TRACEABLE INTEGRAL KERNELS
ON COUNTABLY GENERATED MEASURE SPACES

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Let K be a trace class operator on $L^2(X, \mathcal{M}, \mu)$ with integral kernel $K(x, y) \in L^2(X \times X, \mu \times \mu)$. An averaging process is used to define K on the diagonal in $X \times X$ so that the trace of K is equal to the integral of $K(x, x)$, generalizing results known previously for continuous kernels. This formula is also shown to hold for positive-definite Hilbert-Schmidt operators, thus giving necessary and sufficient conditions for the traceability of positive integral kernels. These results make use of Doob's maximal theorem for martingales and generalize previous results obtained by the author using Hardy-Littlewood maximal theory when $X \subset \mathbf{R}^n$.

1. Introduction. Let K be a compact operator on a Hilbert space, **H**. The positive operator K^*K can be diagonalized by an orthonormal sequence $\{\psi_i\}_{i \in \mathbf{N}}$ of eigenvectors with corresponding eigenvalues $\mu_i > 0$. Define $\lambda_i = \sqrt{\mu_i}$ and $\phi_i = \lambda_i^{-1} K \psi_i$. The numbers λ_i, the eigenvalues of $|K|$, are called the *singular values* of K, and the sequence $\{\phi_i\}_{i \in \mathbf{N}}$ is also orthonormal. K is a Hilbert-Schmidt operator if the singular values are square-summable and a trace class operator if they are absolutely summable. Since $\ker(K) = \ker(K^*K)$, we have

$$(1.1) \qquad \mathrm{cl\,span}\{\psi_i\}_{i \in \mathbf{N}} = \ker(K^*K)^\perp = \ker(K)^\perp.$$

Let $\phi \otimes \overline{\psi}$ denote the rank 1 operator

$$\phi \otimes \overline{\psi} : \theta \mapsto (\theta, \psi)\phi.$$

By expanding $\theta \in \ker(K)^\perp$ in terms of the basis $\{\psi_i\}_{i \in \mathbf{N}}$, we see that

$$K\theta = \sum_{i=1}^{\infty} \lambda_i(\theta, \psi_i)\phi_i.$$

Thus, K has the canonical expansion

$$(1.2) \qquad K = \sum_{i=1}^{\infty} \lambda_i \phi_i \otimes \overline{\psi}_i,$$

norm-convergent in $\mathscr{L}(H)$.

We shall be interested in operators on the Hilbert spaces $L^2(X, \mu)$. In this setting, every Hilbert-Schmidt operator, K, on $L^2(X)$ is given by an integral kernel $K(x, y) \in L^2(X \times X, \mu \times \mu)$:

$$(1.3) \qquad Kf(x) = \int_X K(x, y) f(y) \, d\mu(y).$$

The canonical expansion (1.2) corresponds to an orthonormal expansion of the kernel,

$$(1.4) \qquad K(x, y) = \sum \lambda_i \phi_i(x) \overline{\psi_i(y)},$$

convergent in $L^2(X \times X)$.

In a previous paper [1], it was shown that the ubiquitous trace formula

$$(1.5) \qquad \operatorname{tr} K = \int_X K(x, x) \, d\mu(x)$$

can be extended for arbitrary trace class operators with no continuity assumptions on the kernel function when μ is Lebesgue measure on a set $X \subset \mathbf{R}^n$. Specifically, the diagonal values are defined by averaging the kernel over cubes centered on the diagonal in $\mathbf{R}^{2n}$ and letting the radius of the cubes go to zero. That this limit exists almost everywhere on the diagonal and that the integral of the resulting function furnishes the trace of K follow from the boundedness on $L^2(\mathbf{R}^n)$ of the Hardy-Littlewood maximal operator. These results establish a certain degree of function-theoretic smoothness for traceable integral kernels and strengthen the analogy between noncommutative integration in $\mathscr{L}(\mathbf{H})$ and classical integration theory. They also provide completely general necessary and sufficient conditions for traceability of positive-definite integral operators.

