PACIFIC JOURNAL OF MATHEMATICS
Vol. 150, No. 2, 1991

ON COVERINGS OF FIGURE EIGHT KNOT SURGERIES

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We show that over half of the Dehn surgeries on S^3 along the figure eight knot K yield manifolds having finite covers with positive first Betti number by explicitly constructing these covers and exhibiting their homology.

1. Introduction. Denote by K the figure eight knot, pictured in Figure 1. In his celebrated Notes, [T], Thurston showed that all but finitely many Dehn surgeries along K in S^3 yield hyperbolic non-Haken manifolds—the first such examples. It remains an open question whether or not these manifolds (or every closed, irreducible 3-manifold with infinite π_1) are finitely covered by Haken manifolds, or stronger still, by manifolds with positive first Betti number.

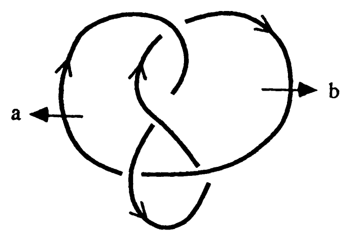

FIGURE 1

In this paper we will show that over half of the Dehn surgeries along K yield manifolds having finite covers with positive first Betti number by explicitly constructing these covers and exhibiting their homology.

Section 2 is devoted to notation and preliminaries. Section 3 contains a statement of our results as well as a summary of previous results on the problem. The method of proof is outlined in §4. Proofs are given in §§5–7.

2. Preliminaries. Throughout this paper K will denote the figure eight knot and M the complement, in S^3, of an open regular neighborhood of K. We will use the fact that M is a bundle over S^1 with fiber a once-punctured torus.

2.1. Let T_0 denote the torus with an open disk removed, pictured in Figure 2. Let D_x denote the left-handed Dehn twist about the loop

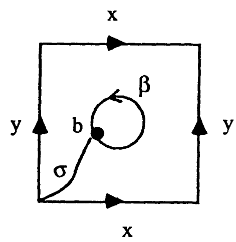

FIGURE 2

x and D_y the right-handed Dehn twist about the loop y in T_0. Then

$$M \cong T_0 \times [0, 1]/(g(s), 0) \sim (s, 1)$$

where $g = D_x \circ D_y$.

We fix a basepoint, b, in ∂T_0 and let x, y be the elements of $\pi_1(T_0, b)$ represented by the loops x, y in T_0 based at b via the arc σ. Then x and y freely generate $\pi_1(T_0, b)$ and D_x, D_y induce the isomorphisms:

$$(D_x)_\#: x \to x, \quad y \to yx,$$
$$(D_y)_\#: x \to xy, \quad y \to y.$$

The loop $\alpha = b \times [0, 1]/\sim$ is a meridian for K and $\beta = \partial T_0$ is a longitude for K. Then

$$\pi_1(M) \cong \langle x, y, \alpha | \alpha^{-1} x \alpha = xyx, \ \alpha^{-1} y \alpha = yx \rangle$$

which is easily seen to be isomorphic to the following Wirtinger presentation for $S^3 \setminus K$:

$$\pi_1(S^3 \setminus K) \cong \langle a, b | (a^{-1}bab^{-1})a(a^{-1}bab^{-1})^{-1}b^{-1} = \mathrm{id} \rangle.$$

Indeed, first eliminate y $(y = x^{-1}\alpha^{-1}x\alpha x^{-1})$ then set $\alpha = a^{-1}$ and $x = ba^{-1}$.

2.2. By Dehn filling on a 3-manifold X with respect to a loop in a boundary torus, we mean attaching a solid torus to ∂X so that this loop bounds a meridional disk in the solid torus.

We say that X has a virtually $\mathbb{Z}$-representable fundamental group if $\pi_1(X)$ contains a finite index subgroup with non-trivial representation to $\mathbb{Z}$. If X is compact, this is equivalent to the existence of a finite cover $\widetilde{X} \to X$ with $\beta_1(\widetilde{X}) \equiv \operatorname{rank} H_1(\widetilde{X}) > 0$.

Given a surface F and a homeomorphism $h: F \to F$, we define the corresponding bundle over S^1 by $F \times I/h \equiv F \times [0, 1]/(h(s), 0) \sim (s, 1)$. Note that the back face $F \times \{1\}$ is attached to the front face $F \times \{0\}$ via h.

Given $M_h = T_0 \times I/h$ with h the identity on ∂T_0, define (as for M) the loops $\alpha_h = b \times I/\sim$, $\beta = \partial T_0$.

DEFINITIONS. (1) $M_h(\mu, \lambda)$ represents the manifold obtained by Dehn filling on M_h with respect to the loop $\alpha_h^\mu \beta^\lambda$.

