\theta| = 1_{s(T)}\}.$$

(b) *Conversely when $(X, \mathscr{F}, \mu) = (Y, \mathscr{G}, \nu)$, given $0 \neq f \in \mathbf{E}^+$ and a sub-σ-ring $\mathscr{F}_1$ with largest element $\operatorname{supp} f$, there exists $T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$ of norm 1 such that $\mathscr{F}'(T) = \mathscr{F}'(T^*) = \mathscr{F}_1$ and (4.1) holds.*

Proof. (a) By Lemma 3.2, there is an $f \in \mathscr{N}^+(T)$ with support $s(T)$. Further, if $0 \neq h \in \mathscr{N}(T)$, then $g \equiv |h| \in \mathscr{N}^+(T)$ and $\theta \equiv \operatorname{sgn} h$ has support $s(T)$ and is $\mathscr{F}'(T)$-measurable. We show that $g = cf$ for a scalar $c > 0$, and thus $h = \theta g$ is in the prescribed set. We may assume $\|T\| = \|f\| = 1 = \|Tf\|$.

Case (1) $q = 1$. We have

$$1 + \|g\| = 1 + \|Tg\| = \|Tf + Tg\| \leq \|f + g\| \leq 1 + \|g\|,$$

whence $\|f + g\| = \|f\| + \|g\|$ and so $g = \|g\|f$.

Case (2) $q > 1$. Proceed as in the proof of Lemma 3.3(b), substituting q for p in the operands $(Tf)^{p-1}$ and $t^{p-2}\,dt$. We get $\|Tg\|^q \leq \int g^q f^{p-q}\,d\mu \leq \|g\|^q$ (Hölder's inequality for p/q). So equalities hold. Hence $g = \|g\|f$.

Conversely by Lemma 2.4(a), the prescribed set is included in $\mathscr{N}(T)$.

(b) By Proposition 3.5 and [**1**, Theorem 4], there is a positive norm-one projection P on $\mathbf{E}$ such that $\mathscr{F}'(P) = \mathscr{F}'(P^*) = \mathscr{F}_1$ and

$$\mathscr{N}(P) = \operatorname{Ker}(I - P) = \{\xi f \in \mathbf{E} \colon \xi \text{ is } \mathscr{F}_1\text{-measurable}\}.$$

Define $Tg = (f/\|f\|)^{p/q-1}Pg \quad (g \in \mathbf{E})$. The rest follows; cf. part (a), Case (2).

An n-dimensional l_p space (on counting measure) is denoted by $l_p(n)$.

THEOREM 4.2. *Let* $O \neq T \in \mathscr{L}(\mathbf{E}^+, \mathbf{F}^+)$. *Then* $\mathscr{N}^+(T)$ *is a closed convex cone if* $\infty > p \geq q \geq 1$. *It may not be so if* $1 \leq p < q < \infty$, *even when* $\mathscr{F}'(T)\backslash\{\varnothing\}$ *is a singleton.*

Proof. If $p > 1$ and $p \geq q$ the first result follows from Theorems 3.4 and 4.1(a), and if $p = q = 1$, from $\mathscr{N}^+(T) = \mathbf{E}^+_{\{T^*1 = \|T\|\}}$. Let $p < q$. Define $T\colon l_p(n) \to l_q(n)$ $(n \geq 2)$ by: $T1_x = 1_{\{x\}^c} + c1_x$, $c =$ constant. (Cf. [**13**, Example 3].) Let $f = (1, \dots, 1) \in l_p(n)$. Then $\|Tf\|/\|f\| = |n - 1 + c|/n^{1/p-1/q} \equiv \alpha(c)$ and $\|T\| \geq \|T(1, 0, \dots, 0)\| = (n - 1 + |c|^q)^{1/q} \equiv \beta(c)$. Choose $c \geq (n-1)/(n^{1/p-1/q} - 1)$. Then $\alpha(c) \leq c < \beta(c)$. So $f \notin \mathscr{N}(T)$. By symmetry of T, permuting the coordinates of a vector $\neq 0$ in $\mathscr{N}^+(T)$ gives like ones. Via summing up these permutants, we infer that $f \in \operatorname{conv}\mathscr{N}^+(T)$, and $\mathscr{N}^+(T)$ is not convex.

