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SOME INFINITE CHAINS IN THE LATTICE OF VARIETIES OF INVERSE SEMIGROUPS

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The relation ν defined on the lattice $\mathscr{L}(\mathscr{I})$ of varieties of inverse semigroups by $\mathscr{U}\nu\mathscr{V}$ if and only if $\mathscr{U}\cap\mathscr{G}=\mathscr{V}\cap\mathscr{G}$ and $\mathscr{U}\vee\mathscr{G}=\mathscr{V}\vee\mathscr{G}$, where $\mathscr{G}$ is the variety of groups, is a congruence. It is known that varieties belonging to the first three layers of $\mathscr{L}(\mathscr{I})$ (those varieties belonging to the lattice $\mathscr{L}(\mathscr{S}\mathscr{I})$ of varieties of strict inverse semigroups) possess trivial ν-classes and that there exist non-trivial ν-classes in the next layer of $\mathscr{L}(\mathscr{I})$. We show that there are infinitely many ν-classes in the fourth layer of $\mathscr{L}(\mathscr{I})$, and also higher up in $\mathscr{L}(\mathscr{I})$, that in fact contain an infinite descending chain of varieties. To find these chains we first construct a collection of semigroups in $\mathscr{B}^1$, the variety generated by the five element combinatorial Brandt semigroup with an identity adjoined. By considering wreath products of abelian groups and these semigroups from $\mathscr{B}^1$, we obtain an infinite descending chain in the ν-class of $\mathscr{U}\vee\mathscr{B}^1$, for every non-trivial abelian group variety $\mathscr{U}$.

1. Introduction. In **[K1]** Kleiman demonstrated that the relation ν defined on the lattice $\mathscr{L}(\mathscr{I})$ of varieties of inverse semigroups by $\mathscr{U}\nu\mathscr{V}$ if and only if $\mathscr{U}\cap\mathscr{G}=\mathscr{V}\cap\mathscr{G}$ and $\mathscr{U}\vee\mathscr{G}=\mathscr{V}\vee\mathscr{G}$, where $\mathscr{G}$ is the variety of groups, is a congruence. He further showed that the lattice $\mathscr{L}(\mathscr{S}\mathscr{I})$ of varieties of strict inverse semigroups is isomorphic to three copies of the lattice $\mathscr{L}(\mathscr{G})$ of varieties of groups and that each of the intervals $[\mathscr{S}, \mathscr{S}\vee\mathscr{G}]$ and $[\mathscr{B}, \mathscr{B}\vee\mathscr{G}]$, where $\mathscr{S}$ is the variety of semilattices and $\mathscr{B}$ is the variety generated by the five element combinatorial Brandt semigroup, is isomorphic to $\mathscr{L}(\mathscr{G})$ (and so, as a result, $\mathscr{L}(\mathscr{S}\mathscr{I})$ is a modular lattice). Consequently, for any variety $\mathscr{V}$ in $\mathscr{L}(\mathscr{S}\mathscr{I})$, the ν-class of $\mathscr{V}$ is trivial. $\mathscr{L}(\mathscr{S}\mathscr{I})$ is sometimes referred to colloquially as the first three layers of the lattice $\mathscr{L}(\mathscr{I})$. The "fourth" layer, $[\mathscr{B}^1, \mathscr{B}^1\vee\mathscr{G}]$, where $\mathscr{B}^1$ is the variety generated by the five element combinatorial Brandt semigroup with an identity adjointed, is not nearly as nice. While it is a modular lattice (the collection of congruences on an inverse semigroup which have the same trace forms a complete modular sublattice of the lattice of congruences on that semigroup), the ν-classes of its members are not all trivial and, as a result, $\mathscr{L}(\mathscr{B}^1 \vee \mathscr{G})$ is not modular, and hence $\mathscr{L}(\mathscr{I})$ is not modular (Reilly **[R2]** provides an example which demonstrates this). In this note we show that the ν-class of $\mathscr{B}^1 \vee \mathscr{A}$, for any abelian group variety $\mathscr{A}$, contains an infinite chain of varieties and so is far from being trivial. The technique used is interesting in that we are only required to know the structure of the $\mathscr{D}$-classes (as reflected by their Schützenberger graphs) of a given collection of words with respect to $\mathscr{B}^1$ (and not the entire $\mathscr{B}^1$-free object on countably infinite X) in order to construct inverse semigroups which are then shown to generate distinct varieties. We remark that the variety $\mathscr{B}^1$ has proved to be rather enigmatic. Even though it is generated by a small (6-element) inverse semigroup and $\mathscr{L}(\mathscr{B}^1)$ is just a 4-element chain, its members are not easily characterized and, as Kleiman proved in **[K2]**, it is not defined by a finite set of identities.