(2) By μ/λ Dehn surgery along K in S^3, we mean Dehn filling on M with respect to $\alpha^\mu \beta^\lambda$. Let $M(\mu, \lambda)$ denote the resulting manifold.

REMARKS. (1) $M(\mu, \lambda) \cong M(\mu, -\lambda)$ since there exists an orientation reversing homeomorphism on M sending α to α and β to β^{-1} (see **[H2]** or **[T]**).

(2) Since $M_h(\mu, \lambda) = M_h(-\mu, -\lambda)$ we will assume that $\mu \geq 1$.

3. Statement of results. $M(\mu, \lambda)$ is known to have a virtually $\mathbb{Z}$-representable fundamental group if:

(i) $\lambda \equiv \pm 2\mu \pmod 7$ (see **[H1]** or **[N]**),
(ii) $\lambda \equiv \pm\mu \pmod{13}$ (see **[H1]**),
(iii) $\mu \equiv 0 \pmod 4$ and $\mu/\lambda \neq \pm 8$ (see **[KL]**).

In §5 below, we will prove:

THEOREM A. *$M(3\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $|\lambda| \notin \{\mu - 1, \mu + 1\}$.*

In §6, we first give a simple proof of (iii) by explicitly constructing covers $N \to M(4\mu, \lambda)$, for which $\beta_1(N) \geq 1$. We show that $M(8, \pm 1)$ has a virtually $\mathbb{Z}$-representable fundamental group, the case not covered in **[KL]**. We then prove virtual $\mathbb{Z}$-representablility for certain $M(2\mu, \lambda)$:

PROPOSITION C. *$M(2\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $\lambda \equiv \pm 7\mu \pmod{15}$.*

In §7, we study singular boundary curve systems for M. In **[H2]**, it is shown that $\{\alpha^3\}$, $\{\alpha\beta\}$ and $\{\alpha\beta^{-1}\}$ are singular boundary curve systems. We prove the following result:

THEOREM D. *$\{\alpha^2\beta\}$, $\{\alpha^2\beta^{-1}\}$, $\{\alpha^3\beta\}$ and $\{\alpha^3\beta^{-1}\}$ are singular boundary curve systems for M.*

REMARK. Our results, combined with (i)–(iii) above, show that approximately two-thirds of the surgeries on K yield manifolds having virtually $\mathbb{Z}$-representable fundamental groups.

4. Construction of covers. For a given (μ, λ), we show that $M(\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group by constructing a finite cover $N \to M(\mu, \lambda)$ with $\beta_1(N) \equiv \operatorname{rank} H_1(N) \geq 1$. The cover N is obtained from a finite cover $\widetilde{M} \to M$ having the following two properties:

(i) The loop $\alpha^\mu \beta^\lambda$ in ∂M lifts to loops in the components of $\partial\widetilde{M}$;

(ii) $\beta_1(\widetilde{M}) > \beta_0(\partial\widetilde{M})$.

Property (i) guarantees that $\widetilde{M} \to M$ extends to an (unbranched) cover $N \to M(\mu, \lambda)$ by Dehn filling on $\widetilde{M}$ and M. Property (ii) guarantees that any manifold obtained by Dehn filling on $\widetilde{M}$ (hence N) has positive first Betti number.

Since M is a bundle over S^1 with fiber T_0 and characteristic homeomorphism g, it follows that $\widetilde{M}$ is also a bundle over S^1 with fiber F a cover of T_0 and characteristic homeomorphism $\tilde{g}$ a lifting of g^n for some integer $n \geq 1$.

It is easy to show (see **[H1]**) that $\widetilde{M}$ satisfies property (ii) above if and only if $\tilde{g}_*: H_1(F) \to H_1(F)$ fixes a non-boundary class in $H_1(F)$. We adopt the terminology of **[H1]** that $\tilde{g}$ is *homology reducible* if it fixes such a non-boundary class in $H_1(F)$.

Thus we will construct $\widetilde{M}$ by constructing a finite cover $F \to T_0$ to which an appropriate power of g lifts to a homeomorphism $\tilde{g}: F \to F$ that is homology reducible.

Since $g = D_x \circ D_y$, it is difficult to tell, given a cover $F \to T_0$, whether or not g^n lifts to a $\tilde{g}$ that is homology reducible (in fact the matter of whether or not a given g^n even lifts is difficult to verify in practice). We will avoid these difficulties by using the fact that g^2, g^3 and g^4 are isotopic to maps that are much easier to work with.

5. In this section we prove the following:

THEOREM A. *$M(3\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $|\lambda| \notin \{\mu - 1, \mu + 1\}$.*

We fix $h = D_x^2 \circ D_y^{-4} \circ D_x \circ D_y^{-4} \circ D_x$. Recall that $M_h = T_0 \times I/h$, $\alpha_h = b \times I/h$, $\beta = \partial T_0$ and $M_h(\mu, \lambda)$ is the manifold obtained by Dehn filling on M_h with respect to the loop $\alpha_h^\mu \beta^\lambda$ (see §2.2).

