SURGERY WITH FINITE FUNDAMENTAL GROUP II: THE OOZING CONJECTURE

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We determine characteristic class formulae for surgery problems over any compact oriented and closed manifold with finite fundamental group. This essentially evaluates the boundary maps in the surgery exact sequences

$$\cdots \xrightarrow{\partial} L_{n+1}^{h}(\mathbf{Z}\pi) \longrightarrow \mathcal{ST}(M^{n}) \longrightarrow [M^{n}, G/TOP] \xrightarrow{\partial} L_{n}^{h}(\mathbf{Z}\pi).$$

(The final step in determining ∂ is carried out in a sequel which represents joint work with I. Hambleton, L. Taylor, and B. Williams.) These formulae involve the L-genus of M, pullbacks of the classes K_{4i} and k_{4i+2} in $H^{*}(G/TOP)$ and characteristic classes of the universal covering of M (coming from $H^{*}(B_{\pi_{1}(M)})$). It turns out that only classes in the first three of these groups, $* = 1, 2, 3$, are needed. This can be interpreted as saying that only codimension 1, 2, and 3 submanifolds are needed to determine the surgery obstruction. In this form our result was originally conjectured twenty years ago and has become known as the oozing conjecture.

In [5] I showed that there are only four types of surgery obstruction possible for surgery problems over closed manifolds with finite π_1 when the problem is of the form

$$M \times K^{4i+2} \longrightarrow M \times S^{4i+2}.$$

In order to do this I introduced an intermediate L group $L_{*}^{h}(\mathbf{Z}(\zeta_3)\pi)$ and maps

$$e_{\zeta_3} : \Omega_{*}(B_\pi) \longrightarrow L_{*}^{h}(\mathbf{Z}(\zeta_3)\pi)$$

which factor the product formula above. Then I showed that there were only four types of classes in the image of e_{ζ_3} which could possibly map non-trivially into the surgery groups.

In this paper I obtain characteristic class formulae for the maps e_{ζ_3}, at least as regards the specific classes described above. First recall the main results of [5]:

THEOREM A. *Let π be a finite 2-group, then there are universal kernels $K_i \subset L_i^{h}(\mathbf{Z}(\zeta_3)\pi)$ with $L_0^{h}(\mathbf{Z}(\zeta_3)\pi)/K_0 = \mathbf{Z}/2$, natural with*

respect to restriction and projection maps, so that, if $e_{\zeta_3}(\alpha) \in K_i$ then the associated surgery problem above has trivial surgery obstruction. Moreover, the images of e_{ζ_3} in the groups $L_i^h(\mathbf{Z}(\zeta_3)\pi)/K_i$ are detected by restriction and then projection to the corresponding groups $L_i^h(\mathbf{Z}(\zeta_3)N)/K_i$ where

   (0) *the quotient $\mathbf{Z}/2$ for $i = 0$ detects the index of M reduced mod (2),*
   (1) *N runs over $\mathbf{Z}/2$-**quotients** of π when $i = 1$,*
   (2) *N runs over $\mathbf{Z}/2 \times \mathbf{Z}/2$ subquotients when $i = 2$, and*
   (3) *N runs over quaternion subquotients of π when $i = 3$,*

In this paper we evaluate the map e_{ζ_3} for the basic classes of groups above. For the set of oriented manifolds we obtain complete answers. In the next three theorems we assume M^n is oriented.

THEOREM B. *When the dimension of M is $4i + 1$, and $\pi_1(M)$ is a finite 2-group then the map e_{ζ_3} is non-trivial if and only if it is non-trivial on projecting to some group $\mathbf{Z}/2$. After projection it is given by the formula*

$$\sigma(\mathrm{id} \times \kappa) = \langle V^2 f^*(e), [M] \rangle \chi.$$

Here χ is the non-zero element in $L_1^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2)/K_1 = \mathbf{Z}/2$ and V is the total Wu class of M. Finally, $f: M \to B_{\mathbf{Z}/2}$ is the classifying map for the universal covering and e is the non-trivial class in $H^1(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$.

In the case $i = 2$ the $\mathbf{Z}/2 \times \mathbf{Z}/2$'s usually occur as subgroups of dihedral groups. For this reason we state our next result in terms of dihedral groups.

THEOREM C. *When the dimension of M is $4i + 2$ then e_{ζ_3} is detected by restriction, projection to a dihedral group or by projection (factoring through π^{ab}) onto a group $\mathbf{Z}/2 \times \mathbf{Z}/2$. In the case of the dihedral group $L_2^h(\mathbf{Z}(\zeta_3)D(2^i, 2))/K_2 = (\mathbf{Z}/2)^i$, but there is a unique class μ_i in this quotient which can be in the image of e_{ζ_3}. Moreover, there is a unique class*

$$w \in H^2(B_{D(2^i, 2)}; \mathbf{Z}/2)$$

which is not in the image of the mod (2) Bochstein and the image of e_{ζ_3} is given by the formula $\langle V^2 f^*(w), [M] \rangle \mu_i$, while in the case $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$ the evaluation is given by the formula

$$\langle V^2 f^*(e \otimes e), [M] \rangle \mu_2.$$

THEOREM D. *If the dimension of* M *is* $4i + 3$*, then the map is detected by restriction and projection onto quaternion subquotients* $\mathcal{Q}_{2^i}$ *where we have* $L_3^h(\mathbf{Z}(\zeta_3)\mathcal{Q}_{2^i})/K_3 = \mathbf{Z}/2$ *with generator* ω_i*. The formula for the surgery obstruction is* $\langle V^2 f^*(b_3), [M]\rangle\omega_i$*, where* b_3 *is the non-zero class in* $H^3(B_{Q(2^i)}; \mathbf{Z}/2) = \mathbf{Z}/2$*.*

Note that in each case the deviation from being a pure characteristic class formula is a class $f^*(c)$ where dimension $(c) \leq 3$. This is the content of the oozing conjecture for oriented manifolds.

COROLLARY E. *The (codimension 3) oozing conjecture is true for oriented manifolds with finite fundamental group. That is to say, if* π *is a finite group, then the formula for the product with Kervaire obstruction above has the form* $\langle V^2 f^*(\kappa_*), [M]\rangle$ *where* $\kappa = 1 + \kappa_1 + \kappa_2 + \kappa_3$*, with the*

$$\kappa_i \in H^i(B_\pi; L_{i+2}^h(\mathbf{Z}\pi) \otimes \mathbf{Z}/2)$$

whenever M *is oriented and* $\pi_1(M) = \pi$*.*

(From [13], [14] it suffices to verify the result for π a finite 2-group, but that is the content of Theorems A–D.)

Of course, Theorems A–D make the determination of the κ's explicit for any finite group π.

Actually, somewhat more is true. The map e_{ζ_3} is associated to a stable map

$$s_\pi : \Sigma^0 B_\pi \longrightarrow \mathscr{L}_*^h(\mathbf{Z}(\zeta_3)\pi)$$

where $\mathscr{L}(\mathbf{Z}(\zeta_3)\pi)$ is the Ranicki spectrum [10] for the L-groups above. At the prime 2, $\mathscr{L}_*(\ )$ is a generalized Eilenberg-MacLane space and the oozing conjecture can be regarded as determining the images of the fundamental classes $\iota \in H^j(\mathscr{L}_*(\ ); \pi_j(\mathscr{L}_*(\ )))$ after restricting to integral homology. But the topological maps have independent interest. Here is what we know in the cases of our model groups.

THEOREM F. *The image of* $s_\pi^* : H^*(\mathscr{L}(\mathbf{Z}(\zeta_3)\pi; \mathbf{Z}/2) \longrightarrow H^*(B_\pi; \mathbf{Z}/2)$ *is always a module over the* mod 2 *Steenrod algebra, and is generated by* e *when* $\pi = \mathbf{Z}/2$*,* $e \otimes 1$*,* $1 \otimes e$*, and* $e \otimes e$ *when* $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$*. When* $\pi = Q(2^i)$ *the image is the set of elements of dimension* < 4*.*

In the case of the dihedral group we have

$$H^*(B_{D(2^i, 2)}; \mathbf{Z}/2) = \mathbf{F}_2[x, y, w]/(xy = 0),$$

with $\dim(x) = \dim(y) = 1$, $\dim(w) = 2$, and $\mathrm{Sq}^1(w) = (x + y)w$ (Theorem 1.25). The generators over the Steenrod algebra of $\operatorname{im}(s^*_{D(2^l, 2)})$ are then x, y, and w.

For example, the module generated over $\mathscr{A}(2)$ by e in $H^*(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$ is the $\mathbf{F}_2$ vector space with generators e, e^2, e^4, ..., e^{2^l}, ...

Our procedure is very close in spirit to that of [3]. There is a commutative diagram

$$\begin{array}{ccc}
\Omega_*(B_\pi) & \xrightarrow{\hat{e}_{\zeta_3}} & L^h_*(\mathbf{Z}(\zeta_3)\pi) \\
\downarrow & & \downarrow{\scriptstyle \otimes \mathbf{Z}/2} \\
\Omega_*(B_\pi;\ \mathbf{Z}/2) & \xrightarrow{e_{\zeta_3}} & L^h_*(\mathbf{Z}(\zeta_3)\pi;\ \mathbf{Z}/2)
\end{array}$$

and exactly as was done in [3], characteristic class techniques can be used to obtain from the evaluation of the map $\hat{e}_{\zeta_3}$, the image of the associated cohomology map s^*_π.

It may be possible to prove an analogue of Theorem A for the groups $L^h_*(\mathbf{Z}(\zeta_3)\pi;\ \mathbf{Z}/2)$, but this is not obvious. There are two families of classes (in even dimensions) which we could ignore in Part I, since they were torsion free. But they cannot be ignored here, and moreover, they do not appear to be detected on any reasonably small set of model groups. Nonetheless they should not matter and we make the

Generalized oozing conjecture. For any finite group π the image of (s^*_π) in mod 2 cohomology is generated over $\mathscr{A}(2)$ by a finite number of elements of dimensions < 4.

In Part 3 of this work [2], using different techniques we obtain more complete information about these mod 2 images, indeed, we obtain sufficient information to prove the conjecture, and this, in turn gives complete characteristic class information for evaluating surgery problems over manifolds with finite fundamental group. This work was completed while the author was a visiting professor at Northwestern University and the University of California at San Diego.

1. Preliminaries. Throughout this paper we assume that all spaces considered have the homotopy types of locally finite CW complexes.

A. *Classifying spaces and extensions for finite 2-groups.* When X is an associative H-space with unit it has a classifying space B_X [4],

which satisfies $\Omega(B_X)$ (the loop space of B_X) is homotopy equivalent to X as a unitary H-space. Moreover, B_X is unique up to homotopy type. Among the more useful properties of this construction are that $B_{X \times Y} = B_X \times B_Y$ when X and Y are both associative H-spaces with units. Additionally, the construction is natural, so that given an associative H-map $f \colon X \to Y$ there is induced a map $B_f \colon B_X \to B_Y$ with all the expected naturality properties.

When X is a commutative topological group then B_X can again be assumed to be a commutative topological group, and in this case the construction can be iterated. (See e.g. [4] for details.)

In particular, if π is an Abelian group, then there is an associative, commutative multiplication

$$(1.1) \qquad B_\pi \times B_\pi \longrightarrow B_\pi$$

making B_π into a topological group. (The circle $S^1 \simeq B_{\mathbf{Z}}$ is a special case of this construction.)

But more is true. Consider a central group extension sequence

$$(1.2) \qquad H \longrightarrow \pi \longrightarrow \pi/H$$

where, of course, $H \subset \mathrm{center}\,(\pi)$. It is standard that the isomorphism classes of exact sequences (1.2) with H, π/H fixed correspond 1-1 with

$$(1.3) \qquad \mathrm{Ext}^2_{\mathbf{Z}(\pi/H)}(\mathbf{Z}; A) = H^2(B_{\pi/H}; H).$$

This correspondence has a geometric interpretation.