The present paper extends these results from $\mathbf{R}^n$ to a general measure-theoretic setting. We shall obtain all the results in [1] with $\mathbf{R}^n$ replaced by a countably generated measure space, $(X, \mathscr{M})$, and Lebesgue measure by a σ-finite measure on $\mathscr{M}$. The canonical expansion (1.2) has been introduced to accommodate an eventual extension of the results of §3 to include nuclear operators on the Banach spaces $L^p(X)$. The diagonal values of an arbitrary trace class kernel, K, will be defined by an averaging process, with pointwise values denoted by $\widetilde{K}(x, x)$. We will then prove the analogue of (1.5):

$$\operatorname{tr} K = \int \widetilde{K}(x, x) \, d\mu(x).$$

Conversely, we get the following generalization of Mercer's theorem: if $P \geq 0$ and $\tilde{P}(x, x)$ exists for almost every x with

$$(1.6) \qquad\qquad \int \tilde{P}(x, x) \, d\mu(x) < \infty,$$

then P is traceable. Traceability also follows if the integrand in (1.6) is replaced by the corresponding maximal function, $MP(x, x)$, generalizing a result of J. Weidmann.

The key device in these proofs, a result from probability theory that replaces the Hardy-Littlewood theorem, is Doob's maximal theorem for martingales. The method of proof employed in [1] will be reformulated in a purely measure-theoretic way that does not depend on the geometric structure of Euclidean space. As an immediate corollary, we shall obtain a new proof of a result of M. Duflo ([3], Théorème V.3.1.1): if $K(x, y)$ is a continuous trace class kernel on a LCH space then (1.5) holds. Our approach avoids the "analytic" hypotheses behind Duflo's proof; specifically, that X be locally compact and σ-compact and that μ be a Radon measure. We shall only require that the topology on X be countably generated and that μ be σ-finite on Borel sets.

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2. Martingales. Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space and $\{\mathscr{S}_n\}_{n \in \mathbf{N}}$ a sequence of sub-σ-algebras increasing to $\mathscr{M}$:

$$\mathscr{S}_n \subset \mathscr{S}_{n+1} \quad \text{and} \quad \mathscr{M} = \bigvee_{n \in \mathbf{N}} \mathscr{S}_n.$$

Assume that μ is σ-finite on each $\mathscr{S}_n$ so that, for $f \in L^p(X)$, the conditional expectations $E(f|\mathscr{S}_n)$ exist. A sequence $\{f_n\}_{n \in \mathbf{N}}$ of functions on X is a *martingale* if each f_n is $\mathscr{S}_n$-measurable and

$$(2.1) \qquad\qquad E(f_n|\mathscr{S}_j) = f_j, \quad \text{whenever } j < n.$$

Of particular interest to us are martingales generated by a single $\mathscr{M}$-measurable function, f, with $f_n = E(f|\mathscr{S}_n)$. The *maximal function* of such a martingale is

$$(2.2) \qquad\qquad Mf(x) = \sup_{n \in \mathbf{N}} E(|f||\mathscr{S}_n)(x).$$

There is a boundedness theorem for the martingale maximal operator analogous to the Hardy-Littlewood maximal theorem. The original

reference is Doob ([**2**], Theorem VII.3.2); for treatments geared more for our purposes, see Stein ([**8**], Theorem IV.1.6), or Edwards and Gaudry ([**4**], Theorem 5.2.7).

THEOREM 2.1. (*Doob's Martingale Maximal Theorem.*) *The maximal operator, M, for martingales is weak-type $(1, 1)$ on $\mathrm{L}^1(X)$ and bounded on $\mathrm{L}^p(X)$ for $1 < p \le \infty$.*

The fact we shall need in §3 is that the martingale maximal operator is bounded on $\mathrm{L}^2(X)$:

$$\|Mf\|_2 \le C_2\|f\|_2 \quad \text{for all } f \in \mathrm{L}^2(X).$$

The analogue of Lebesgue's differentiation theorem is:

THEOREM 2.2. (*Doob's Martingale Convergence Theorem.*) *For each $f \in \mathrm{L}^p(X)$, $1 \le p \le \infty$, the sequence $\mathrm{E}(f|\mathscr{S}_n)$ converges to f a.e. $[\mu]$.*