LEMMA 5.1. *$M_h(\mu, \lambda) \to M(3\mu, \mu+\lambda) \cong M(3\mu, -\mu-\lambda)$ is a 3-fold cover.*

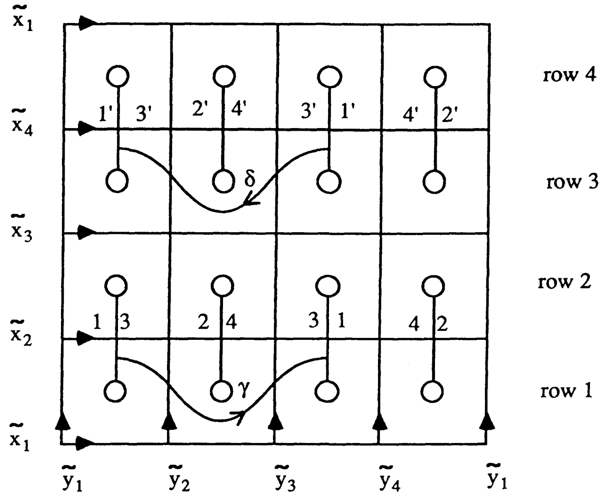

FIGURE 3

Proof. Since h and g^3 both have the same monodromy matrix $\left(\begin{smallmatrix}13 & 8\\ 8 & 5\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ they are isotopic, and hence M_h is bundle equivalent to the 3-fold cyclic cover, M_{g^3}, of M. Moreover the isotopy H from g^3 to h rotates ∂T_0 one turn counter-clockwise, since for any $z \in \pi_1(T_0, b)$,

$$g^3_\#(z) = (xyx^{-1}y^{-1})h_\#(z)(xyx^{-1}y^{-1})^{-1}.$$

(It suffices to check this for x, $y \in \pi_1(T_0, b)$.) Thus the induced bundle isomorphism $H\colon M_h \to M_{g^3}$ sends the pair of loops (α_h, β) to $(\alpha_{g^3}\beta, \beta)$ which projects to $(\alpha^3\beta, \beta)$ in M. □

Now Theorem 1 of **[B]** tells us that $M_h(\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group for $\mu \geq 1$, $|\lambda| \geq 2$ and, if λ is odd, either $\lambda > 2$ or $-4\mu/3 < \lambda < -2$ or $\lambda < -4\mu$. Since $M_h(\mu, \lambda) \to M(3\mu, \mu+\lambda) \cong M(3\mu, -\mu-\lambda)$ is a cover, Theorem A above follows easily.

5.2. We illustrate Theorem A by constructing covers $N \to M(3\mu, \lambda)$, $\beta_1(N) \geq 1$, for μ, λ odd. Consider the 16-fold cover $F \to T_0$ pictured in Figure 3. Let $F' \to T_0$ be the cover corresponding to the kernel of the map $\theta\colon \pi_1(T_0) \to \mathbb{Z}/4 \oplus \mathbb{Z}/4$ defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$. We obtain F by making eight vertical cuts in F' and identifying the left edge of each cut to the right edge of the cut 2 to the right (mod 4). F is a surface of genus 5 with ∂F consisting of eight circles, each projecting 2 to 1 onto β in T_0.

Both D_x and D_y^4 lift to homeomorphisms of F. D_x lifts to $\widetilde{D}_x$ which can be viewed as $1/4$ "fractional" Dehn twists about the $\{\tilde{x}_i\}$. In particular $\widetilde{D}_x$ fixes pointwise rows 1 and 3 while shifting rows 2 and 4 each three squares to the right (mod 4). D_y^4 lifts to $\widetilde{D}_y$ which consists of performing simultaneous Dehn twists about the $\{\tilde{y}_i\}$.

Since both D_x and D_y^4 lift to F, h lifts to a homeomorphism $\tilde{h}\colon F \to F$. It is easy to see that $\tilde{h}$ fixes pointwise ∂F and that $\tilde{h}$ is homology reducible since $\tilde{h}_*$ fixes the nonboundary class $[\gamma] + [\delta]$ in $H_1(F)$.