Note that $B_{(B_H)} = B_H^2$ is an Eilenberg-MacLane space, and consequently, the set of homotopy classes of maps

$$(1.4) \qquad [X, B_H^2] = H^2(X; H).$$

Also, there is a principal B_H fibering

$$(1.5) \qquad \begin{array}{ccc} B_H & \longrightarrow & E(B_H) \\ & & \downarrow{\scriptstyle p} \\ & & B_H^2 \end{array}$$

with contractible total space $E(B_H)$, and we have

THEOREM 1.6. *The map*

$$B_p \colon B_\pi \longrightarrow B_{\pi/H}$$

induced from (1.2) *is a principal fibering with fiber* B_H. *Moreover, the classifying map* γ_p *for* B_p,

$$\gamma_p : B_{\pi/H} \longrightarrow K(H, 2)$$

satisfies $\gamma_p(\iota) = \kappa \in H^2(\mathbf{Z}(\pi/H); H)$ *where* κ *is the class in* (1.3) *corresponding to the extension* (1.2).

In particular, corresponding to the multiplication map $H \times \pi \longrightarrow \pi$ (which is itself a homomorphism of groups in this case) there is a map

$$B_H \times B_\pi \longrightarrow B_\pi$$

and this map is homotopic to the action of B_H on B_π, now regarded as the total space of the *principal* fibering in 1.6.

EXAMPLES 1.7. The sequence

$$\mathbf{Z}/2 \longrightarrow \mathbf{Z}/4 \longrightarrow \mathbf{Z}/2$$

is associated to the non-trivial element $e^2 \in H^2(\mathbf{Z}/2; \mathbf{Z}/2) = \mathbf{Z}/2$. The sequence

$$\mathbf{Z}/2 \longrightarrow D(4, 2) \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is associated to the classes

$$e_1 e_2, \quad e_1(e_1 + e_2), \quad (e_1 + e_2)e_2$$

{all of which are identified via the action of the group of outer automorphisms of $\mathbf{Z}/2 \times \mathbf{Z}/2$,

$$(\mathrm{Out}((\mathbf{Z}/2)^2) = \mathrm{GL}_2(\mathbf{F}_2) = \mathscr{S}_3,$$

where $\mathscr{S}_3$ is the symmetric group on three letters)}. Similarly, the sequence

$$\mathbf{Z}/2 \longrightarrow \mathbf{Z}/4 \times \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is associated with the invariants

$$e_1^2, \quad e_2^2, \quad (e_1 + e_2)^2 = e_1^2 + e_2^2$$

and the sequence

$$\mathbf{Z}/2 \longrightarrow Q(8) \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is associated to the single element

$$e_1^2 + e_1 e_2 + e_2^2.$$

Here $D(2^i, 2)$ represents the dihedral group of order 2^{i+1} with presentation

$$(1.8) \qquad D(2^i, 2) = \{\tau, T \mid \tau^2 = T^{2^i} = (\tau T)^2 = 1\}$$

while $Q(2^i)$ is the quaternion group of order 2^i,

$$(1.9) \qquad Q(2^i) = \{\tau, T \mid \tau^2 = T^{2^{i-2}} = (\tau T)^2\}.$$

B. **Z/2** *manifolds and bordism.* A **Z/2** manifold is a compact manifold with first Stiefel-Whitney class given as the restriction of an integral cohomology class, together with a specified integral lifting of w_1. Thus, a **Z/2** manifold consists of a pair

$$\{M, f\colon M \longrightarrow S^1\}$$

where f is defined up to homotopy and $f^*(\iota)$ is the integral lifting of w_1. If M^n and N^s are **Z/2**-manifolds then

$$w_1(M^n \times N^s) = w_1(M^n) \otimes 1 + 1 \otimes w_1(N^s)$$

is again the restriction of a (specific) integral class, so that $M^n \times N^s$ is also a **Z/2**-manifold. If the boundary of M^n is empty and M^n is a **Z/2**-manifold then it has a unique non-trivial fundamental class in $H^n(M^n; \mathbf{Z}/2) = \mathbf{Z}/2$.

The **Z/2**-bordism group of the space X, denoted $\Omega_*(X; \mathbf{Z}/2)$, is the set of bordism classes of maps

$$f\colon M^n \longrightarrow X$$

where M^n is a **Z/2**-manifold without ∂. Specifically, $(f_1, M_1^n) \sim (f_2, N_2^n)$ if there is a **Z/2**-manifold pair (W^{n+1}, p) with $\partial W = M_1^n \sqcup N_2^n$ and p restricted to M_i^n is f_i. There is a Hurewicz homomorphism

$$h\colon \Omega_*(X; \mathbf{Z}/2) \longrightarrow H_*(X; \mathbf{Z}/2)$$

defined by the rule $h\{(M, f)\} = f_*([M^n]) \in H_n(X; \mathbf{Z}/2)$, where $[M^n] \in H_n(M; \mathbf{Z}/2)$ is the fundamental class.

Likewise, there is an associative pairing

$$e_{X,Y}\colon \Omega_*(X; \mathbf{Z}/2) \otimes \Omega_*(Y; \mathbf{Z}/2) \longrightarrow \Omega_*(X \times Y; \mathbf{Z}/2)$$

defined on generators by

$$e_{X,Y}(\{M_1, f_1\} \times \{M_2, f_2\}) = \{M_1 \times M_2, f_1 \times f_2\}$$

so $\Omega_*(pt; \mathbf{Z}/2) = \mathcal{N}_*$ inherits the structure of a graded ring with unit and $\Omega_*(X; \mathbf{Z}/2)$ is always a graded module over $\mathcal{N}$. $\mathcal{N}_0 = \mathbf{Z}/2$ and there is a natural augmentation $\varepsilon\colon \mathcal{N} \longrightarrow \mathbf{Z}/2$ with kernel the ideal $\mathcal{I} = \bigoplus_{j>0} \mathcal{N}_j$. Then the composition

$$h \circ e_{pt,X}\colon \mathcal{I} \otimes \Omega_*(X; \mathbf{Z}/2) \longrightarrow H_*(X; \mathbf{Z}/2)$$

is the zero map. It follows that there is a well defined homomorphism

$$(1.10) \qquad \hat{h}\colon \Omega_*(X; \mathbf{Z}/2) \otimes_{\mathcal{N}_*} ((\mathcal{N}_*/\mathcal{I}) = \mathbf{Z}/2) \longrightarrow H_*(X; \mathbf{Z}/2).$$

The following result is well known.

THEOREM 1.11. (a) h *is surjective*

(b) $\hat{h}$ *above is an isomorphism. In particular*

$$\Omega_*(X;\mathbf{Z}/2) \cong \mathcal{N}_* \otimes_{\mathbf{Z}/2} H_*(X;\mathbf{Z}/2)$$

(c) $\Omega_*(X \times Y;\mathbf{Z}/2) \cong \Omega_*(X;\mathbf{Z}/2) \otimes_{\mathcal{N}_*} \Omega_*(Y;\mathbf{Z}/2)$.

As an alternate means of studying $\Omega_*(X;\ \mathbf{Z}/2)$, note that we also have the exact sequence

$$(1.12) \quad \cdots \longrightarrow \Omega_n(X) \xrightarrow{\times 2} \Omega_n(X) \longrightarrow \Omega_n(X;\mathbf{Z}/2) \xrightarrow{\partial} \Omega_{n-1}(X) \xrightarrow{\times 2} \cdots$$

where $\Omega_n(X)$ is the usual oriented bordism group of X. In particular, let δM be the transverse inverse image of a point on S^1 under the map $\hat{w}_1 \colon M \to S^1$ corresponding to the particular integral lifting of the first Stiefel-Whitney class of M, then the class of (M, f) in $\Omega_n(X)$ is completely specified by the evaluation of the mod 2 characteristic numbers on $(\delta M, f|_{\delta M})$, and on the oriented manifold with boundary $(M - \delta M, \partial(M - \delta M), f)$ independently.

The most important group for our applications is $\mathbf{Z}/2$ itself, and for this group we now give explicit systems of generators for $\Omega_*(B_{\mathbf{Z}/2};\mathbf{Z}/2)$ as a $\mathcal{N}_*$ module.

Recall that

$$H_i(B_{\mathbf{Z}/2};\mathbf{Z}/2) = \mathbf{Z}/2, \qquad i \geq 0,$$

and, if γ_i denotes the non-zero element in dimension i, then

$$\gamma_i\gamma_j = \binom{i+j}{j}\gamma_{i+j}.$$

Thus, as a graded ring,

$$(1.13) \qquad H_*(B_{\mathbf{Z}/2};\mathbf{Z}/2) \cong E(e_1) \otimes E(e_2) \otimes E(e_4) \otimes \cdots E(e_{2^i}) \cdots$$

an exterior algebra on generators in each dimension a power of 2.

We have an action of $\mathbf{Z}/2$ on $\mathbf{RP}^n$ defined by the rule

$$T(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1}, -x_n).$$

T is orientation reversing when n is odd. Define

$$(1.14) \qquad M^{2n} = S^1 \times \mathbf{RP}^{2n-1}/\{(y, \vec{x}) \sim (-y, T\vec{x})\}.$$

EXAMPLE 1.15. The first of these, $M^2 = S^1 \times_T S^1$ is just the Klein bottle, with fundamental group $\mathbf{Z} \times_T \mathbf{Z} = \{T, \tau \mid \tau T \tau^{-1} = T^{-1}\}$. The remaining manifolds all have Abelian fundamental group $(\mathbf{Z}/2) \times \mathbf{Z}$.

We have

LEMMA 1.16. *M^{2n} is the projectivized sphere bundle of the vector bundle $\omega = (2n-1)\varepsilon \oplus \zeta_1$ over the circle S^1, where ζ_1 is the nontrivial line bundle and ε is the trivial line bundle. Consequently*

$$H^*(M^{2n}; \mathbf{Z}/2) \cong H^*(\mathbf{RP}^{2n-1}; \mathbf{Z}/2) \otimes H^*(S^1; \mathbf{Z}/2)$$

and, if e_1 is the generator dual to the section $y \to (y, (1, 0, \cdots, 0))$ (thought of as the generator on the fiber), then

$$e_1^{2n} = e_1^{2n-1} \cup p^*(f)$$

is the non-trivial class.

(Recall that $e_1^{2n} = \sum w_i(\omega)e^{2n-i}$ {see e.g. [8]} where the $w_i(\omega)$ are the Stiefel-Whitney classes of the bundle ω, and, since $W(\omega) = 1 + p^*(f)$, the result follows.)

COROLLARY 1.17. (a) *M^{2n} is bordant to zero in $\Omega_{2n}(pt; \mathbf{Z}/2)$.*
(b) *The ith Wu class of M^{2n} is given as*

$$V_i = \binom{2n-i-1}{i} e_1^i + \binom{2n-i-1}{i-1} e_1^{i-1} p^*(f).$$

(Indeed,

$$\mathrm{Sq}^i(e_1^{2n-i}) = \binom{2n-i}{i} e_1^{2n},$$

$$\mathrm{Sq}^i(e_1^{2n-i-1} p^*(f)) = \binom{2n-i-1}{i} e_1^{2n}$$

and it is easily checked that V_i above when cupped with these classes gives the same result.)

REMARK 1.18. For future use, let $M^1 = S^1$, and $M(i) = M^{2^i}$. Then for each of the spaces $M(i)$, $i = 0, 1, 2, \ldots$ we have that V^2 is just the class 1 in dimension 0, while $V \mathrm{Sq}^1(V)$, $(\mathrm{Sq}^1 V)^2$ are identically zero.

COROLLARY 1.19. *Let $p_m: M^{2n} \to B_{\mathbf{Z}/2}$ be the map sending the fundamental class in $H^1(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$ to e_1, then the composite map*

$$M^{2n} \times M^{2n} \xrightarrow{p_n \times p_n} B_{\mathbf{Z}/2} \times B_{\mathbf{Z}/2} \xrightarrow{e} B_{\mathbf{Z}/2}$$

represents 0 in $\Omega_{4n}(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$.

Proof.

$$\delta(M^{2n} \times M^{2n}) = (\mathbf{RP}^{2n-1} \times \mathbf{RP}^{2n-1}) \times_T S^1$$

where T acts diagonally on $\mathbf{RP}^{2n-1} \times \mathbf{RP}^{2n-1}$, and freely on S^1.