Section 2 is devoted to preliminary material. In §3 we construct a collection of inverse semigroups each of which belongs to the variety $\mathscr{B}^1$ but not $\mathscr{B}$. From these semigroups we construct in §4 a collection of inverse semigroups belonging to $\mathscr{B}^1 \circ \mathscr{A}_n$, $n \in \omega$, but not $\mathscr{B}^1 \vee \mathscr{A}_n$. In the final section we use the semigroups of §4 to construct an infinite chain of varieties in the interval $[\mathscr{B}^1 \vee \mathscr{A}_n, \mathscr{A}_n \circ \mathscr{B}^1]$ which is the ν-class of $\mathscr{B}^1 \vee \mathscr{A}_n$ (by a theorem due to Reilly **[R1]**). Using this result we can then show that a larger collection of ν-classes which are also intervals in $\mathscr{L}(\mathscr{I})$ possess an infinite descending chain of varieties.

2. Preliminaries. We assume that the reader is familiar with the basic notions of inverse semigroup theory for which Petrich **[P]** is a standard reference. For the basic results concerning varieties we refer the reader to **[BS]**. We will consistently use the following notation:

$\mathscr{I}$— the variety of all inverse semigroups

$\mathscr{G}$— the variety of groups

B_2— the five element combinatorial Brandt semigroup

$\mathscr{B}$— the variety generated by the five element combinatorial Brandt semigroup B_2; it is defined by the identity $xyx^{-1} = (xyx^{-1})^2$

B_2^1— B_2 with an identity adjoined

$\mathscr{B}^1$— the variety generated by B_2^1

$\mathscr{AG}$— the variety of abelian groups

$\mathscr{A}_n$— the variety of abelian groups of exponent n

$F\mathscr{U}(X)$— the $\mathscr{U}$-free object on X in the variety $\mathscr{U}$

$\rho(\mathscr{U})$— the fully invariant congruence on $F\mathscr{I}(X)$ corresponding to the variety $\mathscr{U}$

$c(w)$— for any w over $X \cup X^{-1}$, the *content* of w which is the set $\{x \in X : x \text{ or } x^{-1} \text{ occurs in } w\}$

$w \in E$— for a word w over $X \cup X^{-1}$, the identity $w = w^2$

Throughout this note $X = \{x_i : i \in \omega\}$ is a fixed countably infinite set.

For any congruence ρ on an inverse semigroup S, define the *kernel of* ρ, $\ker\rho$, and the *trace of* ρ, $\operatorname{tr}\rho$, by

$$\begin{aligned}\ker\rho &= \{s \in S : s\rho e \text{ for some idempotent } e \text{ in } S\}\\ &= \{s \in S : s\rho s^2\} = \{s \in S : s\rho = (s\rho)^2\},\\ \operatorname{tr}\rho &= \rho \cap (E_S \times E_S).\end{aligned}$$

Every congruence ρ on an inverse semigroup S is completely determined by its kernel and trace, [**P**; III.1.5].

An inverse semigroup S is *combinatorial* if $\mathscr{H} = \varepsilon$ in S. The variety $\mathscr{V}$ is said to be *combinatorial* if all members of $\mathscr{V}$ are combinatorial. The variety $\mathscr{B}^1$ is a combinatorial variety. Moreover, $\mathscr{B}^1 \subseteq \mathscr{U}^{\max} = [w = w^2 : w = w^2 \text{ is a law in } \mathscr{U}]$ for all group varieties $\mathscr{U}$ (see [**PR**]).