Let $\widetilde{M} = F \times I/\tilde{h}$. All Dehn fillings on $\widetilde{M}$ have positive first Betti number. Moreover, since $\tilde{h}$ fixes pointwise ∂F, it follows that the loops α_h, β^2 in ∂M_h lift to loops $\tilde{\alpha}_i$, $\tilde{\beta}_i$ in the eight components of $\partial\widetilde{M}$. Denote by $\widetilde{M}(\mu, \lambda)$ the manifold obtained by Dehn filling on $\widetilde{M}$ with respect to the curves $\tilde{\alpha}_i^{\mu}\tilde{\beta}_i^{\lambda}$. Then the sequence of covers

$$\widetilde{M}\left(\mu, \frac{\lambda-\mu}{2}\right) \to M_h(\mu, \lambda-\mu) \to M(3\mu, \lambda)$$

gives the desired cover of $M(3\mu, \lambda)$, μ, λ odd.

6. In this section we deal with the manifolds $M(2\mu, \lambda)$. Throughout §6, we fix $h = (R \circ D_y^{-3})$ where R is the homeomorphism of T_0 induced by a $90°$ counter-clockwise rotation of the square in Figure 2.

Let $M_h = T_0 \times I/h$. The loop α_h is represented in $T_0 \times I$ by the image of the curve $b \times I$ under a $90°$ clockwise rotation of $\partial T_0 \times \{1\}$.

LEMMA 6.1. *M_h is bundle equivalent to M, with the pair (α_h, β) mapping to (α, β).*

Proof. Let R' denote R composed with a $90°$ clockwise rotation of ∂T_0. Then R' fixes ∂T_0 and induces on $\pi_1(T_0, b)$ the isomorphism $R'_\#(x) = xyx^{-1}$, $R'_\#(y) = x^{-1}$. A calculation shows that, for any $z \in \pi_1(T_0, b)$,

$$g_\#(z) = (D_x^{-1} \circ R' \circ D_y^{-3} \circ D_x)_\#(z).$$

Thus the isotopy H from g to $D_x^{-1} \circ h \circ D_x$ rotates ∂T_0 only $90°$ counter-clockwise and hence the bundle isomorphism $H \circ (D_x^{-1} \times \mathrm{Id})\colon M_h \to M$ sends (α_h, β) to (α, β). □

Now consider $M' = T_0 \times I/h^4$, the 4-fold cyclic cover of M_h. Note that h^4 fixes ∂T_0, so we define (α', β) for M', where $\alpha' = b \times I/h^4$.

LEMMA 6.2. *$M' \to M$ is a 4-fold cover, sending the pair of loops (α', β) to the pair $(\alpha^4\beta, \beta)$.*

Proof. Note that the lift of α_h^4 to M' winds once around ∂T_0 in the clockwise direction and hence is represented by $\alpha'\beta^{-1}$. Thus α' projects to $\alpha_h^4\beta$ in M_h which maps to $\alpha^4\beta$ in M by Lemma 6.1. □

The following is an immediate consequence of Lemma 6.2 and will be used in §6.1:

COROLLARY 6.3. *$M'(\mu, \lambda) \to M(4\mu, \mu+\lambda) \cong M(4\mu, -\mu-\lambda)$ is a 4-fold cover.*

6.1. Now we prove the following (see also **[KL]**):

THEOREM B. *$M(4\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group.*

We begin by considering the 9-fold cover $S \to T_0$ corresponding to the kernel of the map $\theta\colon \pi_1(T_0) \to \mathbb{Z}/3 \oplus \mathbb{Z}/3$ defined by $\theta([x]) = (1, 0)$ and $\theta([y]) = (0, 1)$. Note that both D_y^{-3} and R lift to S.

Next we construct, for each $d \geq 3$, a cover $F_d \to T_0$ as follows: Let $S_1, \ldots, S_d$ be copies of S, each with eight cuts $\{\tau_i\}$ as pictured in Figure 4. Glue the left edge of τ_1 in S_i to the right edge of τ_1 in S_{i+1} (mod d). Next glue the left edge of τ_2 in S_i to the right edge of τ_2 in S_{i-2} (mod d). Now glue the edges $\tau_3, \ldots, \tau_8$ so that the gluing is compatible with that of τ_1, τ_2 under a simultaneous counter-clockwise rotation by 90° of each S_i. Note that the gluing of τ_1 determines the pattern for τ_3, τ_5, τ_7 while the gluing of τ_2 determines that of the τ_4, τ_6, τ_8. The surface F_3, with identifications for τ_i numbered, is pictured in Figure 5. Some of the properties of the surface F_d are given in:

LEMMA 6.4. *The surface F_d is a $9d$-fold cover of T_0. Each component $\tilde{\beta}_i$ of ∂F_d projects r_i to 1 onto $\beta = \partial T_0$ for $r_i | d$.*

Now the loop x (resp. y) in T_0 is covered by $3d$ loops $\tilde{x}_1, \ldots, \tilde{x}_{3d}$ (resp. $3d$ loops $\tilde{y}_1, \ldots, \tilde{y}_{3d}$) in F_d that project 3 to 1 onto x (resp. 3 to 1 onto y). Thus D_y^{-3} lifts to $\widetilde{D}_y^{-1}$ consisting of simultaneous negative Dehn twists about the $\{\tilde{y}_i\}$. It follows from the construction of F_d that R lifts to $\widetilde{R}$, a simultaneous counter-clockwise rotation by 90° of each of the $S_1 \ldots, S_d$ in F_d. Thus h $(= R \circ D_y^{-3})$ and h^4 lift to $\tilde{h}$ and $\tilde{h}^4$ on F_d. Note that $\tilde{h}^4$ fixes pointwise ∂F_d.