The interchange map on $M^{2n} \times M^{2n}$ restricts to the map on $\delta(M^{2n} \times M^{2n})$ which just interchanges the two copies of $\mathbf{RP}^{2n-1}$ and is fixed on S^1. In both $\delta(M^{2n} \times M^{2n})$ and $M^{2n} \times M^{2n}$ the class $(e(p \times p)^*)\iota = (\tau + 1)A$, where τ is the cohomology map induced by the involution above. From this it follows (since the Stiefel-Whitney classes of $M^{2n} \times M^{2n}$ and $\delta(M^{2n} \times M^{2n})$ are invariant under τ) that every characteristic class in top dimension is of the form $(\tau + 1)B$ for some B, but since $\tau = 1$ in dimension $2n$ all these classes are zero and the result follows.

COROLLARY 1.20. $\Omega_*(B_{(\mathbf{Z}/2)^n}; \mathbf{Z}/2) = \bigotimes_{j=1\cdots n} \otimes_i E(\{M(i), f_j\})$, where f_j maps $M(i)$ into the jth summand of $B_{(\mathbf{Z}/2)^n} = (B_{\mathbf{Z}/2})^n$ as a ring over $\mathcal{N}_*$.

C. *The homology and bordism of the dihedral groups $D(2^n, 2)$ and the quaternion groups $Q(2^{i+2})$.* We have the following central extension sequences for $D(2^i, 2)$, and $Q(2^i)$,

(1.21)
$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow D(2^i, 2) \longrightarrow D(2^{i-1}, 2) \longrightarrow 1$$
$$1 \longrightarrow \mathbf{Z}/2 \longrightarrow Q(2^{i+2}) \longrightarrow D(2^i, 2) \longrightarrow 1.$$

Thus, we proceed by using the first of these sequences to study $D(2^i, 2)$, and then use the second to obtain $Q(2^i)$ from our knowledge of the dihedral groups.

A more symmetric presentation (and one more adapted to the use of 1.21 than that in 1.8) is given by

(1.22) $D(2^i, 2) = \{x, y \mid x^2 = y^2 = (xy)^{2^i} = 1\}$

(set $y = \tau T$, $x = \tau$ in (1.8)), and we have two homomorphisms

(1.23) $\phi_1^i, \phi_2^i \colon \mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow D(2^i, 2),$
$$\phi_j^i(\tau) = (xy)^{2^{i-1}}, \quad \phi_1^i(T) = x, \quad \phi_2^i(T) = y.$$

Note also that $(D(2^i, 2))^{ab} = \mathbf{Z}/2 \times \mathbf{Z}/2$, and let the projection be

denoted by p. We then have the commutative diagram

$$(1.24)$$

$$
\begin{array}{ccc}
\mathbf{Z}/2 & \xrightarrow{\;=\;} & \mathbf{Z}/2 \\
\downarrow & & \downarrow \\
\mathbf{Z}/2 \times \mathbf{Z}/2 \xrightarrow{\;\phi_j^2\;} D(4,2) \xrightarrow{\;\phi_j^i\;} D(2^{i+1},2) \xrightarrow{\;p\;} \mathbf{Z}/2 \times \mathbf{Z}/2 \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow= \\
\mathbf{Z}/2 \times \mathbf{Z}/2 \xrightarrow{\;\phi_j^i\;} D(2^i,2) \xrightarrow{\;p\;} \mathbf{Z}/2 \times \mathbf{Z}/2
\end{array}
$$

and the composite $\phi_j^i \phi_j^2$ is ϕ_j^{i+1}. Specific classes $\langle x \rangle$, $\langle y \rangle$ in $H^1(D(2^i,2);\ \mathbf{Z}/2)$ are given as the composites $p^* \lambda_j^*(e)$ where

$$\lambda_j: \mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2$$

are the projections onto the jth factors (so e.g. $\lambda_1 p(x) = T$, $\lambda_1 p(y) = 1$) and e is the non-trivial class in $H^1(\mathbf{Z}/2;\ \mathbf{Z}/2)$. Then we have

THEOREM 1.25. (a) $H^*(D(2^i,2);\ \mathbf{Z}/2) = \mathbf{F}_2[x,y,w]/xy = 0$, with x, y both 1-dimensional, and w 2-dimensional. Moreover, $Sq^1(w) = (x+y)w$, and the k-invariant for the extension sequence above (with i replaced by $i+1$) is w.

(b) The two inclusions

$$\phi_j^i : \mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow D(2^i,2),$$

together surject in mod 2 homology.

(c) The inclusion of the center $\alpha: \mathbf{Z}/2 \longrightarrow D(2^i,2)$ induces

$$\alpha^* : H^*(D(2^i,2);\ \mathbf{Z}/2) \longrightarrow H^*(\mathbf{Z}/2;\ \mathbf{Z}/2),$$

and $\alpha^*(w) = e^2$, $\alpha^*(x) = \alpha^*(y) = 0$.

(d) The two maps ϕ_j^i satisfy $\phi_j^{i*}(w) = y^2 + xy$, while $\phi_1^{i*}(x) = x$, $\phi_1^{i*}(y) = 0$, $\phi_2^{i*}(x) = 0$, $\phi_2^{i*}(y) = x$.

Proof. The proof is by induction. First some general remarks are required. Assuming that the result is true for $D(2^i,2)$ we note that the k-invariant for $D(2^{i+1},2)$ must be w_i. Indeed, from (1.24) we have that $\phi_j^i(\kappa)$ is a suitable k-invariant for $D(4,2)$. But, by the induction, the only image which is suitable for such a k-invariant is $y^2 + xy$. Moreover, there must be some element γ in $H^2(D(2^{i+1},2);\ \mathbf{Z}/2)$ with composite image under all four of the maps $\phi_j^2 \phi_k^i$ the k-invariant for a dihedral extension.

But, as we just observed, there is only one such image from $H^2(D(4, 2); \mathbf{Z}/2)$. So there is an element in $H^2(D(2^{i+1}, 2); \mathbf{Z}/2)$ which we can take to be our new class w_{i+1}. Moreover, again referring to (1.24) and the definitions of x, y, it is direct to see that the maps ϕ_j^{i+1} behave correctly on x and y. So, to continue the induction we need only show the injectivity of these two maps in cohomology, and the fact that $\mathrm{Sq}^1(w_{i+1}) = (x + y)w_{i+1}$.

For the moment let us assume the injectivity. Then

$$\mathrm{Sq}^1(\phi_j^{i+1}(w_{i+1})) = xy^2 + x^2y = x(\phi_j^{i+1}(w_{i+1})).$$

But the only class which can have this image under both maps is $(x + y)w_{i+1}$, and the proof will be complete once we have demonstrated that $H^*(D(2^{i+1}, 2); \mathbf{Z}/2)$ has the form stated in the theorem.

For this consider the Serre spectral sequence of the fibering associated to the central extension giving $D(2^{i+1}, 2)$. Its $E_2^{i,j}$ term is $H^i(D(2^i, 2); H^j(\mathbf{Z}/2; \mathbf{Z}/2))$ which can be written as

$$E_2^{i,j} = H^i(D(2^i, 2); \mathbf{Z}/2) \otimes H^j(\mathbf{Z}/2; \mathbf{Z}/2).$$

The differential is completely determined by $d^2(e_1) = w_i$, and we have that

$$E_3^{*,*} = \{\mathbf{Z}/2[x, y]/(xy = 0)\} \otimes \mathbf{Z}/2[e^2].$$

But $d^3(e^2) = \mathrm{Sq}^1(w_i) = (x + y)w_i = 0$ in $E_3^{3,0}$, so $E_3 = E_\infty$, and this completes the inductive step.

The proof of 1.25 will be completed once we show we can start the induction. So construct $D(4, 2)$ from the extension sequence

$$\mathbf{Z}/2 \longrightarrow D(4, 2) \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

with k-invariant xy. First, a spectral sequence argument identical to that above shows $H^*(D(4, 2); \mathbf{Z}/2) = \mathbf{Z}/2[x, y, w]/(xy = 0)$. Next, we must check that the image of w_2 under the two maps ϕ_j^2 is as stated. Note first that on going up to $D(8, 2)$ the original center of $D(4, 2)$ is embedded into the extension sequence

$$\mathbf{Z}/2 \longrightarrow \mathbf{Z}/4 \longrightarrow \mathbf{Z}/2$$

This implies that y^2 must be part of the image of w. On the other hand, the image of w must also include xy, since the extension induced on $\mathbf{Z}/2 \times \mathbf{Z}/2$ is $D(4, 2)$. Now the proof is complete.

Thus, the discussion above for $B_{\mathbf{Z}/2 \times \mathbf{Z}/2}$ gives us control of the $\mathbf{Z}/2$-bordism of $B_{D(2^i, 2)}$.

COROLLARY 1.26.

$$H^*(Q(2^i); \mathbf{Z}/2) = \mathbf{Z}/2[x, y, v_i]/(x^3 = y^3 = xy = 0)$$

for $i > 3$, while

$$H^*(Q(8); \mathbf{Z}/2)$$
$$= \mathbf{Z}/2[x, y, v_1]/(x^2 + xy + y^2 = 0, \, xy^2 + x^2 y = 0)$$

where v_i has dimension 4.

Proof. Consider the diagram of central extensions

$$
\begin{array}{ccccc}
\mathbf{Z}/2 & \longrightarrow & Q(8) & \longrightarrow & \mathbf{Z}/2 \times \mathbf{Z}/2 \\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \phi_j^{i-1}} \\
\mathbf{Z} & \longrightarrow & Q(2^{i+2}) & \longrightarrow & D(2^{i-1}, 2)
\end{array}
$$

From 1.25 and its proof the k-invariant for the bottom extension must be $w_{i-1} + x^2 + y^2$. Hence, in the spectral sequence $d^2(e) = x^2 + y^2 + w_{i-1}$, so $d^3(e^2) = (x+y)w_{i-1} = x^3 + y^3$ in $E_3^{3,0}$. But from this $d^5(e^4) = 0$ and $E_4 = E_\infty$. The result follows for $i > 1$. The case $i = 1$ is virtually the same.

In particular the quaternion groups are periodic of period 4, with

$$H^1(Q(2^i); \mathbf{Z}/2) = H^2(Q(2^i); \mathbf{Z}/2) = (\mathbf{Z}/2)^2,$$

and $H^3(Q(2^i); \mathbf{Z}/2) = \mathbf{Z}/2$. Moreover, the central inclusion

$$\mathbf{Z}/2 \longrightarrow Q(2^i)$$

induces a surjection of $H_{4*}(\mathbf{Z}/2; \mathbf{Z}/2)$ onto the periodicity classes in the homology of $Q(2^i)$. Explicit representatives for the classes in dimension 1 are the maps

$$(1.27) \qquad\qquad f_{1,i} \colon S^1 \longrightarrow B_{Q(2^i)}$$

given by surjection onto the subgroups generated by T, τ, respectively. Similarly, explicit representatives for the classes in dimension 2 are given by the two maps

$$f_{2,i} \colon M(1) \longrightarrow B_{Q(2^i)},$$

surjecting $\mathbf{Z} \times_T \mathbf{Z}$ onto $Q(2^i)$, by sending T to T, τ to τ, $T\tau$, respectively. Finally, an explicit representative for the class in dimension 3 is given by the quotient $S^3/Q(2^i)$, where the action is the usual

one. Then we have

THEOREM 1.28.

$$\Omega_*(B_{Q(2^i)}; \mathbf{Z}/2)$$

$$= \bigoplus_{i \geq 2} E(\{M(i), f_{2^i}\}) \otimes \{1, \{S^1, f_1\}, \{S^1, f_2\}, \{M(1), f_{2,1}\},$$

$$\{M(1), f_{2,2}\}, \{S^3/Q^{2^{i+2}}, f\}\}$$

as a module over $\Omega_*(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$.

REMARK 1.29. Since $S^3/Q(2^i)$ is oriented its total Wu class is 1, and all our models for generators over $\Omega_*(pt; \mathbf{Z}/2)$ have the property that all three classes (V^2), $V \operatorname{Sq}^1 V$, $(\operatorname{Sq}^1 V)^2$, are 0 in positive dimensions. As a consequence, the κ_i, τ_j, γ_k of the introduction in $H^*(B_\pi; \mathbf{Z}/2)$ for π one of the groups above, are calculated directly from the evaluation of the images of the $\{M, f\}$ in $L_*^h(\mathbf{Z}(\zeta_3)\pi; \mathbf{Z}/2)$ without the necessity of solving complex equations involving the inversion of the V^2, $V \operatorname{Sq}^1 V$, $(\operatorname{Sq}^1 V)^2$.