Let S be an inverse semigroup. A transformation ρ on S is a *right translation* of S if, for all $x, y \in S$, $(xy)\rho = x(y\rho)$. Likewise, a transformation λ is a *left translation* if $\lambda(xy) = (\lambda x)y$, for all $x, y \in S$. If, in addition, the left translation λ and the right translation ρ satisfy $x(\lambda y) = (x\rho)y$, for all $x, y \in S$, then the two are *linked* and the pair (λ, ρ) is a *bitranslation.* The set of all bitranslations on S under the operation of componentwise composition is an inverse semigroup and is called the *translational hull of* S [**P**; V.1.4]. We denote this semigroup by $\Omega(S)$.

For any $s \in S$, the functions λ_s and ρ_s defined by $\lambda_s x = sx$ and $x\rho_s = xs$, for all $x \in S$, are left and right translations, respectively. In fact, (λ_s, ρ_s) is a bitranslation and so is a member of $\Omega(S)$. The mapping

$$\pi : s \to (\lambda_s, \rho_s) \qquad (s \in S)$$

is a monomorphism of S into $\Omega(S)$ and is called the *canonical homomorphism of S into* $\Omega(S)$.

If S is an ideal of the inverse semigroup V then V is an *ideal extension* of S (by the Rees quotient semigroup V/S).

Let V be an ideal extension of S. For each $v \in V$, define

$$\lambda^v s = vs \quad \text{and} \quad s\rho^v = sv \qquad (s \in S).$$

Then the mapping

$$\tau(V : S) \colon V \to \Omega(S)$$

defined by

$$v\tau(V : S) = (\lambda^v, \rho^v) \qquad (v \in V)$$

is a homomorphism of V into $\Omega(S)$ which extends π. Moreover, $\tau(V : S)$ is the unique extension of π to a homomorphism of V into $\Omega(S)$ [**P**; I.9.2]. We call $\tau(V : S)$ the *canonical homomorphism of* V *into* $\Omega(S)$.

Let S and T be inverse semigroups and suppose that T is an inverse subsemigroup of $\mathscr{I}(I)$, the symmetric inverse semigroup on I. Let ${}^I S$ denote the set of functions (written on the right) from subsets of I into S. For any $\psi \in {}^I S$, denote the domain of ψ by $\mathbf{d}\psi$. Define a multiplication on ${}^I S$ by

$$i(\psi \cdot \psi') = (i\psi) \cdot (i\psi') \qquad [i \in \mathbf{d}\psi \cap \mathbf{d}\psi'].$$

For any $\beta \in \mathscr{I}(I)$ and $\psi \in {}^I S$, we define a mapping ${}^\beta\psi$ by

$$i({}^\beta\psi) = (i\beta)\psi \qquad [i \in \mathbf{d}\beta,\ i\beta \in \mathbf{d}\psi].$$

The (*right*) *wreath product of* S *and* T is the set

$$S \operatorname{wr} T = \{(\psi, \beta) \in {}^I S \times T \colon \mathbf{d}\psi = \mathbf{d}\beta\}$$

with multiplication given by

$$(\psi, \beta) \cdot (\psi', \beta') = (\psi {}^\beta\psi', \beta\beta').$$

If T is an inverse subsemigroup of $\mathscr{I}(I)$, we will sometimes write (T, I) for T if we wish to emphasize the set I on which T acts.