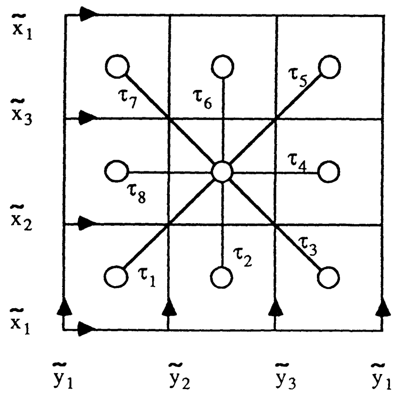

FIGURE 4

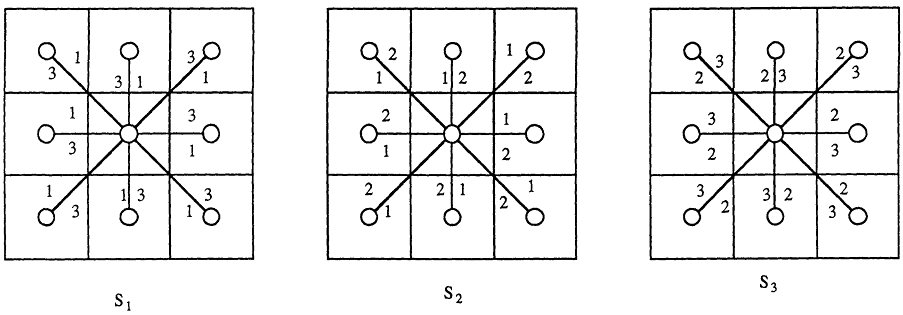

FIGURE 5

LEMMA 6.5. *$\tilde{h}^4$: $F_d \to F_d$ is homology reducible.*

Proof. A portion of F_d is pictured in Figure 6. The non-boundary class $[\gamma] + [\delta]$ in $H_1(F_d)$ corresponding to the loops γ, δ is fixed by $\tilde{h}^4_*$. Indeed, $\widetilde{R}^4 = \mathrm{Id}$ and $[\gamma] + [\delta]$ is fixed by $(\widetilde{D}_y^{-1})_*$ since γ and δ each intersect the same Dehn twist curves in $\{\tilde{y}_i\}$ with opposite orientations. □

Let $\widetilde{M}_d = F_d \times I/\tilde{h}^4$. Now $\widetilde{M}_d$ is, by construction, a $9d$-fold cover of M', the 4-fold cyclic cover of M_h, hence $\widetilde{M}_d \to M$ is a $36d$-fold covering space (see Lemma 6.2). Furthermore, Lemma 6.5 implies that any Dehn filling on $\widetilde{M}_d$ yields a manifold with positive first Betti number.

We complete the proof of Theorem B by constructing, for each $(4\mu, \lambda)$ coprime, a cover $N \to M(4\mu, \lambda)$, $\beta_1(N) \geq 1$, gotten by Dehn filling on an appropriate $\widetilde{M}_d$. Since $M(0, \pm 1)$ itself has positive first Betti number, we exclude this case.