2. The restriction theorem. The discussion of the boundary map b in the exact sequence

$$(2.1) \quad \begin{aligned} \cdots &\longrightarrow \widehat{H}_k(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi)) \longrightarrow L_k^h(\mathbf{Z}(\zeta_3)\pi) \\ &\longrightarrow L_k^p(\mathbf{Z}(\zeta_3)\pi) \xrightarrow{b} \widehat{H}_{k-1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi) \longrightarrow \cdots \end{aligned}$$

given in §6 of [5] (in particular the paragraphs immediately preceding and following 6.2) admits a more intrinsic description. Recall that the essential tool introduced in [5] to study $\widehat{H}_k(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi))$ was the exact sequence valid for an arbitrary finite 2-group

$$(2.2) \quad \begin{aligned} 0 &\longrightarrow U'K_1'(\mathbf{Z}(1/2, \zeta_3)\pi) \xrightarrow{p} UK_1(\widehat{\mathbf{Q}}_2(\zeta_3)\pi)/K_1'(\widehat{\mathbf{Z}}_2(\zeta_3)\pi) \\ &\longrightarrow \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi)_2 \longrightarrow 0. \end{aligned}$$

See in particular Remark 5.8 and Corollary 5.9 there. Since

$$(2.3) \qquad \widehat{H}_*(\mathbf{Z}/2; UK_1(\widehat{\mathbf{Q}}_2(\zeta_3)\pi)/K_1'(\widehat{\mathbf{Z}}_2(\zeta_3)\pi))) \equiv 0$$

the homology of $\widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi)$ is determined by that of $U'K_1'(\mathbf{Z}(\frac{1}{2}, \zeta_3)\pi)$.

Denote by b' the composite map

$$L_k^p(\mathbf{Z}(\zeta_3)\pi) \xrightarrow{b} \widehat{H}_{k-1}(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi))$$

$$\xrightarrow{\cong} \widehat{H}_k(\mathbf{Z}/2; U'K_1'(\mathbf{Z}(1/2, \zeta_3)\pi))$$

then we have

LEMMA 2.4. (a) *The determinant map* $\langle [\mathbf{Z}(1/2, \zeta_3)\pi]^n, \ B \rangle \rightarrow$ $K_1(\mathbf{Z}(1/2, \zeta_3)\pi)$, *when restricted to the quadratic forms coming from* $L_k^p(\mathbf{Z}(\zeta_3)\pi)$ *induces a well defined map (the discriminant),*

$$d_*: L_{2k}(\mathbf{Z}(\zeta_3)\pi) \longrightarrow \widehat{H}_{2k}(\mathbf{Z}/2; U'K_1'(\mathbf{Z}(1/2, \zeta_3)\pi))$$

and d_* *is exactly the map* b' *above.*

(b) *Let* $\alpha \in L_{2k+1}^p(\mathbf{Z}(\zeta_3)\pi)$ *be represented by the formation* $(H \oplus H^*, \ K)$ *with*

$$0 \longrightarrow K \xrightarrow{p_H} H^* \longrightarrow V \longrightarrow 0$$

an exact sequence and V *an odd torsion module over* $\mathbf{Z}(\zeta_3)\pi$. *Let*

$$\{p_H\} = d \in UK_1(\widehat{\mathbf{Q}}_2(\zeta_3)\pi)$$

represent p_H, *then* $\{d^*/d\}$ *in* $UK_1(\widehat{\mathbf{Q}}_2(\zeta_3)\pi)/K_1'(\widehat{\mathbf{Z}}_2(\zeta_3)\pi) = p(a)$ *for some* a *in* $U'K_1'(\mathbf{Z}(1/2, \zeta_3)\pi)$, *and*

$$b'(\alpha) = \{a\} \in H_{2k+1}(\mathbf{Z}/2; K_1'(\mathbf{Z}(1/2, \zeta_3)\pi)).$$

((2.4) is just a reformulation of the remarks in §6 of [5] already referred to.)

Since π is a finite 2-group, all its irreducible representations arise via induction from various subquotients. Specifically, given an irreducible $\mathbf{Q}\pi$ representation, r_i, there is a subgroup π_i and a surjection $p_i: \pi_i \rightarrow M_i$, where M_i is a model group (i.e. one of the groups $\mathbf{Z}/2^j$, $D(2^i, 2)$, $Q(2^i)$, or $SD(2^i)$, {cyclic, dihedral, quaternionic, or semi-dihedral, notation as in [1]}). Then, M_i has a unique faithful $\mathbf{Q}$ representation ϕ_i, and

$$r_i = I_{\pi_i}^\pi p_i^!(\phi_i)$$

where $p_i^!$ is the composition map taking representations on M_i to representations on π_i, and $I_\pi^{\pi_i}$ is the induction map taking representations on π_i to representations on π.

In the situation above we also have the restriction (forgetful) map $R_\pi^{\pi_i}$ which takes representations on π and restricts them to representations on π_i and the push-forward map $p_{i,!}$ taking representations on π_i and pushing them forward ($\otimes$ with $\mathbf{Q}M_i$) to representations on M_i. The induction and restriction maps are related by the Frobenius reciprocity law

(2.5) $$\langle r, I_{\pi_i}^\pi(r_i) \rangle = \langle R_\pi^{\pi_i}(r), r_i \rangle.$$

Now define the total restriction maps

$$R_\pi \colon \mathbf{R}(\mathbf{Q}\pi) \longrightarrow \sum \mathbf{R}(\mathbf{Q}M_i), \quad \text{and}$$

$$R_\pi \colon L^d_*(\mathbf{Z}(\zeta_3)\pi) \longrightarrow \sum L^d_*(\mathbf{Z}(\zeta_3)M_i)$$

($d = h$ or p) as the sum over i of the compositions $p_{i,!} \circ R^{\pi_i}_\pi$, and let p_j be projection onto the summand corresponding to M_j. As a consequence of (2.5) we have

LEMMA 2.6. *Let the irreducible* $\mathbf{Q}$ *representations of* π *be* $r_1, \dots,$ r_n, *then* $p_i \circ R_\pi(r_i) = \phi_i$, *and* $p_j \circ R_\pi(r_i) = \sum n_{i,k} v_k$, *for* $j \neq i$, *where the* v_k *run over the non-faithful* $\mathbf{Q}$ *representations of* M_j. *In particular* R_π *is an injection on representation rings.*

Proof. The second statement follows directly from the first. To see the first statement note that by the reciprocity law

$$p_i \circ R_\pi(r_i) = \phi_i + \sum n_j v_{i,j},$$

and we must show that all the n_j are 0. But the $\mathbf{Q}$-rank of the representation $I^\pi_{\pi_i} p^!_{\pi_i}(v_{i,j})$ is $[\pi : \pi_i] \circ \{\mathbf{Q}\text{-rank } v_{i,j}\}$, and this is strictly smaller than the $\mathbf{Q}$-rank of $r_i = [\pi : \pi_i]\{\mathbf{Q}\text{-rank } \phi_i\}$. Consequently,

$$\langle I^\pi_{\pi_i} \circ p^!_i(v_{i,j}), r_i \rangle = 0,$$

and by the reciprocity rule again $\langle v_{i,j}, p_{!,i} R_\pi(r_i) \rangle = 0$. This completes the proof.

REMARK 2.7. The same result holds (with the same proof) for all fields $\mathbf{F}$ of characteristic different from 2. Additionally, away from 2, $\mathbf{Z}\pi$ is a maximal order, and the result holds for $\mathbf{Z}\pi$ away from 2 as well. In particular, if we consider an odd order $\mathbf{Z}\pi$ torsion module N, we see that $N = \sum N_i$, one direct summand for each irreducible $\mathbf{Q}$ representation of π, $R_\pi(N) = \sum N$, one copy for each i, but after tensoring with $\mathbf{Z}M_i$ we get $N_i \oplus V$ where V is a module over the $V_{i,j}$.

COROLLARY 2.8. *The map* $R_\pi \colon L^p_*(\mathbf{Z}(\zeta_3)\pi) \to \sum L^p_*(\mathbf{Z}(\zeta_3)M_i)$ *is an injection.*

Proof. We first verify 2.8 for the odd groups $L^p_{2i+1}(\mathbf{Z}(\zeta_3)\pi)$. A basis for $L^p_{2i+1}(\mathbf{Z}(\zeta_3)\pi)$ is represented by formations

$$\alpha_j = (H \oplus H^*, K)$$

each satisfying the hypothesis of Lemma 2.4(b). More particularly, the odd torsion module V_α is an irreducible $\mathbf{F}_3$ module corresponding to the irreducible representation r_j, and for $i = 0$, the r_j run over all representations induced up from $\mathbf{Z}/2$'s or $D(2^i, 2)$'s. Similarly for $i = 1$, the r_j run over all the representations induced up from $Q(2^i)$'s. Then the restrictions of these formations represent linearly independent elements in $\sum L_{2i+1}^p(\mathbf{Z}(\zeta_3)M_j)$.

For $L_{2i}^p(\mathbf{Z}(\zeta_3)\pi)$ we note that the natural map

$$\bigotimes \mathbf{Q} \colon L_{2i}^p(\mathbf{Z}(\zeta_3)\pi) \longrightarrow L_{2i}^p(\mathbf{Q}(\zeta_3)\pi)$$

is an inclusion, so it suffices to verify the lemma for these latter groups. But

$$L_{2i}^p(\mathbf{Q}(\zeta_3)\pi) = \bigoplus L_{2i}^p(M_{n_i}(\mathbf{K}_i))$$

where these $M_{n_i}(\mathbf{K}_i)$ are the irreducible representations over $\mathbf{Q}(\zeta_3)$. Once more the restriction and projection maps behave nicely. In particular,

$$p_i \circ R \colon L_{2i}^p(M_{n_i}(\mathbf{K}_i)) \xrightarrow{\ \cong\ } L_{2i}^p(M_{m_i}(\mathbf{K}_i))$$

where the only change is in the size of the matrices, not the field. Consequently, the lemma follows.

Extending 2.8 to the $L^h(\ )$ groups is not quite as direct as one would hope. The difficulty is the classes in $L_2^h(\mathbf{Z}(\zeta_3)\pi)$ coming from $H_*(\mathbf{Z}/2; \widetilde{K}_0(\mathbf{Z}(\zeta_3)\pi))$.

From [5, §6], these classes are associated to elements coming from representations onto dihedral subquotients, and $\mathbf{Z}/2$-subquotients. The dihedral model groups have the desired elements present in their L^h groups, so they present little difficulty, but the $\mathbf{Z}/2$'s are more complex. In case the representation is a matrix ring of the form $M_{2^i}(\mathbf{Q})$ with $i > 0$ then it is always possible to take the associated subquotient to be a dihedral group $D(4, 2)$, and otherwise, the representation is pulled back by the map $\pi \longrightarrow \pi^{\mathrm{ab}}$. This reduces the problem to that for π Abelian.

When π is $\mathbf{Z}/2 \times \mathbf{Z}/2$ the element is already present as we noted in §7 of [5], so the strategy is to replace the $\mathbf{Z}/2$'s by $\mathbf{Z}/2 \times \mathbf{Z}/2$'s.

In particular, it is an easy exercise to show that for Abelian groups π the set of maps $q_j \colon \pi \to \mathbf{Z}/2 \times \mathbf{Z}/2$ give an injection of the exotic elements in $L_2^h(\mathbf{Z}(\zeta_3)\pi)$ into the corresponding elements in the sum of the $L^h(\ )$ for these model groups.

R_π also induces maps of the algebraic K groups $K_i(A\pi) \to \sum K_i(AM_i)$, and in a way virtually identical to the proof above we have

LEMMA 2.9. *The map* $R_\pi : K_1(\mathbf{Z}(1/2, \zeta_3)\pi) \to \sum K_1(\mathbf{Z}(1/2, \zeta_3)M_i)$ *is an injection.*

Let us define a further map

$$(2.10) \qquad S_\pi : L_k^h(\mathbf{Z}(\zeta_3)\pi) \longrightarrow \bigoplus L_k^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2)$$

where the sum above runs over all distinct homomorphisms $\pi \to \mathbf{Z}/2 \times \mathbf{Z}/2$. 2.9, together with 2.4 implies the main technical result of this section.