Our definition of wreath product follows that of Houghton [**H**]. In [**H**] the wreath product $W(S, T)$ of inverse semigroups S and T is, in our notation, $S \operatorname{wr}(T, T)$ where T is given the Wagner representation by partial right translations. Our notation follows Petrich [**P**; V.4]. It is not difficult to verify that if S and (T, I) are inverse semigroups then $S \operatorname{wr}(T, I)$ is also an inverse semigroup. In fact, if $(\psi, \beta) \in S \operatorname{wr}(T, I)$ then

$$(\psi, \beta)^{-1} = (\psi^{-1}, \beta^{-1})$$

where $\psi^{-1} \in {}^I S$ and $\beta^{-1} \in T$ are defined by

$$\begin{gathered}\mathbf{d}\beta^{-1} = \mathbf{d}\psi^{-1} = \{i\beta : i \in \mathbf{d}\beta\}, \\ \beta^{-1} \text{ is the inverse of } \beta \text{ in } T \quad \text{and} \\ i\psi^{-1} = (i\beta^{-1}\psi)^{-1} \qquad (i \in \mathbf{d}\beta^{-1}).\end{gathered}$$

Equivalently, we may define ψ^{-1} by

$$j\beta\psi^{-1} = (j\psi)^{-1} \qquad (j \in \mathbf{d}\beta).$$

For any (ψ, β) belonging to $S\,\mathrm{wr}\,(T, I)$, we have written $(\psi, \beta)^{-1}$ as (ψ^{-1}, β^{-1}) even though the definition of ψ^{-1} depends upon β. This is not to suggest that if (ψ, β') is another member of $S\,\mathrm{wr}\,(T, I)$, then the first coordinate of $(\psi, \beta')^{-1}$ is the same as the first coordinate of $(\psi, \beta)^{-1}$. We use ψ^{-1} to avoid notational difficulties and simply note that when ψ^{-1} is used, the member of (T, I) to which it is paired will be understood.

Let $\mathscr{U}$ and $\mathscr{V}$ be varieties of inverse semigroups. The *Mal′cev product* of $\mathscr{U}$ and $\mathscr{V}$, denoted by $\mathscr{U} \circ \mathscr{V}$, is the collection of those inverse semigroups S for which there exists a congruence ρ on S with the property that $e\rho \in \mathscr{U}$ for all $e \in E_S$ and $S/\rho \in \mathscr{V}$. In general, $\mathscr{U} \circ \mathscr{V}$ is not a variety. For example, if $\mathscr{V}$ is any nontrivial group variety and $\mathscr{U} = \mathscr{S}$ then the five element combinatorial Brandt semigroup B_2 is a member of $\langle \mathscr{U} \circ \mathscr{V} \rangle$ but B_2 is not a member of $\mathscr{U} \circ \mathscr{V}$. However, when $\mathscr{U}$ is a variety of groups, $\mathscr{U} \circ \mathscr{V}$ is a variety (see [**P**; XII 8.3] or [**B**]). Note that, if $\mathscr{V}$ and $\mathscr{W}$ are varieties such that $\mathscr{V} \subseteq \mathscr{W}$ then, for any variety $\mathscr{U}$, $\mathscr{U} \circ \mathscr{V} \subseteq \mathscr{U} \circ \mathscr{W}$ and $\mathscr{V} \circ \mathscr{U} \subseteq \mathscr{W} \circ \mathscr{U}$.

Mal′cev products play an important role in our efforts here, particularly in the context of the congruence ν on $\mathscr{L}(\mathscr{I})$. If $\mathscr{U}$ is a group variety and $\mathscr{V}$ is a combinatorial variety, then $\mathscr{U} \circ \mathscr{V}$ is the maximum variety in the ν-class of $\mathscr{U} \vee \mathscr{V}$, where ν is the congruence on $\mathscr{L}(\mathscr{I})$ defined by $\mathscr{V}_1 \nu \mathscr{V}_2$ if and only if $\mathscr{V}_1 \cap \mathscr{G} = \mathscr{V}_2 \cap \mathscr{G}$ and $\mathscr{V}_1 \vee \mathscr{G} = \mathscr{V}_2 \vee \mathscr{G}$, for all $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{L}(\mathscr{I})$ (see, for e.g., [**P**; XII.2, XII.3]). By a result due to Reilly [**R1**], if $\mathscr{U}$ is a variety of groups and $\mathscr{V}$ is a combinatorial variety, then $[\mathscr{U} \vee \mathscr{V}, \mathscr{U} \circ \mathscr{V}]$ is the ν-class of $\mathscr{V} \vee \mathscr{U}$. For further information on Mal′cev products we refer the reader to [**P**] or [**R1**].