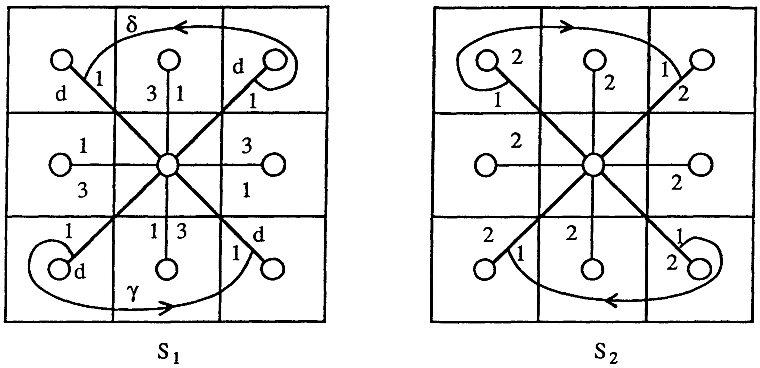

FIGURE 6

Recall that $M'(\mu, \lambda - \mu) \to M(4\mu, \lambda) \cong M(4\mu, -\lambda)$ is a 4-fold cover by Corollary 6.3. Since $(4\mu, \lambda) \neq (0, \pm 1)$, by changing the sign of λ if necessary, we can assume that either $\lambda = \mu = \pm 1$ or $|\lambda - \mu| \geq 3$. In the first case the loop α' in $\partial M'$ lifts to loops $\{c_i\}$ in $\partial \widetilde{M}_d$ for any d. In the second case the loop $(\alpha')^{\mu}\beta^{\lambda-\mu}$ in $\partial \widetilde{M}$ lifts to loops $\{c_i\}$ in $\partial \widetilde{M}_d$ for $d = |\lambda - \mu|$. In both cases we obtain $N \to M(4\mu, \lambda)$ by Dehn filling on $\widetilde{M}_d$ with respect to the loops $\{c_i\}$ in $\partial \widetilde{M}_d$. This completes the proof of Theorem B.

As an example, consider the case $M(8, -1) \cong M(8, 1)$. Then $M'(2, -3) \to M(8, 1)$ and the $(2, -3)$ loop in M' lifts to loops $\{c_i\}$ in the boundary components of $\widetilde{M}_3$. N is gotten by Dehn filling on $\widetilde{M}_3$ with respect to the loops $\{c_i\}$ (see Figure 5).

6.2. PROPOSITION C. *$M(2\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $\lambda \equiv \pm 7\mu$* (mod 15).

Consider the 9-fold cover $S \to T_0$ described in §6.1, and construct a new cover $F \to T_0$ by making eight cuts in S and identifying the edges as shown in Figure 7. The surface F has genus 4 and ∂F consists of 3 circles: $\tilde{\beta}_1$ that projects 5-1 onto β, $\tilde{\beta}_2$ projecting 3-1 onto β, and $\tilde{\beta}_3$ projecting 1-1 onto β. The loop x (resp. y) in T_0 is covered by the three loops $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ (resp. $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$) which project 3-1 onto x (resp. onto y).

It follows from the construction of F that R lifts to $\widetilde{R}$ the homeomorphism induced by a 90° counter-clockwise rotation, and that D_y^{-3} lifts to $\widetilde{D}_y^{-1}$ given by simultaneous negative Dehn twists about the $\{\tilde{y}_i\}$. Hence h $(= R \circ D_y^{-3})$ lifts to $\tilde{h}$.

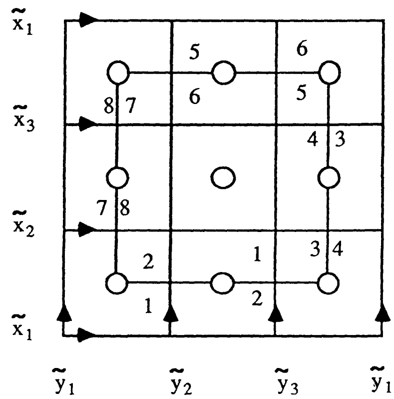

FIGURE 7

LEMMA 6.6. *$\tilde{h}^2$ is homology reducible.*

Proof. $\tilde{h}^2_*$ fixes the non-boundary class $[\tilde{x}_2] - [\tilde{x}_3]$ in $H_1(F)$ (see Figure 7). □

Let $\widetilde{M} = F \times I/\tilde{h}^2$. Then $\widetilde{M}$ is an 18-fold cover of M_h. Since $\tilde{h}^2$ rotates each component $\tilde{\beta}_i$ of ∂F one half turn counter-clockwise, we can choose on each component $T_i \subset \partial\widetilde{M}$ loops $(\tilde{\alpha}_i, \tilde{\beta}_i)$ where $(\tilde{\alpha}_1, \tilde{\beta}_1)$ projects to $(\alpha_h^2\beta^{-2}, \beta^5)$, $(\tilde{\alpha}_2, \tilde{\beta}_2)$ projects to $(\alpha_h^2\beta^{-1}, \beta^3)$ and $(\tilde{\alpha}_3, \tilde{\beta}_3)$ projects to (α_h^2, β) in M_h.

Now our proposition follows, since by the above paragraph any loop in ∂M_h of the form $\alpha_h^{2\mu}\beta^\lambda$, $\lambda \equiv -7\mu \pmod{15}$, lifts to loops $\{c_i\}$ in each component T_i of $\partial\widetilde{M}$. Dehn filling on $\widetilde{M}$ with respect to the loops $\{c_i\}$ provides a cover $N \to M_h(2\mu, \lambda) \cong M(2\mu, \lambda)$, the last isomorphism by Lemma 6.1.