We shall construct sub-σ-algebras for which the conditional expectations $E(f|\mathscr{S}_n)$ are given by averaging f over sets in $\mathscr{S}_n$. For two partitions $\mathscr{P}_i$, $\mathscr{P}_j$ of X, the notation $\mathscr{P}_i \prec \mathscr{P}_j$ will mean that $\mathscr{P}_i$ is a refinement of $\mathscr{P}_j$. Suppose $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ is a sequence of partitions of X into sets of finite measure such that $\mathscr{P}_{n+1} \prec \mathscr{P}_n$. Let $\mathscr{S}_n = \sigma(\mathscr{P}_n)$, and suppose further that $\mathscr{M} = \bigvee_n \mathscr{S}_n$. For each $x \in X$ and each $n \in \mathbb{N}$ there is a unique set $C_n(x) \in \mathscr{P}_n$ containing x. The conditional expectation of f with respect to $\mathscr{S}_n$ is

$$(2.3) \qquad \mathrm{E}(f|\mathscr{S}_n)(x) = \frac{1}{\mu(C_n(x))} \int_{C_n(x)} f \, d\mu,$$

whenever $\mu(C_n(x)) \ne 0$. Assume that each $\mathscr{P}_n$ is a countable partition of X and let $\mathscr{N}$ be the set of all x for which $\mu(C_n(x)) = 0$ for some n (and thus for all $m > n$, since $C_{n+1}(x) \subset C_n(x)$). Then $\mu(\mathscr{N}) = 0$, and for all $x \in \mathscr{N}^c = X\setminus\mathscr{N}$ the formula (2.3) holds.

With these hypotheses we can define an averaging process analogous to the Hardy-Littlewood process:

$$(2.4) \qquad A_n f(x) = \mathrm{E}(f|\mathscr{S}_n)(x)$$
$$= \frac{1}{\mu(C_n(x))} \int_{C_n(x)} f \, d\mu, \quad \text{for } x \in \mathscr{N}^c.$$

It follows that the operators A_n convert L^p convergence into pointwise convergence: if $\sum f_i$ converges to f in $\mathrm{L}^p(X)$ then at each

$x \in \mathcal{N}^c$

$$(2.5) \qquad \sum A_n f_i(x) = A_n f(x).$$

On $X \times X$ we have induced partitions $\mathscr{P}_n \times \mathscr{P}_n$, with each point $(x, y) \in C_n(x) \times C_n(y)$ and $\mathscr{P}_{n+1} \times \mathscr{P}_{n+1} \prec \mathscr{P}_n \times \mathscr{P}_n$. The set $\mathcal{N}^c \times \mathcal{N}^c$ is conull with respect to product measure, and note that almost all of the diagonal lies in $\mathcal{N}^c \times \mathcal{N}^c$:

$$(2.6) \qquad (x, x) \in \mathcal{N}^c \times \mathcal{N}^c \quad \text{a.e. } [\mu].$$

Let $A_n^{(2)}$ denote the averaging operators on $X \times X$. When $(x, y) \in \mathcal{N}^c \times \mathcal{N}^c$,

$(2.7) \quad A_n^{(2)} f(x, y)$

$$= \frac{1}{\mu(C_n(x))\mu(C_n(y))} \int_{C_n(x)} \int_{C_n(y)} f(s, t) \, d\mu(t) \, d\mu(s).$$

Applying (2.7) to elementary tensors,

$$(2.8) \qquad A_n^{(2)}(\phi \otimes \psi)(x, y) = A_n \phi(x) A_n \psi(y),$$

for all $(x, y) \in \mathcal{N}^c \times \mathcal{N}^c$. Consequently,

LEMMA 2.3. *The martingale maximal operator is submultiplicative and subadditive: if* $(x, y) \in \mathcal{N}^c \times \mathcal{N}^c$ *then*

$$M^{(2)}(\phi \otimes \psi)(x, y) \leq M\phi(x) M\psi(y)$$

and

$$M(\phi + \psi)(x) \leq M\phi(x) + M\psi(x).$$

Now let $\mathscr{U} = \{U_n\}_{n \in \mathbb{N}}$ be a countable base for $\mathscr{M}$, and use it to define a sequence of partitions on X. For $\mathscr{P}_0$ take a countable partition of X into sets of finite measure, and define $\mathscr{P}_n$ inductively:

$$(2.9) \qquad \mathscr{P}_n = \{U_n, U_n^c\} \cap \mathscr{P}_{n-1}, \qquad n \in \mathbb{N}.$$