Let $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be disjunctive. Similar to [**8**, Theorems 4.1–4.2], for an $\mathscr{F}$-measurable function $D(T) \geq 0$ with support $s(T)$ and a $\mathscr{G}$-measurable h,

$$Tg = h\Phi^{\#}g \quad \text{and} \tag{4.2}$$
$$\|Tg\|^q = \int |h|^q \Phi^{\#}|g|^q \, d\nu = \int D(T)|g|^q \, d\mu \qquad (g \in \mathbf{E}).$$

THEOREM 4.3. *Let $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be disjunctive. Then when $\infty > p > q \geq 1$, T is norm-attaining and formula* (4.1) *holds with $f \equiv D$ and $D(T)^{1/(p-q)} \in \mathbf{E}$, for which* $\operatorname{supp} f = s(T)$. *When $1 \leq p \leq q < \infty$, T may not attain its norm.*

Proof. The result for $p > q$ follows from the second formula in (4.2) by Hölder's inequality for p/q. For $p \leq q$ take e.g. $T = \operatorname{diag}(1/2, 2/3, \ldots): l_p \to l_q$ and use Theorem 2.2 and Lemma 3.2(ii) (verify the sub-case $p = q$ directly).

5. Norming vectors of regular operators. An operator $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is *regular* if it has a *linear modulus* $|T|$ [**10**, §4], [**20**, Chapter 4]. It is *hyper-regular* if $|T|$ exists with norm $\|T\|$.

THEOREM 5.1. *Let $\infty > p, q \geq 1$ and let $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be hyper-regular and norm-attaining. Let $0 \neq f \in \mathscr{N}(T)$. Then $|f| \in \mathscr{N}^+(|T|)$ and*

$$Tg = \xi|T|(\zeta g) \quad \forall g \in \mathbf{E}_A, \tag{5.1}$$

where $A = \operatorname{supp} f$, $\zeta = \operatorname{sgn} \overline{f}$ and $\xi = \operatorname{sgn} Tf$. Furthermore:

(a) *When $p = q > 1$, there exists a family, reducible to a singleton if μ is σ-finite, $\{f_\alpha\} \subset \mathscr{N}(T)\backslash\{0\}$ with mutually disjoint $\operatorname{supp} f_\alpha \in \mathscr{F}'(T)$ such that*

$$\mathscr{N}(T) = \sum_{(p)} \{\theta f_\alpha \in \mathbf{E} \colon \theta \textit{ is } \mathscr{F}'(T) \cap \operatorname{supp} f_\alpha\textit{-measurable}\}$$

(*direct l_p-sum*) *and is itself a closed linear subspace of* $\mathbf{E}$ *isometrically isomorphic to an L_p space. Moreover, relation* (5.1) *holds for signum functions ζ and ξ with supports $A = \sup\{\operatorname{supp} f \colon f \in \mathscr{N}(T)\} \in \mathscr{F}'(T)$ and ΦA respectively.*

(b) *When $p > q$, formula* (4.1) *holds for an $f \in \mathscr{N}(T)$ with support $s(T)$.*

Proof. As $|Tf| \leq |T|\,|f|$, we have equality. So $|f| \in \mathscr{N}^+(|T|)$ and by Lemma 2.4(b), $T_{YA} = T_{BA}$. Let $S = \overline{\xi} T \circ \overline{\zeta}$. Then $|S| =$

$|\xi|\,|T|\circ|\zeta| = |T|_{BA} = |T|_{YA}$. So

$$S|f| = |Tf| = |T|\,|f| = |S|\,|f|.$$

As $|S| \geq (\operatorname{Re} S)^+$, this gives $-(\operatorname{Re} S)^-|f| = [|S| - (\operatorname{Re} S)^+]|f| = 0$. By Lemma 2.4(b), $(\operatorname{Re} S)^- = O = |S| - (\operatorname{Re} S)^+$. So $|S| = S$. This means (5.1).

We have $\mathscr{F}'(|T|) = \mathscr{F}'(T)$. When $p > q$, for the above f, $\operatorname{supp}|f| = s(|T|) = s(T)$ (Lemma 3.2(i)). By relation (5.1) and Theorem 4.1(a), result (b) follows. For (a), obtain a maximal family of $f_\alpha \in \mathscr{N}(T)\backslash\{0\}$ with disjoint $\operatorname{supp}|f_\alpha| \in \mathscr{F}'(|T|) = \mathscr{F}'(T)$ (Lemma 3.2 and above). The description of $\mathscr{N}(T)$ then follows from (5.1), Theorem 3.4 and p-additivity of the norms. Finally (5.1) holds with $\zeta = \sum \operatorname{sgn} \overline{f}_\alpha$, $\xi = \sum \operatorname{sgn} Tf_\alpha$ and $A = \sup\{\operatorname{supp} f_\alpha\}$.