THEOREM 2.11. *The map*

$$R_\pi \oplus S : L_*^h(\mathbf{Z}(\zeta_3)\pi) \longrightarrow \bigoplus L_*^h(\mathbf{Z}(\zeta_3)M_i) \oplus \sum L_*^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2)$$

is an injection on the subgroups described in Theorem 6.1 of [5].

Proof. The classes of interest in 6.1 all either inject into $L_*^p(\mathbf{Z}(\zeta_3)\pi)$ or are associated to one of the $\mathbf{Z}/2$'s {that is, the elements ± 1 at the faithful representation of M_i} corresponding to $M_i = \mathbf{Z}/2$, a dihedral group, or a quaternion group. So we must show that R_π on such an element is not in the kernel of the map

$$\widehat{H}_k(\mathbf{Z}/2 ; \widetilde{K}_0(\mathbf{Z}(\zeta_3)M_i)) \longrightarrow L_k^h(\mathbf{Z}(\zeta_3)M_i).$$

In the case where k above is even, the situation is quite clear. We consider first the case $k = 3$. Then the image of b' consists exactly of the set of -1's at the quaternion representations. In particular, the -1's at all the other representations are not in the image of b', so the result is true.

When $k = 2$ the critical elements have already been shown to exist for the dihedral restriction models and the $\mathbf{Z}/2 \times \mathbf{Z}/2$ models after projection to the abelianization.

When $k = 1$ the image of b' consists of all the -1's at the representations corresponding to $\mathbf{Z}/2$'s or dihedral groups. Once more there is no overlap onto the faithful representations for the quaternion groups, and so the result follows in this case.

Finally, we need the case $k = 0$. Here, all the elements at all the $\mathbf{Z}/2$ and dihedral groups are hit, but the homomorphism

$$L_3^h(\mathbf{Z}(\zeta_3)Q(2^i)) \xrightarrow{\tau_K} L_1^h(\mathbf{Z}Q(2^i))$$

is non-trivial on the element coming from the -1 at the faithful representation, and this provides a homomorphism out, which proves the injectivity of the map at this level. Thus the proof is complete.

For later use we need the following corollary of 2.11 which reduces the calculation of the map

$$(2.12) \qquad\qquad \nu : \Omega_k(B_\pi) \longrightarrow L_k^h(\mathbf{Z}(\zeta_3)\pi)$$

to the same problem but only for certain of the model groups.

COROLLARY 2.13. *Let* $a \in \Omega_k(B_\pi)$, *then* $\nu(a)$ *is non-zero if and only if there is a* π_i *as above so that* $p_i \circ R_\pi(\nu(a))$ *is non-zero. In other words, restriction to the model groups* $\mathbf{Z}/2$, $D(2^i, 2)$, *or projection onto groups of the form* $\mathbf{Z}/2 \times \mathbf{Z}/2$ *completely determines the map* ν.

3. L theory with coefficients. There is a natural orientation

$$(3.1) \qquad\qquad \hat{e}_{\zeta_3} : \Omega_*(B_\pi ; \mathbf{Z}/2) \longrightarrow L_*^h(\mathbf{Z}(\zeta_3)\pi \times \mathbf{Z}^-)$$

where the involution on $\pi \times \mathbf{Z}^-$ is given by $\tau \leftrightarrow \tau^{-1}$ for $\tau \in \pi$, while $T \leftrightarrow -T^{-1}$ for T the generator of the $\mathbf{Z}$. It is obtained by using the product

$$f \times \{\hat{w}_1\}: M \longrightarrow B_\pi \times S^1$$

where $\hat{w}: M \to S^1$ is the integral lifting of w_1, and $f: M \to B_\pi$ is the map which induces the π covering of M.

For these L groups we have (see e.g. [7])

THEOREM 3.2. *Let* A *be any unitary ring with involution*

$$\tau : A \longrightarrow A,$$

and suppose $u \in \text{center}(A)$ *is a unit fixed under* τ. *Define an involution*

$$\tau^u : A[t, t^{-1}] \longrightarrow A[t, t^{-1}]$$

as τ *on* A *and* $\tau^u(t) = ut^{-1}$, *then there is an exact sequence*

$$\cdots \to L_*^p(A, \tau) \xrightarrow{\langle 1 \rangle - \langle u \rangle} L_*^h(A, \tau) \longrightarrow L_*^h(A[t, t^{-1}], \tau^u) \longrightarrow \cdots$$

$$\cdots \longrightarrow L_{*-1}^p(A, \tau) \xrightarrow{\langle 1 \rangle - \langle u \rangle} L_{*-1}^h(A, \tau) \to \cdots$$

A special case is τ^{-1}, which we abbreviate as τ^-. Consequently, the exact sequence above is extremely efficient for calculating the groups $L_*^h(\mathbf{Z}(\zeta_3)\pi \times \mathbf{Z}^-)$, but, because of the group $L_*^p(\mathbf{Z}(\zeta_3)\pi)$ appearing in the sequence, it is not quite optimally adapted to the study

of the map s_π of the introduction. For that we need a further class of L-groups.

The L groups $L_n^h(A; \mathbf{Z}/2)$ are defined in terms of algebraic models which consist of n-dimensional quadratic Poincaré complexes D with

$$\partial D = \delta D \oplus \delta D$$

where δD is a closed Poincaré complex of dimension $n - 1$. Bordism is given by an $n + 1$ dimensional Poincaré complex W with

$$\partial W = D \cup_{\delta D \oplus \delta D} C \oplus C \quad \text{where } \partial C = \delta D.$$

There is an exact sequence

(3.3)
$$\cdots \longrightarrow L_n^h(A) \xrightarrow{\times 2} L_n^h(A) \longrightarrow L_n^h(A; \mathbf{Z}/2)$$
$$\longrightarrow L_{n-1}^h(A) \xrightarrow{\times 2} L_{n-1}^h(A) \longrightarrow \cdots$$

and (3.1), 3.2 fit together to give the commutative diagram

(3.4)
$$
\begin{array}{ccccccc}
\cdots \longrightarrow & L_*^h(A) & \xrightarrow{\times 2} & L_*^h(A) & \longrightarrow & L_*^h(A; \mathbf{Z}/2) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow v & \\
\cdots \longrightarrow & L_*^p(A) & \xrightarrow{\times 2} & L_*^h(A) & \longrightarrow & L_*^h(A[t, t^{-1}], \tau^-) & \longrightarrow \cdots
\end{array}
$$

We can use the map v in (3.4) to obtain information about

$$\hat{e}_{\zeta_3} : \Omega_*(B_\pi; \mathbf{Z}/2) \longrightarrow L_*^h(\mathbf{Z}(\zeta_3)\pi; \mathbf{Z}/2)$$

because the following diagram commutes

(3.5)
$$
\begin{array}{ccc}
\Omega_*(B_\pi; \mathbf{Z}/2) & \xrightarrow{\hat{e}_{\zeta_3}} & L_*^h(\mathbf{Z}(\zeta_3)\pi, \mathbf{Z}/2) \\
\downarrow & & \downarrow v \\
\Omega_*(B_{\pi \times \mathbf{Z}}; w_1) & \xrightarrow{e_{\zeta_3}} & L_*^h(\mathbf{Z}(\zeta_3)\pi \times \mathbf{Z}^-)
\end{array}
$$

This procedure is especially effective when $\tilde{K}_0(\mathbf{Z}(\zeta_3)\pi) = 0$ which is the case when $\pi = 1$, $\mathbf{Z}/2$ or $\mathbf{Z}/4$ since then v is an isomorphism. The point is that standard techniques with standard closed Poincaré duality chain complexes allow us to calculate the image of the map e_{ζ_3} above, but the map $\hat{e}_{\zeta_3}$ is more complex, since it requires $\mathbf{Z}/2$ type chain complexes, and the interpretation of the image is also more complex. This will be illustrated in the next two sections where we study the explicit images of the map e_{ζ_3} for $\mathbf{Z}/2$.

There is also a product pairing

$$(3.6) \qquad L_h^i(\mathbf{Z};\mathbf{Z}/2) \otimes L_j^h(\mathbf{Z}(\zeta_3)\pi;\mathbf{Z}/2) \xrightarrow{\nu} L_{i+j}^h(\mathbf{Z}(\zeta_3)\pi;\mathbf{Z}/2)$$

which fits into a commutative diagram analogous to (3.5)

(3.7)

$$\begin{array}{ccccc}
\mathscr{N}_* \otimes \Omega_*(B_\pi;\mathbf{Z}/2) & \xrightarrow{e \otimes \hat{e}_{\zeta_3}} & L_j^h(\mathbf{Z};\mathbf{Z}/2) \otimes L_*^h(\mathbf{Z}(\zeta_3)\pi,\mathbf{Z}/2) & \xrightarrow{\nu} & L_*^h(\mathbf{Z}(\zeta_3)\pi,\mathbf{Z}/2) \\
\downarrow & & \downarrow{\scriptstyle 1 \otimes v} & & \downarrow{\scriptstyle v} \\
\mathscr{N}_* \otimes \Omega_*(B_{\pi \times \mathbf{Z}};w_1) & \xrightarrow{e \otimes e_{\zeta_3}} & L_j^h(\mathbf{Z}(\mathbf{Z}^-)) \otimes L_*^h(\mathbf{Z}(\zeta_3)\pi \times \mathbf{Z}^-) & \xrightarrow{\nu} & L_*^h(\mathbf{Z}(\zeta_3)\pi \times \mathbf{Z}^-)
\end{array}$$

We have from ([**10**, page 173]) that

$$L_h^j(\mathbf{Z};\mathbf{Z}/2) = L_h^j(\mathbf{Z}(\mathbf{Z}^-)) = \begin{cases} \mathbf{Z}/2 & \text{generator } I, & j \equiv 0 \bmod 4 \\ \mathbf{Z}/2 & \text{generator } \gamma, & j \equiv 1 \bmod 4 \\ \mathbf{Z}/2 & \text{generator } \tau, & j \equiv 2 \bmod 4 \\ 0 & & j \equiv 3 \bmod 4 \end{cases}$$

Moreover, from [**3**] or [**9**], the map

$$\tilde{e}_{\zeta_3} : \mathscr{N}_* \longrightarrow L_h^*(\mathbf{Z})$$

is given by the formula

$$\tilde{e}_{\zeta_3}\{M\} = \langle V^2,[M]\rangle I + \langle V\,\mathrm{Sq}^1 V,[M]\rangle \gamma + \langle (\mathrm{Sq}^1 V)^2,[M]\rangle \tau.$$

Thus, we obtain the general form of the evaluation formulae similar to that given in [**13**], as

COROLLARY 3.8. *There are well defined elements* $\iota_{*,\kappa}$, $\gamma_{*,\kappa}$, $\tau_{*,\kappa}$ *in* $H^*(B_\pi;\mathbf{Z}/2)$ *which have the property that the image of* $\{M,f\} \in \Omega_*(B_\pi,\mathbf{Z}/2)$ *in* $L_*^h(\mathbf{Z}(\zeta_3)\pi;\mathbf{Z}/2)$ *is given by a collection of formulae*

$$(\langle V^2 f^*(\iota_{*,\kappa}),[M]\rangle + \langle V\,\mathrm{Sq}^1 V f^*(\gamma_{*,\kappa}),[M]\rangle$$
$$+ \langle (\mathrm{Sq}^1 V)^2 f^*(\tau_{*,\kappa}),[M]\rangle)\kappa,$$

one for each generator κ *in* $L_*^h(\mathbf{Z}(\zeta_3)\pi,\mathbf{Z}/2)$.

In our explicit case it turns out that the terms involving γ are 0, since we have

THEOREM 3.9. *The maps*

$$e_{\zeta_3}(\gamma *) : \Omega_*(B_\pi) \longrightarrow L_{*+1}^h(\mathbf{Z}(\zeta_3)\pi)$$

$$\hat{e}_{\zeta_3}(\gamma *) : \Omega_*(B_\pi;\mathbf{Z}/2) \longrightarrow L_{*+1}^h(\mathbf{Z}(\zeta_3)\pi;\mathbf{Z}/2)$$

are identically zero while the map

$$e_{\zeta_3}(\tau *): \ \Omega_*(B_\pi; \mathbf{Z}/2) \longrightarrow L^h_{*+2}(\mathbf{Z}(\zeta_3)\pi; \mathbf{Z}/2)$$

is multiplication by the non-trivial element in $L^h_2(\mathbf{Z}(\zeta_3); \mathbf{Z}/2) = \mathbf{Z}/2$.