6.3. REMARK. By similar arguments, we can show that $M(2\mu, \lambda)$ has a virtually $\mathbb{Z}$-representable fundamental group if $\lambda \equiv \pm 3\mu \pmod 7$. These cases have been done in **[H1]** and **[N]** by different methods (see §3). Consider the cover $F \to T_0$ in Figure 8, obtained from 3 copies of S by removing the interiors of the four shaded regions in each copy of S and identifying the edges as numbered. The reader should check the following: ∂F consists of 3 circles, each projecting 7 to 1 onto $\beta = \partial T_0$; h^2 lifts to a homology reducible map $\tilde{h}^2 \colon F \to F$; and the loop $\alpha^{2\mu}\beta^\lambda$, $\lambda \equiv \pm 3\mu \pmod 7$, in M lifts to loops in $\widetilde{M} = F \times I/\tilde{h}^2$.

7. Singular boundary curve systems for M. In this section we study singular incompressible surfaces in M. Given a cover $N \to M(\mu, \lambda)$

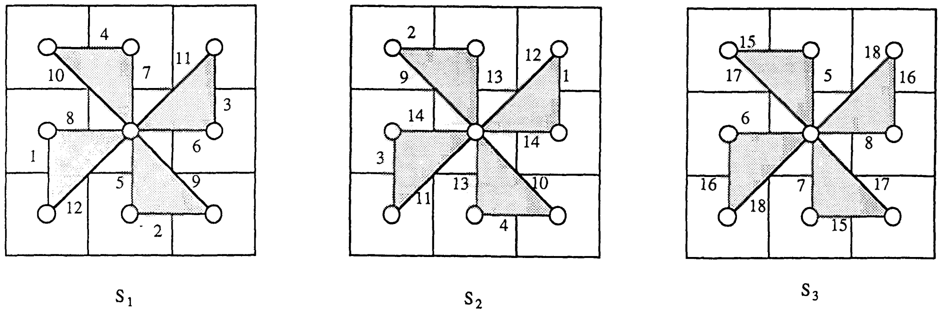

FIGURE 8

obtained by Dehn filling on $\widetilde{M} \to M$, then $\beta_1(N) \geq \beta_1(\widetilde{M}) - \beta_0(\partial\widetilde{M})$. Hempel shows ([H2]) that this inequality is strict if and only if there is an incompressible, boundary incompressible surface F in $\widetilde{M}$ such that ∂F consists of a non-empty collection of Dehn filling curves. This surface F projects to a singular surface in M whose boundary curves are $\alpha^\mu\beta^\lambda$, and we say that $\{\alpha^\mu\beta^\lambda\}$ is a singular boundary curve system for M.

In [H2] the curves $\{\alpha^3\}$, $\{\alpha, \beta\}$, and $\{\alpha\beta^{-1}\}$ are shown to be singular boundary curve systems. We show:

THEOREM D. *The curves* $\{\alpha^2\beta\}$, $\{\alpha^2\beta^{-1}\}$, $\{\alpha^3\beta\}$ *and* $\{\alpha^3\beta^{-1}\}$ *are singular boundary curve systems for* M.

(a) *The curves* $\alpha^3\beta^{\pm 1}$: We use the 3-fold cover $M_h \to M$ for $h = D_x^2 \circ D_y^{-4} \circ D_x \circ D_y^{-4} \circ D_x$ described in §5.

By Lemma 5.1, $M_h(1, 0) \to M(3, 1)$ is a 3-fold covering. Now consider the 8-fold cover $F \to T_0$, pictured in Figure 9, to which h lifts (see §5). Denote this lift by $\tilde{h}$. Note that $\tilde{h}$ fixes pointwise the eight components of ∂F and that $\tilde{h}$ is not homology reducible.

Let $\widetilde{M} = F \times I/\tilde{h}$. By construction, the loop α_h in M_h lifts to eight loops $\tilde{\alpha}_1, \dots, \tilde{\alpha}_8$ in $\partial\widetilde{M}$—indexed so that the loops $(\tilde{\alpha}_i, \tilde{\beta}_i)$ lie in the ith boundary torus of $\widetilde{M}$. Thus the loops $\tilde{\alpha}_i$ project to $\alpha^3\beta$ in ∂M and Dehn filling on $\widetilde{M}$ with respect to the $\{\tilde{\alpha}_i\}$ gives a cover $N \to M(3, 1) \cong M(3, -1)$.

LEMMA 7.1. *There exist relations among* $\{[\tilde{\alpha}_i]\}$ *in* $H_1(\widetilde{M})$; *hence* $\beta_1(N) > \beta_1(\widetilde{M}) - \beta_0(\widetilde{M})$.