REMARK 5.2. (i) For non-norm-attaining hyper-regular T, (5.1) may not hold for any signum functions $\xi, \zeta \neq 0$ with $\operatorname{supp}\zeta = A \in \mathscr{F}'(T)$. For $1 \leq p \leq q < \infty$, take $T\colon l_p \to l_q$ defined on each $f \in l_p$ by:

$$Tf(0) = f(0) - f(1) + f(2),$$
$$Tf(n) = f(n) + f(n+1) + f(n+2) \quad (n \geq 1).$$

Formally $|T| = I + S + S^2$ where $S =$ unit shift operator; each summand is bounded (use Theorem 2.2 if $p < q$). The assertion can then be easily verified.

(ii) All results on T (not involving T^*) valid for $p > q \geq 1$, e.g. Theorem 5.1(ii), extend to the case $p > 1 > q > 0$. The proofs adapt themselves readily. Thus, for Lemma 3.2(i), just replace q in the computations by 1. For Theorem 4.1(a), replace the "=" sign by "$\leq$" in the displayed relations in Case (1).

When $p = 1 \leq q$, each $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is hyper-regular [**10**, Remark 4.3(i)]. For $p = 1 = q$, Theorem 5.1 and the fact $\mathscr{N}^+(|T|) = \mathbf{E}^+_{\{|T|^*1 = \|T\|\}}$ amply describe $\mathscr{N}(T)$.

THEOREM 5.3. *Let $p = 1 < q < \infty$ and let $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ be norm-attaining. If $0 \neq f \in \mathscr{N}(T)$ and $A = \operatorname{supp} f$, then*

$$Tg = \langle g, \operatorname{sgn}\overline{f}\rangle Tf/\|f\| \quad \forall g \in \mathbf{E}_A, \quad \textit{and}$$
$$\mathscr{N}(T) \cap \mathbf{E}_A = \{ch.\operatorname{sgn} f\colon c \textit{ is a scalar and } h \in \mathbf{E}^+_A\}.$$

Proof. We may assume $\|T\| = \|f\| = 1$. Let A be decomposed into $B, C \in \mathscr{F}$. Then

$$\|Tf\| \leq \|Tf_B\| + \|Tf_C\| \leq \|f_B\| + \|f_C\| = 1.$$

Thus we have equalities. So $f_B \in \mathscr{N}(T)$ and as $\mathbf{F}$ is strictly convex,

$$Tf_B = \|Tf_B\|Tf = \|f_B\|Tf = \langle f_B, \operatorname{sgn}\overline{f}\rangle Tf.$$

By a routine process, we get the equation for T on $\mathbf{E}_A$. The rest follows.

Using [**18**, Theorem A2], Johnson and Wolfe [**7**, Proposition 4.2] showed that every $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ is norm-attaining if and only if $p > q$ and either (i) $p > 2$ and μ is (purely) atomic or (ii) $q < 2$ and ν is atomic. We extend this below (we could even take $p > 1 > q$ for the implications). In a similar attempt, for $\infty > p > q > 1$ and $T \in \mathscr{L}(l_p^+, l_q^+)$ Koskela [**12**, Theorem 1] indicated that there is an $f \in l_p^+$ with support $s(T)$ satisfying relation (3.1) with the " = " sign replaced by " ≤ ". A sharper result follows from Theorem 5.4, Lemmas 3.1 and 3.2(i).

THEOREM 5.4. *Let* $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ *be regular. Consider the statements*:

(a) μ *or* ν *is purely atomic*;
(b) T *is compact*;
(c) $\mathscr{N}(T) \neq \{0\}$.

If $\infty > p > q \geq 1$, *then* (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *If* $1 \leq p \leq q < \infty$, *then* (c) *may be false even if* T *is positive,* μ *and* ν *are purely atomic, and* $\mathscr{F}'(T)\backslash\{\varnothing\}$ *is a singleton.*

Proof. Let $p > q$. Assume (a) μ (resp. ν) (purely) atomic. For some increasing sequence of finite subsets $A^n \subset X$ (resp. $C^n \subset Y$), $\|T_{YA^n} - T\| = \|T_{Y(A^n)^c}\| \leq \|\,|T|_{Y(A^n)^c}\| \to 0$ (resp. $\|T_{C^nX} - T\| = \|T_{(C^n)^cX}\| \leq \|\,|T|_{(C^n)^cX}\| \to 0$), from which (b) follows as each T_{YA^n} (resp. T_{C^nX}) is of finite rank. When μ is atomic, the claim on $S = |T|$ follows if we choose A^n with $\|S_{YA^n}\| \to \|S\|$. Here we have used this fact: for any $A \in \mathscr{F}$ and $B = A^c$, with $r = pq/(p-q)$,