The $\cap$ in (2.9) means "all pairwise intersections." This is an increasing sequence of partitions, $\mathscr{P}_{n+1} \prec \mathscr{P}_n$, so $\mathscr{S}_n = \sigma(\mathscr{P}_n)$ is an increasing sequence of sub-σ-algebras of $\mathscr{M}$, and $\mathscr{M} = \bigvee_n \mathscr{S}_n$. For $f \in \mathrm{L}^p(X)$ define

$$(2.10) \qquad \tilde{f}(x) = \lim_{n \to \infty} A_n f(x).$$

Then

$$\tilde{f}(x) = f(x) \quad \text{a.e. } [\mu]$$

by Theorem 2.2. Call $x \in \mathcal{N}^c$ a *regular point* for f if the limit (2.10) exists. Note that the set Y_f of regular points for f is conull in X. Y_f should be thought of as the "Lebesgue set" of f with respect to the measure μ.

If X is a second-countable topological space we shall let $\mathcal{U}$ be a countable base for the topology on X. The σ-algebras $\sigma(\mathcal{P}_n)$ will then converge to the Borel σ-algebra. Because the averaging process for this martingale is based on the topology of X, we have the following result:

THEOREM 2.4. *Let μ be a σ-finite Borel measure on a second-countable space, X. If $f \in L^p(X, \mu)$, $1 \leq p \leq \infty$, then each point of continuity $x \in \mathcal{N}^c$ is a regular point for f, with $\tilde{f}(x) = f(x)$.*

The same result holds in $X \times X$ with regard to the process (2.7). Because almost all of the diagonal lies in $\mathcal{N}^c \times \mathcal{N}^c$, if $f(x, y)$ is continuous almost everywhere along the diagonal then

$$(2.11) \qquad A_n^{(2)} f(x, x) \to f(x, x) \quad \text{a.e. } [\mu],$$

even though the diagonal is null with respect to product measure. Without any continuity hypotheses, we only have almost everywhere existence with respect to product measure:

$$(2.12) \qquad \tilde{f}(x, y) = \lim_{n \to \infty} A_n^{(2)} f(x, y) \quad \text{a.e. } [\mu \times \mu].$$

Of course, if μ is a Borel measure on $\mathbf{R}^m$ then the construction (2.9) is unnecessary and one can define the partitions $\mathcal{P}_n$ by dyadic decompositions of $\mathbf{R}^m$ into half-open cubes:

$$[j2^{-n}, (j+1)2^{-n}) \times \cdots \times [k2^{-n}, (k+1)2^{-n}).$$

These partitions satisfy $\mathcal{P}_{n+1} \prec \mathcal{P}_n$ and generate the Borel σ-algebra, so all of the above results apply, including Theorem 2.4. Note that the averaging process defined by these partitions satisfies Theorems 2.1 and 2.2 even though the Hardy-Littlewood maximal theorem may fail for averaging over balls with respect to the measure μ. For certain μ, however, one can smooth by averaging over balls with respect to Lebesgue measure; Muckenhoupt [5] has classified the weight functions $w(x)$ (the "A_p weights") for which the ordinary Hardy-Littlewood maximal operator is bounded on the weighted space $L^2(\mathbf{R}^m, w(x)\,dx)$. For more on this and related results see Torchinsky [9].