Proof. $\{\gamma\}$ is represented by the formation

$$(3.10) \qquad \gamma = \left\{ \begin{array}{ccc} \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2} \\ \mathbf{Z} & \xrightarrow{1} & \mathbf{Z} \end{array} \right\}$$

with generators A, B where $\partial B = 2A$, and $\phi_1: B^* \to B$.

This can be verified by observing that the formation associated to the complex above (see [10, p. 162], for details on how to obtain this correspondence) is

$$\left\{ \mathbf{Z} \oplus \mathbf{Z}, (0, 1), (2, 1), \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

But as in ([10, pp. 169-173]) this is the boundary in the (symmetric) localization sequence of the torsion form $\{\mathbf{Z}/2, 1/2\}$ which represents the deRham invariant.

$\gamma \oplus \gamma$ is an explicit boundary as follows. Let $\{V, \partial, \Psi\}$ be the complex

$$(3.11) \qquad \{V, \partial, \Psi\} = \begin{array}{ccc} \mathbf{Z} & & \\ {\scriptstyle 2}\downarrow & & \\ \mathbf{Z} & \xrightarrow{\Psi=1} & \mathbf{Z} \\ & & \downarrow{\scriptstyle 2} \\ & & \mathbf{Z} \end{array}$$

with generator a in dimension 0, b in dimension 1. Define

$$f: \gamma \oplus \gamma \longrightarrow \{V, \partial, \Psi\} \quad \text{by} \quad f(A_i) = a, \ f(B_i) = b.$$

Then we easily verify the formulae

$$\Psi + \Psi^* = f\phi_1 f^* = \{b^* \mapsto 2b\},$$

$$\partial\Psi - \Psi\delta = \varepsilon f\phi_0 f^*,$$

so $\{\gamma \oplus \gamma, V, f, \phi_i, \Psi\}$ is a symmetric pair. We check that it is, in fact, a Poincaré pair. Let Y be the mapping cylinder of f, so

$$Y = V \oplus \Sigma(\gamma \oplus \gamma) \quad \text{with } \partial = \begin{pmatrix} \partial & 0 \\ -\varepsilon f & \partial \end{pmatrix}$$

and consider the map

(3.12) $(\phi f^*, \Psi): V^{2-*} \longrightarrow Y.$

It is easily checked that (3.12) is a chain homotopy equivalence, but this is the requirement that we have a Poincaré pair.

Now we check the image $e_{\zeta_3}(\gamma)$. It is given by the diagram

(3.13)
$$\begin{array}{ccc}
\mathbf{Z}(\zeta_3) & \xrightarrow{\zeta_3 - \zeta_3^2} & \mathbf{Z}(\zeta_3) \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
\mathbf{Z}(\zeta_3) & \xrightarrow{\zeta_3 - \zeta_3^2} & \mathbf{Z}(\zeta_3)
\end{array}$$

From [5] it follows that $L_1^h(\mathbf{Z}(\zeta_3)) = 0$, so $e_{\zeta_3}(\gamma)$ is a boundary. Indeed, in the localization sequence the form

$$\Psi = \begin{pmatrix} \zeta_3 - (\zeta_3)^2 & 2 \\ -2 & 2(\zeta_3 - (\zeta_3)^2) \end{pmatrix}$$

has boundary the torsion form $\langle \mathbf{F}_4, 1 \rangle$. Then it is easily checked that the 2-dimensional Poincaré pair $\{\partial W, W\}$ where $W = \{W_1 = (\mathbf{Z}(\zeta_3))^2, \Psi\}$ has boundary $e_{\zeta_3}(\gamma)$.

This proves the first two statements in 3.9, so it remains to check $\hat{e}_{\zeta_3}(\tau)$. Consider the union of the Poincaré pairs

$$Y' = W \cup_{e_{\zeta_3}(\gamma_0)} e_{\zeta_3}(\gamma) \cup_{e_{\zeta_3}(\gamma_1)} W,$$

(see [10, pages 135–136] for details of this construction). This is a closed Poincaré complex in $L_2^h(\mathbf{Z}(\zeta_3))$, and its image in $L_2^h(\mathbf{Z}(\zeta_3); \mathbf{Z}/2)$ represents $\hat{e}_{\zeta_3}(\tau)$. On the other hand,

$$\mathbf{Z} = L_2^h(\mathbf{Z}(\zeta_3)) \longrightarrow L_2^h(\mathbf{Q}(\zeta_3))$$

is an injection, so, in order to evaluate the class of Y' in $L_2^h(\mathbf{Z}(\zeta_3))$ we need only evaluate its image in $L_2^h(\mathbf{Q}(\zeta_3))$. But

$$Y' \otimes \mathbf{Q} \sim W \otimes \mathbf{Q} \oplus W \otimes \mathbf{Q},$$

since $e_{\zeta_3}(\gamma)$ and $e_{\zeta_3}(\ )$ are rationally trivial.

From the definition of $\{W, \Psi\}$ it follows that

$$W \otimes \mathbf{Q} = \{\mathbf{Q}(\zeta_3) \oplus \mathbf{Q}(\zeta_3), \Psi\}$$

diagonalizes to

$$\begin{pmatrix} \zeta_3 - (\zeta_3)^2 & 0 \\ 0 & 2(\zeta_3 - (\zeta_3)^2) \end{pmatrix}$$

in $L_2^h(\mathbf{Q}(\zeta_3))$, and twice this is the image of the generator of $L_2^h(\mathbf{Z}(\zeta_3))$. The proof of 3.9 is complete.

REMARK 3.14. In particular 3.9 implies that the only coefficients appearing in the general product formulae have the form $\langle (V)^2 f^*(\kappa), [M] \rangle$ or $\langle (\mathrm{Sq}^1 V)^2 f^*(\kappa'), [M] \rangle$. The first terms are unavoidable, but the second terms are highly undesirable. For our model groups we will return to these second terms in §6.

4. The products $S^1 \times_T \mathbf{RP}^{4n-1}$. The fundamental group of $S^1 \times_T \mathbf{RP}^{4n-1}$ is $\mathbf{Z} \times \mathbf{Z}/2 = \{\tau, T \mid T^2 = 1, T\tau = \tau T\}$ and the associated involution of the group ring $\mathbf{Z}(\mathbf{Z} \times \mathbf{Z}/2)$ is generated by $\tau \leftrightarrow -\tau^{-1}$, $T \leftrightarrow T$, moreover its universal cover is $\mathbf{R} \times S^{4n-1}$.

An equivariant cell decomposition is given as the product of the usual decomposition of S^{4n-1} into $+$ and $-$ hemispheres and the line into the union of its integer points and the intervals $[m, m+1]$, as m runs over the integers. Moreover, on cells of the form $e_i \otimes e_j$, the operation of the fundamental group is just the product action as long as $j \neq 4n-1$, but for $j = 4n-1$ we have

$$T(e_i \times e_{4n-1}^+) = e_i \times e_{4n-1}^-, \quad \text{while} \quad \tau(e_i \otimes e_{4n-1}^+) = -\tau(e_i) \times e_{4n-1}^-.$$

Our object in this section is to study the class of the image $\langle \mathbf{Z}, -(\zeta_3) \rangle \otimes \{\mathscr{C}, \phi\}$ in $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$, where $\{\mathscr{C}, \phi\}$ is the Ranicki symmetric structure induced on the complex for $S^1 \times_T \mathbf{RP}^{4n-1}$ via a suitable diagonal approximation to the lifting of the geometric diagonal map

$$\widehat{\Delta}: S^1 \times_T \mathbf{RP}^{4n-1} \longrightarrow (\mathbf{R} \times S^{4n-1}) \times_{\mathbf{Z}/2 \times \mathbf{Z}} (\mathbf{R} \times S^{4n-1}).$$

Specifically, since the first Stiefel-Whitney class of $S^1 \times_T \mathbf{RP}^{4n-1}$ is non-trivial, the generating chain associated to the chain map $\mathscr{C}^{4n-*} \longrightarrow \mathscr{C}_*$ actually occurs in the complex $(\mathscr{C}_* \otimes \mathscr{C}_*) \otimes_{\mathbf{Z}(\mathbf{Z} \times \mathbf{Z}/2)} \mathbf{Z}^\phi$ where $\mathbf{Z}^\phi$ is the $\mathbf{Z}(\mathbf{Z} \times \mathbf{Z}/2)$ module induced by the homomorphism $\phi: \mathbf{Z} \times \mathbf{Z}/2 \longrightarrow \mathbf{Z}/2$ associated to the the first Stiefel-Whitney class.

But it is explicitly given by taking the usual equivariant diagonal approximation

(4.1) $$\tilde{\Delta}\colon \mathscr{C}_* \longrightarrow \mathscr{C}_* \otimes \mathscr{C}_*,$$

tensoring it with $\mathbf{Z}^\phi$, and taking the image of the cycle associated to a generator of the homology group

(4.2) $$H_{4n}(\mathscr{C}_* \otimes_{\mathbf{Z}(\mathbf{Z}\times\mathbf{Z}/2)} \mathbf{Z}^\phi) = \mathbf{Z}.$$

Recall, that an equivariant diagonal approximation on $\mathscr{C}(S^{4n-1})$ is given by the formla

(4.3) $$\hat{\Delta}_{\#}(e_i^+) = \sum e_j \otimes (T^j)e_{i-j}.$$

(see e.g. [11] for details), and a $\mathbf{Z}$-equivariant diagonal approximation on $\mathbf{R}$ is given by

$$e_0 \longrightarrow e_0 \otimes e_0, \quad e_1 \longrightarrow e_0 \otimes e_1 + e_1 \otimes \tau e_0.$$

On tensoring these maps together we get a suitable diagonal approximation on the $4n-2$-skeleton. It remains to extend these maps over the $4n-1$ and $4n$ skeletons.

On the $4n-1$ skeleton everything is defined except for the cell $e_0 \times e_{4n-1}^+$. But on this cell the previous diagonal approximation preserves the boundary, so extend the map by equivariance to its translates under $\mathbf{Z} \times \mathbf{Z}/2$. The carrier of the image is correct (and acyclic), hence the map is correct on the $4n-1$ skeleton.

The $4n$ skeleton is a single copy of $\mathbf{Z}(\mathbf{Z} \times \mathbf{Z}/2)$ with generator $e_1 \times e_{4n-1}^+$, and a long but direct calculation gives

$$
\begin{aligned}
\Delta(\partial e_1 & \times e_{4n-1}^+) \\
&= \partial \Big\{ 1 \times T\tau(e_1 \times e_{4n-1}^+ \otimes 1) + 1 \otimes e_1 \times e_{4n-1}^+ \\
&\quad + \sum_{n-1 \geq i \geq 1} e_1 \times e_i \otimes T^i \tau \varepsilon_0 \times e_{4n-i-1} \\
&\quad + \sum (-1)^i e_0 \times e_i \otimes T^i e_1 \times e_{4n-i-1} \\
&\quad - \sum_{0 \leq i < n} \tau e_0 \times e_i \otimes \tau T^i e_0 \times e_{4n-i-1} - e_1 \times e_0 \otimes T\tau e_0 \times e_{4n-1}^+ \\
&\quad - e_0 \times e_{4n-1}^+ \otimes Te_1 \times e_0 + \tau T \times \tau T(e_0 \times e_{4n-1}^+ \otimes e_0 \times e_1) \Big\}.
\end{aligned}
$$

Once more, the class above has the correct acyclic carrier, hence provides (after extending equivariantly) a suitable diagonal approximation.