$$\|S\|^r \geq \|S_{YA}\|^r + \|S_{YB}\|^r. \tag{5.2}$$

To get (5.2), observe that for all unit vectors $f \in \mathbf{E}_A^+$ and $g \in \mathbf{E}_B^+$, $\|S\|^r \geq \|Sf\|^r + \|Sg\|^r$. This was shown in the proof of Lemma 3.2(i) when $Sf \neq 0$; else it is trivial. When ν is atomic, we choose C^n such that $\|S_{C^nX}\| \to \|S\|$ and use the dualized analog of (5.2): if $C \in \mathscr{G}$ and $D = C^c$, then with $r = pq/(p-q)$,

$$\|S\|^r \geq \|S_{CX}\|^r + \|S_{DX}\|^r.$$

This follows from (5.2) by considering S^* when $q > 1$. When $q = 1$, $\|S_{CX}\| = \|S^* 1_C\|$, which is r-super-additive on $C \in \mathscr{G}$. (Or: extend (5.2) to $p = \infty$, with $r = q$. Then use dualization.)

Assume (b). For some unit vectors $f_n \in \mathbf{E}$ $(n \geq 1)$, $\|Tf_n\| \to \|T\|$. As $\mathbf{E}$ is reflexive, a subsequence f_{n_k} (weakly) $\to f \in \mathbf{E}$ with $\|f\| \leq 1$ and $Tf_{n_k} \to Tf$ in norm. Hence $\|Tf\| = \|T\|$ and so $\|f\| = 1$. This gives (c).

When $1 < p \leq q$, $T\colon l_p \to l_q$ defined by $Tf(n) = f(n) + f(n+1)$, $n = 0, 1, \ldots$ $(f \in l_p)$, is bounded; cf. Remark 5.2(i). As $\|Tf\| < \|T(0, f(0), f(1), \ldots)\| \leq \|T\| \cdot \|f\|$ if $f(0) \neq 0$, $\mathscr{N}(T) = \{0\}$ by Lemma 3.2. When $1 = p \leq q$, $T\colon l_1 \to l_q(1)$ given by $Tf = f(0)/2 + 2f(1)/3 + 3f(2)/4 + \cdots$ $(f \in l_1)$ has norm 1 and $\mathscr{N}(T) = \{0\}$.

6. Norming vectors and extreme contractions. Theorem 6.1 extends Lemma 3.2. We could allow for $0 < p$, $q \leq \infty$ and quaternion scalars; the extended proofs involve modification of inequalities (6.1). (To extend (b) and (c) for $q < 1$, we replace the integral term in (6.1) by one (< 0) of order $o(t^q)$. To extend (a) for $q < 1$ and scalar field not the reals, utilize average$\{H(h\zeta)\colon |\zeta| = 1\}$ instead of $H(h)$ and obtain lower bounds 0 $(|h| \geq 1)$ and $K|h|^2$ $(|h| < 1)$, $K > 0$. We leave details to the interested reader. For parts (b) and (c), note that

$$\begin{aligned} &\int_B \{|Tf + tTg|^q + |Tf - tTg|^q - 2|Tf|^q\}\, d\nu \\ &\quad \geq \int_B \{(|Tf| + t|Tg|)^q + \big||Tf| - t|Tg|\big|^q - 2|Tf|^q\}\, d\nu \\ &\quad \geq -\int_{(B^t)^c \cap B} |Tf|^q d\nu - \int_{B^t} \psi_t \, d\nu = o(t^q). \end{aligned}$$

Here $B^t = \{t|Tg| < |Tf|\}$ and $\psi_t = |Tf|^q - (|Tf| - t|Tg|)^q$. Observe that $1_{B^t}\psi_t/t^q$ converges a.e. to 0 as $t \to 0+$ and is majorized by $|Tg|^q$ since $a^q/t^q - (a/t - b)^q$ increases with $t \in (0, a/b)$ for $a, b > 0$, to b^q at a/b.)

THEOREM 6.1. *Suppose* $1 \leq p, q < \infty$, $O \neq T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ *and* $0 \neq f \in \mathscr{N}(T)$. *Let* $A = \operatorname{supp} f$ *and* $B = \operatorname{supp} Tf$. *Then*:

(a) *if* $p > 2$, $T_{BA^c} = O$,

(b) *if* $p > q$, $T_{B^cA^c} = O$,

(c) *if* $q < 2$ *or* $p = 1$, $T_{B^cA} = O$,

except that the sub-case (a_1) $p > 2$ *and* $q = 1$ *of* (a) *may fail if the scalar field is the reals. Furthermore the indicated ranges of* (p, q) *are in general optimal* (*broadest possible*).