Define the binary operator Wr on the lattice of varieties of inverse semigroups by

$$\mathrm{Wr}(\mathscr{U}, \mathscr{V}) = \langle S\,\mathrm{wr}\,(T, I) : S \in \mathscr{U} \text{ and } T \in \mathscr{V} \rangle \qquad (\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{I})).$$

If $\mathscr{U}$ is a group variety and $\mathscr{V}$ is a variety of inverse semigroups then $\mathrm{Wr}(\mathscr{U}, \mathscr{V}) = \mathscr{U} \circ \mathscr{V}$ (see **[C]**).

We find it convenient in our investigations to make use of the graphical representation of inverse semigroups introduced by Stephen **[S]**, which he calls the Schützenberger representation of an inverse semigroup with presentation. Schützenberger graphs are defined as follows:

Let $P = (X; R)$ be a fixed presentation of the inverse semigroup S with τ the corresponding congruence on $F\mathscr{I}(X)$, the free inverse semigroup on S. Let $w \in S$ and R_w the $\mathscr{R}$-class of w in S. The *Schützenberger graph of R_w with respect to P* is the labelled digraph $\Gamma(w)$, where

$$\begin{aligned} V(\Gamma(w)) &= R_w, \\ E(\Gamma(w)) &= \{(v_1, x, v_2) : v_1, v_2 \in R_w, \ x \in X \cup X^{-1} \\ &\qquad\qquad \text{and } v_1(x\tau) = v_2\}. \end{aligned}$$

The *Schützenberger representation of w* (with respect to P) is the birooted labelled digraph $(ww^{-1}, \Gamma(w), w)$, where ww^{-1} is the *start* vertex and w is the *end* or *terminal* vertex. The Schützenberger representation of the semigroup S is the family of birooted graphs $\{(ww^{-1}, \Gamma(w), w) : w \in S\}$. Schützenberger graphs enjoy the following properties:

Let $v \in S$, $\Gamma(v)$ be its Schützenberger graph with respect to P, $v_1, v_2, v_3 \in R_v$ and $w \in (X \cup X)^+$ (see **[S]**).

(a) if (v_1, x, v_2) is an edge in $\Gamma(v)$ then (v_2, x^{-1}, v_1) is also an edge in $\Gamma(v)$;

(b) if (v_1, x, v_2) and (v_1, x, v_3) are edges in $\Gamma(v)$ then $v_2 = v_3$;

(c) if (v_2, x, v_1) and (v_3, x, v_1) are edges in $\Gamma(v)$ then $v_2 = v_3$;

(d) $v_1(w\tau) = v_2$ if and only if w labels a $v_1 - v_2$ walk;

(e) $(w\tau) \geq v$ if and only if w labels an $e - v$ walk;

(f) $v_1 \mathscr{D} v_2$ if and only if $\Gamma(v_1)$ is isomorphic to $\Gamma(v_2)$;

(g) $v_1 \mathscr{R} v_2$ if and only if there exists an isomorphism from $\Gamma(v_1)$ to $\Gamma(v_2)$ such that $v_1 v_1^{-1}$ is mapped to $v_2 v_2^{-1}$;

(h) $v_1 \mathscr{L} v_2$ if and only if there exists an isomorphism from $\Gamma(v_1)$ to $\Gamma(v_2)$ such that v_1 is mapped to v_2.

We will only be considering Schützenberger graphs of the $\mathscr{B}^1$-free inverse semigroup on (countably infinite) X with respect to the presentation $P = (X; \rho(\mathscr{B}^1))$. For further properties and a detailed discussion of Schützenberger graphs we refer the reader to Stephen **[S]**.

