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AUTOMORPHISM GROUPS OF CERTAIN DOMAINS IN $\mathbf{C}^n$ WITH A SINGULAR BOUNDARY

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In this paper, we show how to use the so-called scaling technique to prove the compactness of the automorphism groups of bounded strictly convex circular domains in $\mathbf{C}^n$ whose boundaries are not entirely smooth, in case the singular locus of the boundary is globally complicated but locally simple in some topological sense.

1. Introduction. We develop a certain scheme of computing the automorphism groups of the bounded circular convex domains in $\mathbf{C}^n$ whose boundary is not entirely smooth. As an application, we compute the automorphism group of the unit open ball with respect to the minimal complex norm in $\mathbf{C}^n$ introduced by K. T. Hahn and P. Pflug [3], thus answering their question raised there. In this paper, we restrict ourselves to the automorphism groups of Hahn-Pflug examples. However, we believe that all the ideas and complexity of our technique are clearly shown in this somewhat special case.

Hahn and Pflug ([3]) showed that the complex norm N^* in $\mathbf{C}^n$ defined by

$$N^*(z_1, \ldots, z_n) = \frac{1}{\sqrt{2}}\sqrt{\sum_{j=1}^{n} |z_j|^2 + \left|\sum_{j=1}^{n} z_j^2\right|}$$

is the smallest complex norm in $\mathbf{C}^n$, that extends the real Euclidean norm in the following sense: *For any complex norm N in $\mathbf{C}^n$ that extends the real Euclidean norm and satisfies the inequality $N(z) \le |z|$ for any $z \in \mathbf{C}^n$, $N^*(z) \le N(z)$ holds for all $z \in \mathbf{C}^n$.*

Denote by

$$B_n^* := \{z \in \mathbf{C}^n \mid N^*(z) < 1\}$$

the unit open ball in $\mathbf{C}^n$ with respect to the norm N^*. Let $O(n, \mathbf{R})$ denote the set of all $n \times n$ real orthogonal matrices. Notice that the boundary ∂B_n^* is not entirely smooth. It was shown in [3] that this domain is not homogeneous, but no explicit description beyond that was known except when $n = 2$. Moreover, the method used in [3] to show that

$$\operatorname{Aut} B_2^* = \{e^{i\theta}A \mid \theta \in \mathbf{R},\ A \in O(2, \mathbf{R})\}$$

is indeed very special to the case of $n = 2$. However, our method in this paper applies in all dimensions. Consequently we are able to give an explicit description of $\operatorname{Aut} B_n^*$ for any $n \geq 2$.

We would like to point out that our method here is closely related to the results of **[1]**, **[5]** and **[11]**. Moreover, we express our special thanks to A. Browder, K. T. Hahn, P. Pflug and J. Wermer for their interest and helpful comments.

2. Compactness of certain automorphism groups. In this section, for simplicity, we will work on compactness of Aut B_3^* in most of our arguments with respect to the usual topology of uniform convergence on compact subsets. Then, at the end of our arguments, one ought to be able to observe that the same method will work for any $n \geq 2$. Furthermore, one can also observe that the technique we introduce here can be applied to a broader class of domains than the one consisting only of B_n^*, $n \geq 2$.

PROPOSITION 1. *Aut B_3^* is compact.*

To prove the statement, we first observe the following facts on B_n^* for $n \geq 2$ (see **[3]**, e.g.):

(2.1) B_n^* is not biholomorphic to the open ball

$$\{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

for $n \geq 2$.

(2.2) B_n^* is convex.

(2.3) ∂B_n^* is smooth (C^∞) strongly pseudoconvex everywhere except along

$$\partial Q = \partial B_n^* \cap \left\{ z \mid \sum_{j=1}^{n} z_j^2 = 0 \right\}.$$

(2.4) ∂B_n^* does not admit any non-trivial analytic subset.