Proof. Let $H(h) = |1+h|^q + |1-h|^q - 2$ (h = scalar). When $q \geq 2$, simple calculus gives $H(h) \geq H(i|h|) = 2(1+|h|^2)^{q/2} - 2 \geq q|h|^2$, via the mean value theorem. When $2 > q \geq 1$, $H(h) \geq H(|h|)$. Hence $H(h) \geq H(1) = 2^q - 2 \geq 0$ ($|h| \geq 1$) and $H(h) \geq q(q-1)|h|^2$ ($|h| < 1$), via Taylor's formula.

We may assume $\|T\| = \|f\| = 1 = \|Tf\|$. Consider any $g \in \mathbf{E}_{A^c}$. Let $t > 0$. Apply the inequalities for $H(h)$ to $h = tTg/Tf$ on B, multiply by $|Tf|^q$ and integrate the result. Then add $2t^q\|1_{B^c}Tg\|^q$. We get

$$\begin{aligned}(6.1)\qquad 2t^q\|1_{B^c}Tg\|^q &+ Kt^2\int_{B^t}|Tf|^{q-2}|Tg|^2\,d\nu \\ &\leq \|Tf+tTg\|^q + \|Tf-tTg\|^q - 2 \\ &\leq \|f+tg\|^q + \|f-tg\|^q - 2 \\ &= 2(1+t^p\|g\|^p)^{q/p} - 2 \\ &= O(t^p) \quad \text{as } t \to 0+ .\end{aligned}$$

Here (K, B^t) is (q, B) $(q \geq 2)$ or $(q(q-1), \{t|Tg| < |Tf|\})$ $(1 \leq q < 2)$. Hence $1_{B^c}Tg = 0$ when $p > q$ and $1_BTg = 0$ when $p > 2$, $q > 1$. This proves (b) and for $q > 1$, (a).

Let $q = 1$. Let ζ be any complex scalar with $|\zeta| = 1$ and s any real number. Let $D(s, \zeta) = H(s\zeta)$. Then $D(0, \zeta) = D_s(0, \zeta) = 0$ and for $0 < s < 1$,

$$\begin{aligned}D_{ss}(s, \zeta) &= (|1+s\zeta|^{-3} + |1-s\zeta|^{-3})|\mathrm{Im}\,\zeta|^2 \\ &\geq 2(1+s^2)^{-3/2}|\mathrm{Im}\,\zeta|^2 \\ &\geq |\mathrm{Im}\,\zeta|^2/\sqrt{2}.\end{aligned}$$

Hence for $|h| < 1$, $H(h) \geq 2^{-3/2}|\mathrm{Im}\,h|^2$, by Taylor's formula. With this new estimate the method above gives (6.1), now with $K = 2^{-3/2}$, B^t unchanged and Tg in the integrand replaced by $\mathrm{Im}(Tg \cdot \mathrm{sgn}\,\overline{Tf})$. Hence the last is 0 when $p > 2$. We may replace g by ig. So $1_BTg = 0$. This proves (a_1) for complex scalars.

To prove (c), take $g = f_C$, with any $\varnothing \neq C \in \mathcal{F} \cap A$, instead. Then the inequalities (6.1) hold with the last two lines replaced by

$$= \sum_{z=\pm 1}(1 + \|f_C\|^p(|1+zt|^p - 1))^{q/p} - 2 = O(t^2) \qquad (t \to 0+).$$

So $1_{B^c}Tf_C = 0$ if $2 > q \geq 1$. Hence $T_{B^cA} = O$. If $p = 1 \leq q$, then T is hyper-regular [**10**, Remark 4.3(i)]. So the same result follows from equation (5.1).

Now the optimality. For (b), $T = \operatorname{diag}(1, 1)\colon l_p(2) \to l_q(2)$ is a nonexample if $p \le q$ (use Lemma 3.2(ii) if $p < q$). Optimality for (c) follows from that for (a), by Lemma 3.1. For (a), let $T\colon l_p(2) \to l_q(2)$ be defined by $T(x, y) = (x + ty, x - ty)$, with $t > 0$. Let $p \le 2 < q$. With $r = |y|$ we have

$$\|T(1, y)\|/\|(1, y)\| \le [(1 + tr)^q + |1 - tr|^q]^{1/q}/(1 + r^p)^{1/p} \le 2^{1/q}$$