3. The trace of an integral kernel. Let $(X, \mathcal{M}, \mu)$ be a countably generated, σ-finite measure space and suppose that K is a trace class operator on $L^2(X, \mu)$ with kernel $K(x, y)$ in $L^2(X \times X, \mu \times \mu)$. Consider an expansion of the form (1.4):

$$(3.1) \qquad K(x, y) = \sum \lambda_i \phi_i(x)\overline{\psi_i(y)}.$$

We will say that $x \in \mathcal{N}^c$ is a *regular point of the expansion* (3.1) if, for all $i \in \mathbf{N}$, the limits

$$(3.2) \qquad \lim_{n \to \infty} A_n \phi_i(x) = \tilde{\phi}_i(x) \quad \text{and} \quad \lim_{n \to \infty} A_n \psi_i(x) = \tilde{\psi}_i(x)$$

exist at x. That is, the set Y_Σ of regular points for (3.1) is

$$Y_\Sigma = \bigcap_{i \in \mathbf{N}} (Y_{\phi_i} \cap Y_{\psi_i}).$$

Thus, almost every point in X is a regular point by Theorem 2.2. From now on we will drop the tildes and consider ϕ_i and ψ_i to be defined pointwise by (3.2) at each $x \in Y_\Sigma$. By (2.5) and (2.8) we have

$$(3.3) \qquad A_n^{(2)} K(x, y) = \sum \lambda_i A_n \phi_i(x) A_n \overline{\psi_i}(y)$$

at every point $(x, y) \in \mathcal{N}^c \times \mathcal{N}^c$. $\widetilde{K}(x, y)$ is defined as above:

$$\widetilde{K}(x, y) = \lim_{n \to \infty} A_n^{(2)} K(x, y),$$

which exists pointwise a.e. $[\mu \times \mu]$.

We are now ready to state and prove the following generalization of Duflo's theorem:

THEOREM 3.1. *Let K be a trace class operator on $L^2(X, \mu)$; then the canonical expansion (3.1) for $K(x, y)$ converges absolutely a.e. $[\mu \times \mu]$ and a.e. $[\mu]$ on the diagonal, $M^{(2)} K(x, x) \in L^1(X)$, $\widetilde{K}(x, x)$ exists a.e. $[\mu]$, and*

$$(3.4) \qquad \operatorname{tr} K = \int \widetilde{K}(x, x) \, d\mu(x).$$

Proof. We have a canonical expansion for K of the form (3.1),

$$K(x, y) = \sum \lambda_i \phi_i(x)\overline{\psi_i(y)},$$

with $\lambda_i > 0$ and $\sum \lambda_i < \infty$. Since K is traceable,

$$\int \sum \lambda_i |\phi_i(x)|^2 = \int \sum \lambda_i |\psi_i(x)|^2 = \sum \lambda_i < \infty$$

so the series $\sum \lambda_i |\phi_i(x)|^2$ and $\sum \lambda_i |\psi_i(x)|^2$ are finite almost everywhere. The inequality

$$\left[\sum |\lambda_i \phi_i(x) \overline{\psi_i(y)}| \right]^2 \leq \sum \lambda_i |\phi_i(x)|^2 \sum \lambda_i |\psi_i(y)|^2$$

shows that the canonical expansion converges absolutely a.e. $[\mu \times \mu]$ to $K(x, y)$ and that $\sum \lambda_i \phi_i(x) \overline{\psi_i(x)}$ converges absolutely a.e. $[\mu]$.

Using the results of §2, we have the following calculation:

$$\int M^{(2)} K(x, x) \, dx \leq \int \sum \lambda_i M \phi_i(x) M \overline{\psi_i}(x) \, d\mu(x)$$

$$\leq \sum \lambda_i \|M\phi_i\|_2 \|M\overline{\psi_i}\|_2$$

$$\leq C_2^2 \sum \lambda_i \|\phi_i\|_2 \|\overline{\psi_i}\|_2$$

$$< \infty.$$

This proves the second assertion in the theorem and shows the convergence of the nonnegative series

$$\sum \lambda_i M \phi_i(x) M \overline{\psi_i}(x) < \infty \quad \text{a.e. } [\mu].$$

Choose a conull set $Y \subset Y_\Sigma$ of regular points for the canonical expansion so that for all $x \in Y$ each of the following three series is finite:

$$\sum \lambda_i |\phi_i(x)|^2, \quad \sum \lambda_i |\psi_i(x)|^2, \quad \sum \lambda_i M \phi_i(x) M \overline{\psi_i}(x).$$