Due to the theorem of B. Wong (**[12]**) and J.-P. Rosay (**[13]**), we may deduce that there do not exist a point $q \in B_n^*$ and a sequence $\{f_j\} \subset \operatorname{Aut} B_n^*$ such that

$$\lim_{j \to \infty} f_j(q) \in \partial B_n^* \setminus \partial Q.$$

Consequently, if we suppose that Aut B_n^* is non-compact, then there exist a point $p_0 \in B_n^*$ and a sequence $\{g_j\} \subset \text{Aut } B_n^*$ such that

$$(1) \qquad \lim_{j\to\infty} g_j(p_0) \in \partial Q.$$

Then from this we expect to derive a contradiction to prove Proposition 1.

Let us denote by

$$p := \lim_{j\to\infty} g_j(p_0) \in \partial Q.$$

Then the version of the scaling technique used in [5] applies as follows:

LEMMA 1. *Assuming that* (1) *above holds, there exists a sequence* $\{A_j\} \subset \mathrm{GL}(n, \mathbf{C})$ *with* $A_j \to 0$ *as* $j \to \infty$ *such that the sequence* $A_j^{-1}(B_n^* - p)$ *of convex sets converges to a convex domain in* $\mathbf{C}^n$, *say* $\widehat{B}_n^*$, *which is biholomorphic to* B_n^*, *with respect to the local Hausdorff set convergence.*

Now we apply this scaling technique on B_3^*. We will first try to scale B_3^* at $p = (1, i, 0) \in \partial Q$. The notation $B_n^* - p$ stands for the Euclidean parallel translation of B_n^* by $-p$ in $\mathbf{C}^n$. Hence it can be represented by the inequality

$$|z_1 + 1|^2 + |z_2 + i|^2 + |z_3|^2 + |(z_1 + 1)^2 + (z_2 + i)^2 + z_3^2| < 2$$

i.e.,

$$(2) \quad 0 > 2 \text{ Re}\,(z_1 - iz_2) + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1^2 + z_2^2 + z_3^2 + 2(z_1 + iz_2)|.$$

We perform a $\mathbf{C}$-linear change of coordinates by

$$\zeta_1 = z_1 - iz_2, \quad \zeta_2 = z_1 + iz_2, \quad \zeta_3 = z_3.$$

Then the domain $B_3^* - p$ is still convex and bounded and is represented by the inequality

$$(3) \qquad 0 > 2 \text{ Re } \zeta_1 + \tfrac{1}{2}(|\zeta_1|^2 + |\zeta_2|^2) + |\zeta_3|^2 + |\zeta_1\zeta_2 + \zeta_3^2 + 2\zeta_2|$$

with $p = (0, 0, 0)$ the reference point for scaling. Now the domain

$$A_j^{-1}(B_3^* - p)$$

is represented by

$$\begin{aligned}(4) \quad 0 > 2 \text{ Re}\,&(a_j^{11}z_1 + a_j^{12}z_2 + a_j^{13}z_3) \\ &+ \tfrac{1}{2}(|(a_j^{11}z_1 + a_j^{12}z_2 + a_j^{13}z_3|^2 + |a_j^{21}z_1 + a_j^{22}z_2 + a_j^{23}z_3)|^2) \\ &+ |a_j^{31}z_1 + a_j^{32}z_2 + a_j^{33}z_3|^2 \\ &+ |(a_j^{11}z_1 + a_j^{12}z_2 + a_j^{13}z_3) \cdot (a_j^{21}z_1 + a_j^{22}z_2 + a_j^{23}z_3) \\ &\quad + (a_j^{31}z_1 + a_j^{32}z_2 + a_j^{33}z_3)^2 + 2(a_j^{21}z_1 + a_j^{22}z_2 + a_j^{23}z_3)|\end{aligned}$$

where $A_j = (a_j^{lk})$.

Without loss of generality, we may assume

$$a_j^{1l}/a_j^{11} \to \alpha^{1l} \quad \text{as } j \to \infty \tag{5a}$$

and

$$a_j^{2l}/a_j^{2k} \to \tilde{\alpha}^{2l} \quad \text{as } j \to \infty \tag{5b}$$

for some k fixed, and for any $l = 1, 2, 3$.