for some $t \in (0, 1]$. For the last inequality notice that the middle expression is strictly increasing, to ∞, in $t \ge 0$ for each $r > 0$. So it equals $2^{1/q}$ for a unique $t = s(r) > 0$ for each $r > 0$ and the said inequality holds for $t = \inf s(r)$. We have $0 < t \le 1$ as $s(\infty) = 1$, $0 < s(0+)$ ($= \infty$ if $p < 2$ or $1/\sqrt{q-1}$ if $p = 2$) and $s(r)$ is continuous (implicit function theorem). When $p, q \le 2$, let $t = 1$. Then

$$\|T(1, y)\|/\|(1, y)\| \le 2^{1/q}(1 + |y|^2)^{1/2}/(1 + |y|^p)^{1/p} \le 2^{1/q}.$$

In either case, $\|T\| = 2^{1/q}$, $(1, 0) \in \mathcal{N}(T)$ but $\operatorname{supp} T(0, 1) = \operatorname{supp} T(1, 0)$. So the range $p > 2$ is optimal in (a). Let now the scalars be the reals and assume (a_1). Take $t = 1$. Then $\|T(x, y)\| = 2\max\{|x|, |y|\}$. The conclusions as before follow. So the result (a) may fail in the sub-case (a_1) for real scalars.

In the proof below the analysis is similar to the case $\infty > p = q > 2$ given in [**11**, §2].

LEMMA 6.2. *Suppose* $2 \le q < p < \infty$, $0 < a, b < 1$, *and* $a^p + b^p = 1$. *There is a unique* $t = t(a)$ *in* $(0, 1)$ *such that the operator* $\tau\colon l_p(2) \to l_q(2)$ *defined by* $\tau(x, y) = (a^{p-1}x + b^{p-1}y, t(bx - ay))$ *has norm* 1 *and also has two distinct directions of isometry, one of which is* (a, b). *Moreover,* $t(a)$ *is continuous in* a.

Proof. For $r \ge 0$ and $|\zeta| = 1$, define $f(r, \zeta) = (b^{p-1}, -a^{p-1}) + r\zeta(a, b)$. Then for a given positive t, $\|\tau f(r, \zeta)\|^q = \|(r\zeta, t)\|^q = r^q + t^q$. Since $\|(a, b)\| = 1 = \|\tau(a, b)\|$, we have $\|\tau\| = 1$ provided $\Delta \equiv \|f(r, \zeta)\|^q - r^q \ge t^q$. Equality must hold for some (r, ζ) in order that τ be isometric in another direction. The problem therefore is equivalent to proving that $\min\{\Delta\colon r \ge 0,\ |\zeta| = 1\}$ exists, lies in $(0, 1)$ and is continuous in a; $t(a)$ then is $(\min \Delta)^{1/q}$ and is unique.

When $q = 2$, let $t = (ab)^{p/2-1}$. Then

$$\tau = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \operatorname{diag}(a^{p/2-1}, b^{p/2-1}),$$

where $\alpha = a^{p/2}$ and $\beta = b^{p/2}$. The first factor is an $l_2(2)$ isometry. The second, by Theorem 4.3, is a norm-one operator from $l_p(2)$ to $l_2(2)$ isometric on $(a, \theta b)$ ($|\theta| = 1$). So, directly, we get this t as the one required.

Let $q > 2$. If we write $c = a^p$, $d = b^p$, $u = abr$ and $z = \operatorname{Re}\zeta$, then

$$\begin{aligned}\Delta &\equiv \Delta(u, z)\\ &= \{(u^2 + 2zud + d^2)^{p/2}/d + (u^2 - 2zuc + c^2)^{p/2}/c\}^{p/q} - u^q/(ab)^q.\end{aligned}$$

Assume $a > b$. Simple calculus shows that for a fixed $u \geq u_0 \equiv (c-d)/2$, Δ is minimized at $z = u_0/u$ to

$$\Delta(u, u_0/u) = [(u^2 + cd)^{q/2} - u^q]/(ab)^q,$$

which for these u, is in turn minimized at $u = u_0$ to $\Delta(u_0, 1)$. On the other hand for $0 < u < u_0$, Δ is a decreasing function on $z \in [-1, 1]$ while $\Delta(0, z) \equiv \Delta(0, 1)$. These imply that Δ has a minimum, which is attained on the compact subset $[0, u_0] \times \{1\}$. Now if $0 < u \leq u_0$, then $\Delta_u(u, 1) = qu^{q-1}U(u)$, where

$$U(u) = \frac{(d/u + 1)^{p-1}/d - (c/u - 1)^{p-1}/c}{\{(d/u + 1)^p/d + (c/u - 1)^p/c\}^{1-q/p}} - (ab)^{-q}.$$