Since Y consists of regular points,

$$\lim_{n \to \infty} A_n \phi_i(x) = \phi_i(x) \quad \text{and} \quad \lim_{n \to \infty} A_n \psi_i(x) = \psi_i(x),$$

for all $x \in Y$ and all $i \in \mathbf{N}$. Now,

$$|A_n \phi_i(x) A_n \overline{\psi_i}(x)| \leq M \phi_i(x) M \overline{\psi_i}(x)$$

for $x \in Y$ so the series

$$\sum \lambda_i A_n \phi_i(x) A_n \overline{\psi_i}(x)$$

converges absolutely for each $x \in Y$. From (3.3) we know that

$$A_n^{(2)} K(x, x) = \sum \lambda_i A_n \phi_i(x) A_n \overline{\psi_i}(x)$$

so, letting $n \to \infty$, the dominated convergence theorem on $\mathbf{N}$ yields

$$\widetilde{K}(x, x) = \lim_{n \to \infty} A_n^{(2)} K(x, x)$$

$$= \sum \lambda_i \lim_{n \to \infty} A_n \phi_i(x) A_n \overline{\psi_i}(x)$$

$$= \sum \lambda_i \phi_i(x) \overline{\psi_i(x)},$$

for each $x \in Y$. Note that this series converges absolutely in $L^1(X)$. Using (1.1), we compute the trace:

$$
\begin{aligned}
\operatorname{tr} K &= \sum (K\psi_i, \psi_i) \\
&= \sum \int (\lambda_i \phi_i(x)) \overline{\psi_i(x)} \, d\mu(x) \\
&= \int \sum \lambda_i \phi_i(x) \overline{\psi_i(x)} \, d\mu(x) \\
&= \int \widetilde{K}(x, x) \, d\mu(x). \qquad \qquad \square
\end{aligned}
$$

In general, the function $\widetilde{K}(x, x)$ cannot be computed effectively because of the abstract nature of the averaging process defined in §2. The existence and integrability of $\widetilde{K}(x, x)$ is a completely general necessary condition for traceability, but Theorem 3.1 is most effective for actually computing the trace when $K(x, y)$ is continuous. Theorem 2.4 implies

COROLLARY 3.2. *Let μ be a σ-finite Borel measure on a second-countable space, X, and let K be a trace class operator on $L^2(X, \mu)$. If the kernel $K(x, y)$ is continuous at (x, x) for almost every x then*

$$
\operatorname{tr} K = \int K(x, x) \, d\mu(x).
$$

Now consider the trace formula given by factoring a trace class operator as a product of Hilbert-Schmidt operators: if $K = LJ$, where L, J are Hilbert-Schmidt operators, then

$$
(3.5) \qquad \operatorname{tr} K = \int L * J(x, x) \, d\mu(x).
$$

Although the integrand in (3.5),

$$
L * J(x, x) = \int L(x, z) J(z, x) \, d\mu(z),
$$

is a well-defined element of $L^1(X)$, it is not clear that this function is independent of the particular factors L and J, nor is it related in an obvious manner to the diagonal values of the kernel $K(x, y)$. As in [1], we can prove that, for *any* factorization $K = LJ$, the function $L * J(x, x)$ coincides almost everywhere with $\widetilde{K}(x, x)$, so (3.5) agrees with the trace formula in Theorem 3.1.

THEOREM 3.3. *If $K = LJ$ is a factorization of the trace class oper-*
ator K into a product of Hilbert-Schmidt operators, then

$$\widetilde{K}(x, x) = L * J(x, x) \quad a.e. \ [\mu].$$

The proof is identical to the proof of Theorem 3.5 in [1], using the
averaging process constructed in §2 in lieu of the Hardy-Littlewood
process.

4. Traceability of positive integral operators. Our final goal is to
prove a version of Mercer's theorem (see [7] or [10]) that does not re-
quire continuity of the kernel. Along the way, we shall also generalize
a theorem of Weidmann ([11], Satz 1) and obtain its converse imme-
diately from Theorem 3.1. This will give us two integral conditions,
each of which is both necessary and sufficient for traceability of an
arbitrary positive-definite kernel. We begin with a very easy lemma:

LEMMA 4.1. *Suppose that*

$$(4.1) \qquad\qquad P = \sum_{i=1}^{\infty} \lambda_i \phi_i \otimes \overline{\phi_i}, \qquad \lambda_i > 0,$$

is the eigenfunction expansion of a positive compact operator on $L^2(X)$.
If x is a regular point for this expansion, then

$$(4.2) \qquad\qquad \sum \lambda_i |\phi_i(x)|^2 \leq \liminf_{n \to \infty} \sum \lambda_i |A_n \phi_i(x)|^2.$$

Proof. Since x is a regular point, $A_n \phi_i(x) \to \phi_i(x)$ for all $i \in \mathbf{N}$,
so (4.2) is just Fatou's lemma on $\mathbf{N}$.   □

Now for Weidmann's theorem and its converse in a more general
setting than $\mathbf{R}^n$:

THEOREM 4.2. *Let $MP(x, y)$ be the maximal function of a positive*
Hilbert-Schmidt kernel. Then P is traceable if and only if

$$\int MP(x, x) \, d\mu(x) < \infty.$$

Proof. As in §3,

$$(4.3) \qquad\qquad \sum \lambda_i |A_n \phi_i(x)|^2 = A_n^{(2)} P(x, x)$$

at each regular point, x. Apply Lemma 4.1

$$\sum \lambda_i |\phi_i(x)|^2 \leq \liminf_{n \to \infty} \sum \lambda_i |A_n \phi_i(x)|^2$$
$$\leq MP(x, x)$$

and integrate:

$$\operatorname{tr} P = \sum \lambda_i$$

$$= \int \sum \lambda_i |\phi_i(x)|^2 \, d\mu(x) \le \int MP(x, x) \, d\mu(x).$$

Thus, P is traceable if $MP(x, x) \in L^1(X)$. The converse follows by Theorem 3.1.   □

Next, we obtain a generalization of Mercer's theorem with no compactness or continuity hypotheses:

THEOREM 4.3. *Suppose that P is a positive Hilbert-Schmidt operator and that $\widetilde{P}(x, x)$ exists a.e. $[\mu]$. Note that we allow the extended real value $\widetilde{P}(x, x) = +\infty$. If $\widetilde{P}(x, x) < \infty$ a.e. $[\mu]$ then expansion (4.1) converges pointwise:*

(4.4) $$\widetilde{P}(x, y) = \sum \lambda_i \phi_i(x)\overline{\phi_i(y)} \quad a.e. \ [\mu \times \mu].$$

Moreover,

(4.5) $$\operatorname{tr} P = \int \widetilde{P}(x, x) \, dx.$$

Thus, P is traceable if and only if $\widetilde{P}(x, x)$ exists for almost every x and the integral (4.5) is finite.

Proof. First note that $\widetilde{P}(x, x) \ge 0$ a.e. $[\mu]$, by (4.3). For all regular points x, y at which both $\widetilde{P}(x, x)$ and $\widetilde{P}(y, y)$ are finite, Lemma 4.1 implies

$$\left[\sum |\lambda_i \phi_i(x)\overline{\phi_i(y)}| \right]^2 \le \sum \lambda_i |\phi_i(x)|^2 \sum \lambda_i |\phi_i(y)|^2$$

$$\le \widetilde{P}(x, x)\widetilde{P}(y, y) < \infty,$$

so (4.4) converges absolutely a.e. $[\mu \times \mu]$, and the pointwise limit necessarily agrees with $P(x, y)$ almost everywhere. Lemma 4.1 and (4.3) also imply

$$\operatorname{tr} P = \int \sum \lambda_i |\phi_i(x)|^2 \, d\mu(x)$$

$$\le \int \liminf_{n \to \infty} A_n^{(2)} P(x, x) \, d\mu(x)$$

$$\le \int \widetilde{P}(x, x) \, d\mu(x).$$

Thus, if this integral is finite then so is $\operatorname{tr} P$, and the trace formula (4.5) follows from Theorem 3.1. Conversely, if P is not traceable then both sides of (4.5) are infinite.   □

Since $\widetilde{P}(x, y)$ agrees with $P(x, y)$ at all regular points of continuity, Theorem 4.3 extends two previous generalizations of Mercer's theorem for continuous kernels due to Duflo ([3], Théorème V.3.3.1) and Reed and Simon ([6], Lemma to Theorem XI.31.)

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