Comparing (5a) and (5b) above, we may assume further that

$$a_j^{2l}/a_j^{11} \to \alpha^{2l} \quad \text{as } j \to \infty \tag{5c}$$

holds together with (5a). Moreover, replacing A_j by $(\overline{a}_j^{11}/|a_j^{11}|)A_j$ and choosing a subsequence if necessary, we may assume that $a_j^{11} > 0$ for any j.

Now consider the speed of convergence (or, divergence) of each coefficient. Then Lemma 1 above forces us to conclude that $\widehat{B}_3^*$ is defined by the inequality

$$\begin{aligned} 0 > 2 \operatorname{Re}\, (z_1 + \alpha^{12}\zeta_2 + \alpha^{13}\zeta_3) &+ |\alpha^{31}\zeta_1 + \alpha^{32}\zeta_2 + \alpha^{33}\zeta_3|^2 \\ &+ |(a^{31}\zeta_1 + a^{32}\zeta_2 + \alpha^{33}\zeta_3)^2 + 2(a^{21}\zeta_1 + a^{22}\zeta_2 + \alpha^{23}\zeta_3)| \end{aligned}$$

where

$$\alpha^{3l} = \lim_{j\to\infty} \frac{a_j^{3l}}{\sqrt{a_j^{11}}} \quad \text{for } l = 1, 2, 3.$$

This follows because it is the only possibility that the local Hausdorff set limit of the sequence of the convex domains $A_j^{-1}(B_3^* - p)$ represented by (4) can be a domain in $\mathbf{C}^3$ which could be hyperbolic in the sense of Kobayashi ([**6**]). Again, since $\widehat{B}_3^*$ is biholomorphic to a bounded domain, it cannot contain a complex line. Hence in particular

$$\det \begin{pmatrix} 1 & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} \neq 0.$$

Hence, by an obvious change of coordinates, we have a new defining inequality for $\widehat{B}_3^*$:

$$0 > 2 \operatorname{Re}\, z_1 + |z_3|^2 + |z_3^2 + z_2|. \tag{6a}$$

Now we apply the biholomorphic mapping $\varphi\colon \widehat{B}_3^* \to \mathbf{C}^3$ defined by

$$\varphi(z_1, z_2, z_3) = \left(\frac{1+2z_1}{1-2z_1}, \frac{4z_2}{(1-2z_1)^2}, \frac{2z_3}{(1-2z_1)}\right)$$

to deduce that $\widehat{B}_3^*$ is biholomorphic to the domain, which we again call $\widehat{B}_3^*$, defined by

$$\{(z_1, z_2, z_3) \mid |z_1|^2 + |z_3|^2 + |z_3^2 + z_2| < 1\} \tag{6b}$$

which is again biholomorphic to the domain defined by

$$|z_1|^2 + |z_2|^2 + |z_3| < 1. \tag{6c}$$

For an arbitrary $n \geq 3$, one obtains that $\widehat{B}_n^*$ is biholomorphic to the domain defined by

$$|z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n| < 1 \tag{6c}$$

by an identical argument. According to Lemma 1, this domain has to be biholomorphic to B_n^*, since we assumed that B_n^* admits a non-compact automorphism group. We will try to derive a contradiction from this to complete the proof of Proposition 1. First, we have

LEMMA 2. *The set*

$$\partial \widehat{Q} := \{z \in \mathbf{C}^n \mid |z_1|^2 + |z_3|^2 + \cdots + |z_{n-1}|^2 = 1 \text{ and } z_n = 0\} \tag{7}$$

is homeomorphic to the real $2n-3$*-dimensional sphere for any* $n \geq 3$.

The proof of this is trivial. Now we look at the points where ∂B_n^* is not smooth. They form a set

$$\partial Q = \{z \in \mathbf{C}^n \mid z_1^2 + \cdots + z_n^2 = 0\} \cap \partial B_n^*$$

which turns out to be topologically different from the sphere as follows:

LEMMA 3. ∂Q, *for any dimension* $n \geq 3$, *is diffeomorphic to the Stiefel manifold* $O(n)/O(n-2)$.

Proof. It follows directly from the fact that ∂Q is in fact homeomorphic to the unit tangent bundle of the $2n-1$ dimensional sphere.