U is an increasing function (the numerator in the fraction is increasing and the denominator, decreasing), changing from $-\infty$ at $0+$ to

$$[(c-d)^{-(q-2)} - 1]/(ab)^q > 0$$

at u_0. It follows that Δ attains a strict minimum at $(w, 1)$, with w uniquely defined by $0 < w < u_0$ and $U(w) = 0$. From this, if we write $W = (d+w)^p/d + (c-w)^p/c$, we get

$$\begin{aligned}\min\Delta &= W^{q/p} - w \cdot w^{q-1}/(ab)^q\\ &= \{(d+w)^{p-1} + (c-w)^{p-1}\}/W^{1-q/p} > 0.\end{aligned}$$

Also, the last equation implies that $W >$ the last numerator. Hence

$$\begin{aligned}\min\Delta &< \{(d+w)^{p-1} + (c-w)^{p-1}\}^{q/p}\\ &< \{(d+w) + (c-w)\}^{q/p} = 1.\end{aligned}$$

By symmetry, similar results hold when $a < b$. When $a = b$, the argument gets simplified (change u_0 to 0) and we have $\min\Delta = \Delta(0, 1) = 2^{-(1-2/p)q} \in (0, 1)$. The continuity of $\min\Delta$ is now an easy consequence of all these. (Use implicit function theorem on $U(w) = 0$ when $c > d$ to obtain continuity of w in a, etc.)

THEOREM 6.3. *Suppose* $\infty > p > q > 1$. *Let* $T \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ *be of norm* 1.

(a) *When* $p > 2$, *if the norm closed linear span of* $\mathscr{N}(T)$ *is* $\mathbf{E}_A$ *for some* $\emptyset \neq A \in \mathscr{F}$, *then* T *is an extreme point of the unit ball of* $\mathscr{L}(\mathbf{E}, \mathbf{F})$ *and* $A = s(T)$.

(b) *Suppose that* T *is disjunctive. When* $p > 2$, *it is extreme. When* $p \leq 2$, *it is extreme if and only if either* $s(T) = X$ *or* T^* *is also disjunctive.*

Proof. (a) Let $R \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ with $\|T \pm R\| \leq 1$. As $\mathbf{F}$ is strictly convex, $R = O$ on $\mathscr{N}(T)$, whence on $\mathbf{E}_A$. By Theorem 6.1(a) (b), $T_{YA^c} = R_{YA^c} = O$. So $A = s(T)$, $R = O$ and T is extreme.

(b) By Theorem 4.3, $\overline{\operatorname{span}}\,\mathscr{N}(T) = \mathbf{E}_{s(T)}$. So T is extreme if $p > 2$, by (a), or if $s(T) = X$, by the argument in (a). Let $p \leq 2$. If T^* is disjunctive, then T^*, and so T, is extreme by the case $p > 2$.

Assume $s(T) \neq X$ and T^* not disjunctive. There is $B \in \mathscr{G} \cap \Phi s(T) \backslash \Phi\mathscr{F}$ (Theorem 2.1). Let ΦA, $A \in \mathscr{F} \cap s(T)$, be its $\Phi\mathscr{F}$-measurable cover. We may assume the like cover of $B' \equiv \Phi A \backslash B$ to be also ΦA (else intersect B with this cover to get a new B). With notation as in formulas (4.2), $\eta \equiv D(T_{YA}) = D(T)_A$. There are disjunctive $U, V \in \mathscr{L}(\mathbf{E}, \mathbf{F})$ such that $D(U) = D(V) = \eta$, $T_{BA} = U \circ \xi$ and $T_{B'A} = V \circ \zeta$ for $\mathscr{F}$-measurable functions $\xi, \zeta \geq 0$ with support A. By (4.2), $\xi^q + \zeta^q = 1_A$. By Lemma 6.2 and taking dual there is $t(a) \in (0, 1)$ continuous on $a \in (0, 1)$ such that with $b = (1 - a^q)^{1/q}$ the operator $(x, y) \mapsto (a, b)x + t(a)(b^{q-1}, -a^{q-1})y$ from $l_p(2)$ to $l_q(2)$ has norm 1. Take unit vectors $u \in \mathbf{E}'_{o(T)}$ and $g \in \mathbf{E}_A$. Define $Wf = \langle f, u\rangle g$ $(f \in \mathbf{E})$. Then $\|W\| = 1$ and

$$O \neq R \equiv (U \circ t(\xi)\zeta^{q-1} - V \circ t(\xi)\xi^{q-1}) \circ W \in \mathscr{L}(\mathbf{E}, \mathbf{F}).$$