3. The variety $\mathscr{B}^1$. In this section we construct inverse semigroups which belong to the variety $\mathscr{B}^1$ which, in subsequent sections, will be used to construct inverse semigroups in $\mathrm{Wr}(\mathscr{U}, \mathscr{B}^1)$, where $\mathscr{U}$ is a variety of abelian groups. These semigroups will be used to define an infinite collection of varieties in the interval $[\mathscr{U} \vee \mathscr{B}^1, \mathrm{Wr}(\mathscr{U}, \mathscr{B}^1)]$. Throughout the remainder of this note ρ will denote the fully invariant congruence on $F\mathscr{I}(X)$ corresponding to $\mathscr{B}^1$.

Before we proceed, we require some notation. For any word $w \in X \cup X^{-1}$, denote by w_A the word obtained from w by deleting all occurrences of variables not in A. For example, $(x_1x_2x_1^{-1}x_3x_2x_1)_{\{x_1\}}$ is the word $x_1x_1^{-1}x_1$.

LEMMA 3.1. *Let w and v be words over $X \cup X^{-1}$. Then $w\,\rho\, v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \varnothing$, $w_A\rho(\mathscr{B})v_A$.*

Proof. $w\,\rho\, v$ if and only if B_2^1 satisfies the equation $w = v$. Since B_2^1 possesses an identity, B_2^1 satisfies the equation $w = v$ if and only if B_2 satisfies $w_A = v_A$ for all $A \subseteq c(w_A) = c(v_A)$. This is equivalent to $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \varnothing$, $w_A\rho(\mathscr{B})v_A$. □

COROLLARY 3.2. *Let w and v be words over $X \cup X^{-1}$. Then $w\,\rho\, v$ if and only if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \varnothing$, $w_A\rho v_A$.*

Proof. If $w\,\rho\, v$ then by Lemma 3.1, $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \varnothing$, $w_A\rho(\mathscr{B})v_A$. But then for any $A \subseteq c(w) = c(v)$, for all $B \subseteq A$, $B \neq \varnothing$, $w_B\rho(\mathscr{B})v_B$ and so by Lemma 3.1, $w_A\,\rho\, v_A$. On the other hand, if $c(w) = c(v)$ and for all $A \subseteq c(w)$, $A \neq \varnothing$, $w_A\,\rho\, v_A$, then in particular, $w = w_{c(w)}\,\rho\, v_{c(w)} = v_{c(v)} = v$. □

LEMMA 3.3. *If $S \in \mathscr{B}^1$ then $S^1 \in \mathscr{B}^1$.*

Proof. Suppose that $\mathscr{B}^1$ satisfies the equation $w = v$, where $c(w) = c(v) = \{x_1, \ldots, x_n\}$. Let $s_1, \ldots, s_n$ be arbitrarily chosen elements of S^1 with repetitions allowed. Suppose that $s_{i_1}, \ldots, s_{i_k}$ are those s_i that are the identity of S^1. Then S^1 satisfies $w[s_1, \ldots, s_n] = v[s_1, \ldots, s_n]$ if S satisfies $w_A[s_1, \ldots, s_n] = v_A[s_1, \ldots, s_n]$, where $A = \{x_1, \ldots, x_n\}\backslash\{x_{i_1}, \ldots, x_{i_k}\}$. Since $S \in \mathscr{B}^1$, S does satisfy $w_A[s_1, \ldots, s_n] = v_A[s_1, \ldots, s_n]$ by Corollary 3.2 and so, as a result, $w[s_1, \ldots, s_n] = v[s_1, \ldots, s_n]$ is true in S^1. Since the s_i were chosen arbitrarily, S^1 satisfies the equation $w = v$. Therefore, $S^1 \in \mathscr{B}^1$. □

We require some further notation for this section. Let $w \in (X \cup X^{-1})^+$. We write $w \equiv v$ to mean w and v are identical words, letter for letter, over a common alphabet (in this case $X \cup X^{-1}$). We say that the word v is a *cyclic shift* of w if $w \equiv u_1 u_2$ and $v \equiv u_2 u_1$ for words u_1, u_2 over the alphabet of w. For each $n \in \omega$, we denote by τ_n the equation $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \in E$. Observe that if w is the word $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1}$ then any cyclic shift of w can be written $y_1 y_2 \cdots y_n y_1^{-1} y_2^{-1} \cdots y_n^{-1}$ (where the y_i all belong to $\{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$).