Now, notice that both B_n^* and $\widehat{B}_n^*$ are completely circular. Hence, they are linearly equivalent ([**5**]). However, two lemmas above then yield a contradiction. Consequently, we obtain

THEOREM 1. Aut B_n^* *is compact for any* $n \geq 2$.

One may also notice that the argument we used above to show the compactness of Aut B_n^*, $n \geq 3$, could lead us to obtain the compactness of the automorphism group of any strictly convex bounded circular domain in $\mathbf{C}^n$ with a singular boundary in case its singular locus of the boundary possesses a topology globally complicated but locally simple.

3. An explicit description of Aut B_n^***.** Now we focus more into Aut B_n^*, $n \geq 3$. We begin with the following statement:

PROPOSITION 2. *Let* Ω *be a convex bounded domain of holomorphy in* $\mathbf{C}^n$ *that is circular, meaning that* Ω *is invariant under the circular action*

$$(z_1, \ldots, z_n) \mapsto (e^{i\theta} z_1, \ldots, e^{i\theta} z_n), \quad \forall \theta \in \mathbf{R}.$$

Assume further that Aut Ω *is compact. Then every automorphism of* Ω *is complex linear.*

To deduce this, we start with the following result due to L. Lempert ([**10**]).

THEOREM A. *For any convex bounded domain in* $\mathbf{C}^n$, *every Kobayashi metric ball is convex.*

Then following the proof of Cartan's fixed point theorem (e.g., see [7], p. 111), we get

THEOREM B. *Every compact biholomorphic group action on a convex, bounded and complete hyperbolic domain in* $\mathbf{C}^n$ *has a common fixed point.*

Therefore, all the automorphisms of Ω have a common fixed point. On the other hand, note that the circular action is a part of Aut Ω. It has one and only one common fixed point that is the origin. Consequently, every automorphism of Ω fixes the origin $(0, \ldots, 0)$. Then Proposition 2 directly follows from the following classical theorem by H. Cartan (e.g., see [**8**]):

THEOREM C. *Let* Ω *be a circular domain in* $\mathbf{C}^n$ *containing the origin. Then every* $f \in$ Aut Ω *with* $f(0) = 0$ *is complex linear.*

Therefore, we have

COROLLARY. Aut B_n^*, *for any* $n \geq 2$, *consists of linear maps only.*

In fact, one can say more than Corollary above. Since all the automorphisms of B_n^* are complex linear, they extend smoothly across the boundary of B_n^*, which is singular. Hence the singular locus ∂Q

of ∂B_n^* must be preserved by all the linear automorphisms. On the other hand, the singular locus ∂Q is precisely the set

$$\{z \in \mathbf{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 2\} \cap \{z \in \mathbf{C}^n \mid z_1^2 + \cdots + z_n^2 = 0\}.$$

Then we have the following lemma:

LEMMA 4. *Any $n \times n$ unitary matrix of complex numbers which preserves the quadric*

$$\{z \in \mathbf{C}^n \mid z_1^2 + \cdots + z_n^2 = 0\}$$

is in fact $\lambda \cdot A'$ for some $A' \in O(n, \mathbf{R})$ and some $\lambda \in \mathbf{C}$ with $|\lambda| = 1$.

Proof. Let B^T denote the transpose of B for any $m \times n$ matrix B of complex numbers. Then the fact that A preserving the quadric given above is nothing but

$$z^T A^T A z = 0, \text{ for any column vector } z = (z_1, \dots, z_n)^T \text{ with } z^T z = 0.$$

Now let $U = A^T A$. Then it is a symmetric matrix satisfying the relation above. Applying the values of z such as

$$(0, \dots, 1, \dots, \pm i, \dots, 0)$$

to the relation, one easily gets the conclusion that $U = \lambda \cdot I$. Thus the lemma follows.

Therefore, we can deduce the following

THEOREM 2. Aut $B_n^* = \{e^{i\theta} \cdot A \mid \theta \in \mathbf{R},\ A \in O(n, \mathbf{R})\}$, *for any $n \geq 2$.*

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