Let $A' = s(T) \backslash A$. If $f \in \mathbf{E}$, then

$$\begin{aligned}
\|Tf \pm Rf\|^q &= \int (|\xi f \pm t(\xi)\zeta^{q-1} Wf|^q + |\zeta f \mp t(\xi)\xi^{q-1} Wf|^q)\eta \, d\mu \\
&\quad + \|Tf_{A'}\|^q \\
&\leq \int (|f|^p + |Wf|^p)^{q/p} \eta \, d\mu + \|Tf_{A'}\|^q \\
&= \|T(|f_A|^p + |Wf|^p)^{1/p}\|^q + \|Tf_{A'}\|^q \\
&= \|T[(|f_A|^p + |Wf|^p)^{1/p} + f_{A'}]\|^q \\
&\leq (\|f_A\|^p + \|Wf_{o(T)}\|^p + \|f_{A'}\|^p)^{q/p} \\
&\leq \|f\|^q .
\end{aligned}$$

Hence $\|T \pm R\| \leq 1$. So T is not extreme.

REMARK 6.4. (i) Let $p = q > 1$ or $p > q \geq 1$. If T is not hyperregular Theorem 5.1(a) (b) may fail. Even when $\mathscr{F}'(T) = \{\varnothing, s(T)\}$, $\mathscr{N}(T)$ may not be linear if $(p, q) \neq (2, 2)$, and if $p = q = 2$ it is linear, being $\operatorname{Ker}(T^*T - \|T\|^2 I)$ (use Hilbert space adjoint) by Lemma 3.1, but may not be as given in Theorem 5.1(a) with $\mathscr{F}'(T)$ replaced by any sub-σ-ring of $\mathscr{F}$. Take T in the proof of Theorem 4.2. Use its notation, with $n \geq 3$ and $c < 0$ close enough to $-(n-1)$ so that $\alpha(c) < \beta(c)$. So $(1, \ldots, 1) \notin \mathscr{N}(T)$. Take $0 \neq g \in \mathscr{N}(T)$. If no coordinate of g is 0, interchange any two with unequal values. Else swap a zero with a non-zero one. We get $g' \in \mathscr{N}(T)$ not a scalar multiple of g. $\mathscr{S} \equiv \operatorname{span}\{g, g'\}$ contains a vector $\neq 0$ with a zero coordinate. So does $T\mathscr{S}$, as $\operatorname{Ker} T = \{0\}$. So $\mathscr{N}(T)$ is not linear if $p > 2$ and $q > 1$, by Theorem 6.1, part (a), or if $q < 2$, by its part (c). This proves the claim for $(p, q) \neq (2, 2)$. Finally the orthoprojector P from $l_2(3)$ onto $\mathscr{S} = \operatorname{span}\{(1, 1, 1), (1, 0, -1)\}$ has a 2-dimensional $\mathscr{N}(P) = \mathscr{S}$ not containing any coordinate vector. Our claim for $p = q = 2$ follows.

(ii) Consider complex scalars and $T\colon l_4(2) \to l_4(3)$ defined by $T(x, y) = (1, 1, 1)x + (e^{i\pi/3}, e^{-i\pi/3}, -1)y$. (This example originates from a perturbation of [**10**, Example 7.1], up to a scalar factor.) For $r \geq 0$ and ζ with $|\zeta| = 1$, routinely we get

$$9\|(1, r\zeta)\|^4 - \|T(1, r\zeta)\|^4 = 6(1 - r^2)^2.$$

Thus T is not a scalar multiple of an isometry, $\mathscr{F}'(T)\backslash\{\varnothing\}$ is a singleton which is finite, but T has infinitely many norming directions: $(1, e^{i\theta})$. These remain true for quaternion scalars (add $6r^2[(\operatorname{Re}\zeta j)^2 + (\operatorname{Re}\zeta k)^2]$ to the R.H.S. of the equation). Such a phenomenon does not seem to occur in real spaces.

Note. In the case of σ-finite measures, Theorems 3.4 and 4.1(a) were presented (in "Norm-attaining vectors of operators on L_p spaces") at the International Mathematical Conference [**23**] held at National University of Singapore, Singapore, June 1–13, 1981. Some of the results, among other things, are contained in the author's unpublished manuscripts *On norming vectors and norm structures of linear operators between L_p spaces*, I, II, Nat. Univ. of Singapore Mathematics Research Report nos. 151, 171 (1984).

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