THEOREM 4.4. *The class of*

$$\{\mathscr{C}^{4n-*} \rightarrow \mathscr{C}_*\} \otimes \langle 1, \zeta_3 \rangle$$

in $L^h_{4n}(\mathbf{Z}(\zeta_3)\mathbf{Z}^- \times \mathbf{Z}/2)$ *is the image of the class* $\langle 1 \rangle \perp \langle T \rangle$ *coming from* $L^h_{4n}(\mathbf{Z}(\zeta_3)\mathbf{Z}/2)$ *under the usual inclusion.*

Proof. After tensoring with $\mathbf{Z}^\phi$ the chain above gives the map ϕ as follows:

(0) $(e_1 \times e^+_{4n-1})^* \longrightarrow T\tau(e_0 \times e_0)$

(1) $(e_1 \times e_{4n-2})^* \longrightarrow -\tau e_0 \times e_1$

   $(e_0 \times e_{4n-1})^* \longrightarrow T(e_1 \times e_0) + e_0 \times e_1$

(2) $(e_1 \times e_{4n-3})^* \longrightarrow -e_0 \times T\tau e_2$

   $(e_0 \times e_{4n-2})^* \longrightarrow -e_1 \times e_1 - e_0 \times e_2$

(3) $(e_1 \times e_{4n-4})^* \longrightarrow -e_0 \times \tau e_3$

   $(e_0 \times e_{4n-3})^* \longrightarrow T e_1 \times e_2 - T e_0 \times e_3$

(4) $(e_1 \times e_{4n-5})^* \longrightarrow e_0 \times T\tau e_4$

   $(e_0 \times e_{4n-4})^* \longrightarrow e_1 \times e_3 + e_0 \times e_4$

(5) $(e_1 \times e_{4n-6})^* \longrightarrow e_0 \times \tau e_5$

   $(e_0 \times e_{4n-5})^* \longrightarrow -T(e_1 \times e_4) + T(e_0 \times e_5)$

and from here on it is periodic. Thus we have the array

	$e_0 \times 2_{2n}$	$e_1 \times e_{2n-1}$	$(e_1 \times e_{2n})^*$	$(e_0 \times e_{2n+1})^*$
$(e_0 \times e_{2n})^*$	1	$-\zeta + \tau^{-1} T\zeta^2$	$-(\tau^{-1}+1)$	$T - 1$
$(e_1 \times e_{2n-1})^*$	$-T\tau\zeta - \zeta^2$	0	$-(T+1)$	0
$e_1 \times e_{2n}$	$\tau - 1$	$-(T+1)$	0	0
$e_0 \times e_{2n+1}$	$T - 1$	0	0	0

Do row reduction so as to replace the second row by the sum

$$R_2 + \zeta\tau R_4 + \zeta R_3 = R'_2.$$

Then the image of R_2 is replaced by

$$(1, \ -\zeta(T+1), \ -(T+1), \ 0)$$

and it is easily checked that R'_2 and R_1 map onto generators for the image. Thus we choose $\langle R'_2, \ R_1 \rangle$ as a basis for the free complex and calculate the resulting non-singular form as

$$\begin{pmatrix} 1 & 1 \\ 1 & 1+T \end{pmatrix}.$$

But this is equivalent to

(4.5) $\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$

and the result follows.

5. The case $\mathbf{Z} \times_T \mathbf{Z}$. $\mathbf{Z} \times_T \mathbf{Z} = \{T, \tau \mid \tau^{-1}T\tau T = 1\}$, and a resolution is given by the complex

(5.1) $$C[A] \longrightarrow C[e] \oplus C[f] \longrightarrow C[1]$$

where

(5.2)
$$\partial[e] = (T - 1)[1],$$
$$\partial[f] = (\tau - 1)[1],$$
$$\partial[A] = (T^{-1} + \tau^{-1})[e] + (\tau^{-1}T - \tau^{-1})[f].$$

We define an involution on the group ring $\mathbf{Z}(\mathbf{Z} \times_T \mathbf{Z})$ by $T \leftrightarrow T^{-1}$, $\tau \leftrightarrow (-\tau^{-1})$. Then we have

PROPOSITION 5.3. *Given a homomorphism* $\lambda : \mathbf{Z} \times_{*T} \mathbf{Z} \to \pi$, *there is an involution preserving homomorphism*

$$\hat{\lambda} : \mathbf{Z} \times_T \mathbf{Z} \longrightarrow \pi \times \mathbf{Z}^-,$$

which, on projection to π *is just* λ.

(Indeed, $\mathbf{Z} \times_T \mathbf{Z}/[\mathbf{Z} \times_T \mathbf{Z}, \mathbf{Z} \times_T \mathbf{Z}] = \mathbf{Z}$ and the projection p: $\mathbf{Z} \times_T \mathbf{Z} \to \mathbf{Z}$, of $\mathbf{Z} \times_T \mathbf{Z}$ onto its abelianization is involution preserving provided $\mathbf{Z}$ is given the involution $\tau \leftrightarrow -\tau^{-1}$. Then the map $\hat{\lambda}$ is the product $\lambda \times p$, and it is direct to see that it preserves involutions.)

We now describe the explicit quadratic form associated to this group, resolution, and involution. With respect to the involution, the dual complex is

$$C[1]^* \longrightarrow C[e]^* \oplus C[f]^* \longrightarrow C[A]^*$$

where

$$\delta[e]^* = (T - \tau)[A]^*,$$
$$\delta[f]^* = (-T^{-1}\tau + \tau)[A]^*,$$
$$\delta[1]^* = (T^{-1} - 1)[e]^* - (\tau^{-1} + 1)[f]^*.$$

A chain equivalence $\phi : C^{2-*} \to C_*$ homotopic to the equivalence induced from the geometric diagonal approximation (see the remarks preceding 4.1, 4.2 in §4) is given by the formulae

(5.4)
$$[A]^* \longrightarrow [1]$$
$$[e]^* \longrightarrow [e] - [f]$$
$$[f]^* \longrightarrow (1 - T^{-1})[f] + T^{-1}[e]$$
$$[1]^* \longrightarrow -T[A].$$

In the mapping cone of ϕ_*, when we look at the boundary map from dimension 2 to dimension 1 we get the array

$$
(5.5) \quad
\begin{array}{cccc}
 & [e] & [f] & [A]^* \\
[e]^* & \zeta - \zeta^2 & -(\zeta + T\zeta^2) & -T + \tau \\
[f]^* & \zeta T^{-1} + \zeta^2 & \zeta - \zeta^2 + \zeta^2 T - \zeta T^{-1} & T^{-1}\tau - \tau \\
[A] & (\tau^{-1} + T^{-1}) & \tau^{-1}T - \tau^{-1} & 0
\end{array}
$$

When we apply row reduction to the matrix above we find that the kernel in dimension one of the boundary map ∂_1 is two copies of $\mathbf{Z}(\mathbf{Z} \times_T \mathbf{Z})$, and it is the image of ∂_2. Moreover, two generators which surject onto $\ker(\partial_1)$ can be chosen as

$$(1 - T)[e]^* - T[f]^* \longrightarrow (T - \zeta^2(1 - T))[e] + T(1 - T)[A]^*$$

and

$$[e]^* + [f]^* - \zeta T^{-1}\tau[A] \longrightarrow \zeta(1 - T^{-2}\tau)[e] + f + (\tau T^{-1} - T)[A]^*.$$

From this we can easily calculate the matrix of the associated skew symmetric quadratic form as

$$
(5.6) \quad Q_K = \begin{pmatrix} \zeta^2(T^{-1} - 1) - \zeta(T - 1) & T + \zeta^2(1 + T^2\tau^{-1})(T - 1) \\ -T^{-1} - \zeta(1 - T^{-2}\tau)(T^{-1} - 1) & -(\zeta T^{-2}\tau + \zeta^2 T^2 \tau^{-1}) \end{pmatrix}
$$

(Here Q_K denotes the invariant for the Klein bottle.) Now, consider the surjection

$$\nu : \mathbf{Z} \times_T \mathbf{Z} \longrightarrow \mathbf{Z}/2$$

defined by $\nu(T) = T$, $\nu(\tau) = 1$. As above, this gives the involution preserving homomorphism

$$\hat{\nu} : \mathbf{Z} \times_T \mathbf{Z} \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}^-,$$

and we have

$$
(5.7) \quad \hat{\nu}(Q_K) = \begin{pmatrix} (\zeta^2 - \zeta)(T - 1) & T + \zeta^2(1 + \tau^{-1})(T - 1) \\ -T - \zeta(1 - \tau)(T - 1) & -(\zeta\tau + \zeta^2\tau^{-1}) \end{pmatrix}.
$$

However, the following equivalent form will be more useful to us $\hat{\nu}(Q_K) \sim$

$$
\begin{pmatrix} 1 & 0 \\ \zeta T\tau & 1 \end{pmatrix} \begin{pmatrix} (\zeta^2 - \zeta)(T - 1) & T + \zeta^2(1 + \tau^{-1})(T - 1) \\ -T - \zeta(1 - \tau)(T - 1) & -(\zeta\tau + \zeta^2\tau^{-1}) \end{pmatrix}
$$
$$
\cdot \begin{pmatrix} 1 & -\zeta^2 T\tau^{-1} \\ 0 & 1 \end{pmatrix}
$$
$$
= \begin{pmatrix} (\zeta^2 - \zeta)(T - 1) & T + (\zeta^2 + 2\tau^{-1})(T - 1) \\ -T - (\zeta + 2\tau)(T - 1) & -(\tau + \tau^{-1} + \zeta^2 - \zeta)(T - 1) \end{pmatrix}.
$$

In particular we have

PROPOSITION 5.8. $\langle T \rangle \otimes \hat{\nu}(Q_K) \sim \langle -1 \rangle \otimes \hat{\nu}(Q_K)$.

(Indeed, using the conjugate form of $\hat{\nu}(Q_K)$ given above we see that
$$(\langle 1 \rangle \oplus \langle T \rangle) \otimes \hat{\nu}(Q_K)$$
has a kernel with a basis of the form
$$(\langle 1 \rangle \otimes e_1, \ \langle T \rangle \otimes e_1), \ (\langle 1 \rangle \otimes e_2, \ \langle T \rangle \otimes (-T)e_2),$$
and the result follows.)

Finally, we need to point out the result of using the product in $\Omega_*(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$, at least as regards its effect on the map λ.

COROLLARY 5.9. *The product map*
$$\gamma: \{S^1 \times_T S^1\} \times \{S^1 \times_T S^1\} \to B_{\mathbf{Z}/2} \times B_{\mathbf{Z}/2} = B_{\mathbf{Z}/2 \times \mathbf{Z}/2}$$
is represented in $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2; \mathbf{Z}/2)$ *either by* 0 *or by the image of the class* $\langle 1 \rangle \perp \langle T_1 \rangle \perp \langle T_2 \rangle \perp \langle T_1 T_2 \rangle$ *in* $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2)$.

Proof. Denote the product map $B_{\mathbf{Z}/2} \times B_{\mathbf{Z}/2} \to B_{\mathbf{Z}/2}$ by p, then the composite map $p\gamma$ is null cobordant as an element in the bordism group $\Omega_4(B_{\mathbf{Z}/2}; \mathbf{Z}/2)$. Consequently, $e_{\zeta_3}(p\gamma)$ is also 0 in $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$. But, at the same time the class of $\{S^1 \times_T S^1\}$ in $\Omega_*(pt; \mathbf{Z}/2)$ is also 0. Thus all three non-trivial projections $B_{\mathbf{Z}/2 \times \mathbf{Z}/2} \to B_{\mathbf{Z}/2}$ take $e_{\zeta_3}(\gamma)$ to 0. Now, from [5], we know that $L_3^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2) = 0$, while $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2) = \mathbf{Z}^4$ with generators $\langle 1 \rangle$, $\langle 1 \rangle \perp \langle T_1 \rangle$, $\langle 1 \rangle \perp \langle T_2 \rangle$, and $\langle 1 \rangle \perp \langle T_1 \rangle \perp \langle T_2 \rangle \perp \langle T_1 T_2 \rangle$. But it is direct to check that after tensoring with $\mathbf{Z}/2$, the intersection of the kernels of the various projections is a single copy of $\mathbf{Z}/2$ generated by the element in the corollary. The result follows.

6. The proof of the main theorems.

LEMMA 6.1. *Let* $(\mathscr{C}, \phi_*)$, $(\mathscr{D}, \phi'_*)$ *be two symmetric Poincaré duality complexes of even dimensions* $2n$, $2m$ *respectively. Then*
$$e_{\zeta_3}(\mathscr{C} \otimes \mathscr{D}, \ \phi_* \otimes \phi'_*) = -g(e_{\zeta_3}(\mathscr{C}, \ \phi_*) \otimes e_{\zeta_3}(\Delta, \ \phi'_*))$$
where $g : \mathbf{Z}(\zeta_3) \to \mathbf{Z}(\zeta_3)$ *is the Galois automorphism*.