Proof. We have $[\tilde{\alpha}_2] - [\tilde{\alpha}_1] = [\tilde{\alpha}_6] - [\tilde{\alpha}_5]$ in $H_1(\widetilde{M})$. One computes $[\tilde{\alpha}_j] - [\tilde{\alpha}_i]$ as follows. Let σ_{ij} be a simple path in $F \times \{0\}$ from

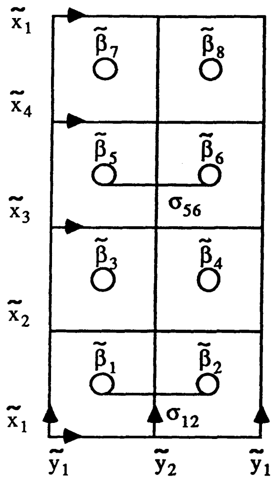

FIGURE 9

$\tilde{\alpha}_i \cap F$ to $\tilde{\alpha}_j \cap F$. Then the disk $\sigma_{ij} \times I \subset F \times I$ provides the relation $[\tilde{\alpha}_j] - [\tilde{\alpha}_i] = [\tilde{h}(\sigma_{ij}) * \sigma_{ij}^{-1}]$ where $*$ denotes path composition.

Now σ_{12} and σ_{56} can be chosen as in Figure 9, and $[\tilde{h}(\sigma_{12}) * \sigma_{12}^{-1}] = [\tilde{h}(\sigma_{56}) * \sigma_{56}^{-1}]$ in $H_1(\widetilde{M})$ since $\widetilde{D}_x$ fixes σ_{12} and σ_{56} pointwise and they both intersect the Dehn twist curve $\tilde{y}_2$. □

(b) *The curves* $\alpha^2\beta^{\pm 1}$: Consider the bundle $M_f = T_0 \times I/f$ for $f = (D_x^{-1} \circ D_y^5)^2$.

LEMMA 7.2. *M_f is a 2-fold cover of M. The pair (α_f, β) maps to the pair $(\alpha^2\beta^{-1}, \beta)$.*

Proof. Let $g' = (D_x^{-1} \circ D_y^2 \circ D_x^{-1}) \circ g^2 \circ (D_x^{-1} \circ D_y^2 \circ D_x^{-1})^{-1}$. We have, for any $z \in \pi_1(T_0, b)$,

$$g'_{\#}(z) = (xyx^{-1}y^{-1})^{-1} f_{\#}(z)(xyx^{-1}y^{-1}).$$

Thus the isotopy H between g' and f rotates ∂T one full turn clockwise, so that the bundle isomorphism $\{(D_x^{-1} \circ D_y^2 \circ D_x^{-1})^{-1} \times \mathrm{Id}\} \circ H: M_f \to M_{g^2}$ sends the pair (α_f, β) to $(\alpha_{g^2}\beta^{-1}, \beta)$ which projects to $(\alpha^2\beta^{-1}, \beta)$ in M. □

By Lemma 7.2 $M_f(1, 0) \to M(2, -1)$ is a 2-fold cover. Now consider the 10-fold cover $F \to T_0$, pictured in Figure 10 to which f

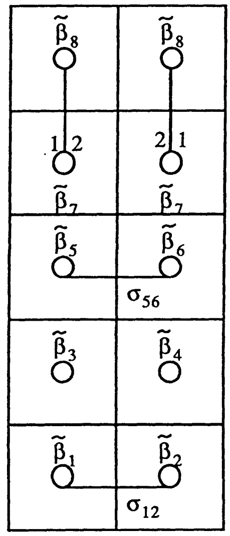

FIGURE 10

lifts. Denote the lift of f by $\tilde{f}$ and let $\widetilde{M} = F \times I/\tilde{f}$. Now $\tilde{f}$ fixes pointwise the eight boundary circles of F. Denote by $\tilde{\alpha}_i$ a lift of α_f to $\partial\widetilde{M}$, indexed so that the loops $(\tilde{\alpha}_i, \tilde{\beta}_i)$ lie on the ith boundary torus of $\widetilde{M}$. Thus the loops $\tilde{\alpha}_i$ project to $\alpha^2\beta^{-1}$ in M and Dehn filling on $\widetilde{M}$ with respect to $\{\tilde{\alpha}_i\}$ gives a cover $N \to M(2, -1) \cong M(2, 1)$.

LEMMA 7.3. *There exist relations among* $\{[\tilde{\alpha}_i]\}$ *in* $h_1(\widetilde{M})$; *hence* $\beta_1(N) > \beta(\widetilde{M}) - \beta_0(\widetilde{M})$.

Proof. We have $[\tilde{\alpha}_2]-[\tilde{\alpha}_1] = [\tilde{\alpha}_6]-[\tilde{\alpha}_5]$ in $H_1(\widetilde{M})$ by an argument identical to that in Lemma 7.1. □

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Received February 26, 1990.

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