(The proof is direct. The associated class for $e_{\zeta_3}(\mathscr{C}, \ \phi_*)$ is $(\mathscr{C}, \ -\phi_*\zeta_3)$, and similarly for $\mathscr{D}$. When we tensor them together

we get the complex $(\mathscr{C} \otimes \mathscr{D}, \ \phi_* \otimes \phi'_*(\zeta_3)^2)$. Now, apply the instant surgery construction, as was done in §4 and §5. It is easily checked that when the dimensions are both even, exchanging $-\zeta_3$ for $(\zeta_3)^2$ has the effect of exchanging $(\zeta_3)^2$ for ζ_3 and switches the signs on all the terms in the matrix of the associated non-singular quadratic form.)

We now consider the case of $B_{\mathbf{Z}/2}$. From §1 we need only consider the images

$$e_{\zeta_3}(S^1), \ e_{\zeta_3}(S^1 \times_T S^1), \ e_{\zeta_3}(S^1 \times_T \mathbf{RP}^3), \ldots, e_{\zeta_3}(S^1 \times_T \mathbf{RP}^{2^i-1}) \cdots,$$

and we have

LEMMA 6.2. *Each of the generators above has non-trivial image in* $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$, *but every product maps to* 0.

Proof. The results of §4 and §5 imply the non-triviality of the images of the generators. Moreover, from 4.4 we have that $e_{\zeta_3}(M(i))$ is the image of $\langle 1 \rangle \perp \langle T \rangle$ for $i > 1$. Now, 5.8, 5.9 show that the product of the generator in dimension 2 with any of the above generators gives 0 in $L_2^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$. Likewise, the product

$$(\langle 1 \rangle \perp \langle T \rangle)^2 \ = \ \langle 1 \rangle \perp \langle 1 \rangle \perp \langle T \rangle \perp \langle T \rangle,$$

and this certainly goes to 0 in $L_2^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$. Finally, the results of [5] give that $(\langle 1 \rangle \perp \langle T \rangle) \otimes e_{\zeta_3}(S^1) = 0$, and 6.2 follows.

COROLLARY 6.3. *The κ's for $B_{\mathbf{Z}/2}$ are non-zero precisely when the dimension has the form* 2^i, *for* $i = 0, 1, 2, \ldots$.

(Since the model manifolds have trivial Wu-product classes V^2, $\sum V_i \mathrm{Sq}^1(V_i)$, $(\mathrm{Sq}^1 V)^2$ it follows that the homomorphism into $L_2^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2; \mathbf{Z}/2)$ directly calculates the κ's. See e.g. [13], [3], [14], and especially Theorem 3.9.)

There are some additional difficulties for the group $G = \mathbf{Z}/2 \times \mathbf{Z}/2$.

LEMMA 6.4. (a) $e_{\zeta_3}(M(i) \times 1)$, $e_{\zeta_3}(1 \times M(i))$, $e_{\zeta_3}(M(0) \times M(0))$ *are all non-zero.*

(b) $e_{\zeta_3}(M(i) \times M(j)) = e_{\zeta_3}(M(j) \times M(i))$ *all* i, j.

(c) $e_{\zeta_3}(M(0) \times M(i)) = 0$ *for* $i > 1$, *while* $e_{\zeta_3}(M(i) \times M(j)) = \langle 1 \rangle \perp \langle T_1 \rangle \perp \langle T_2 \rangle \perp \langle T_1 T_2 \rangle = V \neq 0$ *for* $i, j > 1$.

(d) $e_{\zeta_3}(M(1) \times M(1))$ *is either* 0 *or* V.

Proof. (a) $L_2^h(\mathbf{Z}(\zeta_3)G) = \mathbf{Z}/2 \oplus \mathbf{Z}^4$ and in [7], [5] we showed that $S^1 \times S^1$ represents the non-trivial torsion element in this dimension. Likewise, to see the rest of (a), use the projection $B_{\mathbf{Z}/2 \times \mathbf{Z}/2} \to B_{\mathbf{Z}/2}$.

(b) From 6.2 the composite

$$p_i(e_{\zeta_3}(M(i) \times M(j))) \in L_*^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2\,;\,\mathbf{Z}/2) = 0$$

for each of the three projections $\mathbf{Z}/2 \times \mathbf{Z}/2 \to \mathbf{Z}/2$. Moreover, in dimension 0, the intersection of the three kernels is generated by V (in $L_0^h(\mathbf{Z}(\zeta_3)\mathbf{Z}/2)$ we have that $p_i(V) = 2X_i$ for each p_i), while in dimension 2 the intersection of the kernels is the image of the torsion element in $L_2^h(\mathbf{Z}(\zeta_3)G)$.

(c) The symmetric complex associated to S^1 $(\pi_1 = \mathbf{Z})$ is

(6.5)

$$
\begin{array}{ccc}
\mathbf{Z}(\mathbf{Z}) & \xrightarrow{\quad -T^{-1} \quad} & \mathbf{Z}(\mathbf{Z}) \\
{\scriptstyle T^{-1}-1}\big\downarrow & & \big\downarrow{\scriptstyle T-1} \\
\mathbf{Z}(\mathbf{Z}) & \xrightarrow{\quad 1 \quad} & \mathbf{Z}(\mathbf{Z})
\end{array}
$$

Its product with $-\zeta_3 T_2$ gives the formation (over G)

(6.6) $(\mathbf{Z}G \oplus \mathbf{Z}G, \; K = \mathbf{Z}G\langle T_1 - 1, \; \zeta_3 T_1 T_2 + (\zeta_3)^2 T_2\rangle)$.

Similarly, the product of (6.5) with $-\zeta_3$ gives the formation

(6.7) $(\mathbf{Z}G \oplus \mathbf{Z}G, \; K = \mathbf{Z}G\langle T_1 - 1, \; \zeta T_1^{-1} + (\zeta_3)^2\rangle)$.

In each case projection to the second factor gives $F_3^{-+} \oplus F_3^{--}$ as quotient. Moreover, the induced torsion forms on these two modules are isomorphic, so standard local-global arguments show that their sum is equivalent to a torsion form with kernel $F_3^{-+} \oplus F_3^{--}$ which has Euler characteristic zero. This implies $e_{\zeta_3}(M(i) \times M(0)) = 0$. On the other hand 6.1, 4.5 show that $e_{\zeta_3}(M(i) \times M(j)) = V$ for $i, j > 1$.

d) follows as in the proof of (b).

For $D(2^i, 2)$ we have

LEMMA 6.8. *The maps* $\phi_j^i : \mathbf{Z}/2 \times \mathbf{Z}/2 \to D(2^i, 2)$ *satisfy* $(\phi_j^i)(V) \neq 0$ *in the groups*

$$L_0^h(\mathbf{Z}(\zeta_3)D(2^i, 2)\,;\,\mathbf{Z}/2).$$

Proof. The ring $\mathbf{Q}(D(2^i, 2))$ splits as

$$M_2(\mathbf{Q}(\lambda_i)) \oplus M_2(\mathbf{Q}(\lambda_{i-1})) \oplus \cdots \oplus M_2(\mathbf{Q}) \oplus \mathbf{Q}^{++} \oplus \mathbf{Q}^{+-} \oplus \mathbf{Q}^{-+} \oplus \mathbf{Q}^{--},$$

where $\lambda_j = \zeta_{2^j} + (\zeta_{2^j})^{-1}$, and $M_2(\mathbf{F})$ is the 2-matrix ring with coefficients in $\mathbf{F}$. Let I be the induction map from the representation ring for G to that for $D(2^i, 2)$, and let V^j be the irreducible representation associated to the component $M_2(\mathbf{Q}(\lambda_j))$ above, then we have

$$I(\mathbf{Q}^{-\varepsilon}) = V^i,$$
$$I(\mathbf{Q}^{+\varepsilon}) = V^{i-1} \oplus \cdots \oplus V^1 \oplus \mathbf{Q}^{++} \oplus \mathbf{Q}^{-+} \oplus \mathbf{Q}^{+-} \oplus \mathbf{Q}^{--}.$$

Hence,

$$\phi_j^i(V) = 4(\perp_{j<i} \langle -1 \rangle_{V_j} \perp \langle -1 \rangle_{+-} \perp \langle -1 \rangle_{++} \perp \langle -1 \rangle_{-+} \perp \langle -1 \rangle_{--})$$

in $L_0^h(\mathbf{Q}(\zeta_3)D(2^i, 2))$, and the results of [5] show that the corresponding element in

$$L_0^h(\mathbf{Z}(\zeta_3)D(2^i, 2))$$

is not divisible by 2.

For the quaternion group $Q(2^i)$ we have

Lemma 6.9. *The generators (in 1.28) all have trivial images in $L_*^h(\mathbf{Z}(\zeta_3)Q(2^i); \mathbf{Z}/2)$ if their dimension are 4 or greater, while those of dimension < 4 all have non-trivial (and independent) images.*

Proof. The non-triviality of all the generators above is clear for dimensions ≤ 3. (In dimension 3, this is just the Cappell-Shaneson example as described in [5, §8], while in dimensions 1, 2 project onto the dihedral group). The triviality of the classes $e_{\zeta_3}(\{M(i), f_i\})$ follows from an argument completely analogous to that in the proof of 6.8. The point is that the image of $\langle 1 \rangle^+$ for $\mathbf{Z}/2$ is the class $\langle 1 \rangle$ in $L_0^h(\mathbf{Q}(\zeta_3)Q(2^i))$, but $\langle 1 \rangle \perp \langle T \rangle$ is represented by $2\langle 1 \rangle^+$. The proof is complete.

Finally, to complete the proof of the results in the introduction we need to consider the extra terms $\langle (\mathrm{Sq}^1 V)^2 f^*(\kappa'), [M] \rangle$ which could possibly appear in the product formulae for the map e_{ζ_3}. For oriented M it suffices, as usual to analyze the situation for the model groups, and we have

Theorem 6.10. *For the model groups the coefficients above are identically 0 whenever M is an oriented manifold.*

Proof. For $\pi = \mathbf{Z}/2$ or $\mathbf{Z}/4$ we know that $L_3^h(\mathbf{Z}(\zeta_3)\pi; \mathbf{Z}/2) = 0$, so $e_{\zeta_3}(\tau \circ M(0)) = 0$ in both these cases. Likewise, when $i > 1$, 6.1

implies $e_{\zeta_3}(\tau \circ M(i)) = 0$. But in any case $e_{\zeta_3}(\tau \circ M(i))$ will involve detection by coefficients $\langle (Sq^1 V)^2 f^*(b), [M] \rangle$, and since $Sq^1(b) = 0$, it follows that

$$(6.11) \qquad (Sq^1 V)^2 f^*(b) = Sq^1(V \circ Sq^1 V \circ f^*(b))$$

and so will vanish in top dimension on any oriented manifold.

When $\pi = \mathbf{Z}/2 \times \mathbf{Z}/2$ we have that $e_{\zeta_3}(\tau \circ M(1) \times M(1))$ is represented by either $\tau \circ V$ or 0 from 6.4. But 6.1 implies that $\tau \circ V = 0$ so this class vanishes. Otherwise,

$$e_{\zeta_3}(\tau \circ (M(i) \times M(j))) = e_{\zeta_3}((\tau \circ M(i)) \times M(j)) = e_{\zeta_3}(M(i) \times (\tau \circ M(j)))$$

and unless both i, $j = 1$ the proof for $\mathbf{Z}/2$ implies that the above class gives 0 in $L^h_*(\mathbf{Z}(\zeta_3)\mathbf{Z}/2 \times \mathbf{Z}/2; \mathbf{Z}/2)$.

When $\pi = Q(2^i)$ we dispose of the classes in dimension 1 by noting that they factor through $\mathbf{Z}/4$, hence $e_{\zeta_3}(\tau \circ M(0))$ represents 0 in $L^h_3(\mathbf{Z}(\zeta_3)Q(2^i); \mathbf{Z}/2)$. The remaining classes in dimensions 2 and 3 on producting with τ give formulae involving only the classes in dimensions 2 and 3 in $H^*(Q(2^i); \mathbf{Z}/2)$, but these classes are all in the kernel of Sq^1, so 6.11 implies the vanishing of these classes on oriented manifolds and 6.10 follows.

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