The remainder of this section is devoted to a construction of a family of inverse semigroups $\{S(\tau_n) : n \in \omega\}$ each of which belongs to the variety $\mathscr{B}^{-1}$. For each $n \in \omega$, $S(\tau_n)$ is obtained from the $\mathscr{B}^1$-free inverse semigroup by first identifying the ideal consisting of those elements whose $\mathscr{D}$-class does not lie above the $\mathscr{D}$-class of $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \rho$ (which results in an ideal extension of the principal factor of the $\mathscr{D}$-class of $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \rho$, a Brandt semigroup) and then mapping this semigroup into the translational hull of the principal factor corresponding to the $\mathscr{D}$-class of $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \rho$. In order to do this we require some knowledge of the $\mathscr{D}$-class of $x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1} \rho$.

LEMMA 3.4. *Let* $w = x_1 x_2 \cdots x_n x_1^{-1} x_2^{-1} \cdots x_n^{-1}$ *and suppose that* $v = y_1 y_2 \cdots y_n y_1^{-1} y_2^{-1} \cdots y_n^{-1}$ *is a cyclic shift of* w. *Let* $a \in X \cup X^{-1}$.
(a) $v\rho$ *is an idempotent*;
(b) $(va\rho)\mathscr{R}(v\rho)$ *if and only if* $a = y_1$ *or* $a = y_n$.

Proof. (a) As we remarked in §2, $\mathscr{B}^1$ is contained in $\mathscr{A}_2^{\max}$ (because it has E-unitary covers over the variety $\mathscr{A}_2$ of abelian groups of exponent two; see **[PR]**). Since $\mathscr{A}_2$ satisfies the equation $v = v^2$, $\mathscr{A}_2^{\max}$ and hence $\mathscr{B}^1$ satisfies $v = v^2$. Thus, $v\rho$ is an idempotent.

(b) Since $v\rho$ is an idempotent, if $a = y_1$ or $a = y_n$ then $(va\rho)\mathscr{R}(v\rho)$. On the other hand, suppose that $(va\rho)\mathscr{R}(v\rho)$. Then $vaa^{-1}v^{-1}\rho vv^{-1}$ and so $c(va) = c(v)$. Thus, $a \in c(v)$. But $(va\rho)\mathscr{R}(v\rho)$ also implies that $vaa^{-1}\rho v$. If $a = y_i^{-1}$ for some i, then $(vaa^{-1})_{\{y_i\}} = y_i y_i^{-1} y_i^{-1} y_i \rho(\mathscr{B}) y_i^2$, while $v_{\{y_i\}} = y_i y_i^{-1} \not\rho(\mathscr{B}) y_i^2$ and so, by Lemma 3.2, $vaa^{-1} \not\rho v$. Therefore, $a = y_i$ for some i. If $1 < i < n$ then $(vaa^{-1})_{\{y_1, y_i, y_n\}} = y_1 y_i y_n y_1^{-1} y_i^{-1} y_n^{-1} y_i y_i^{-1}$ and $v_{\{y_1, y_i, y_n\}} = y_1 y_i y_n y_1^{-1} y_i^{-1} y_n^{-1}$. If b is any non-idempotent element of B_2, then substituting b for y_1 and y_n and substituting b^{-1} for y_i, yields that $(vaa^{-1})_{\{y_1, y_i, y_n\}} \not\rho(\mathscr{B}) v_{\{y_1, y_i, y_n\}}$. As a consequence, y_i must be either y_1 or y_n. $\square$

LEMMA 3.5. *Let* $w = x_1x_2\cdots x_nx_1^{-1}x_2^{-1}\cdots x_n^{-1}$ *and suppose that* u *is a proper initial segment of* w *with* $w \equiv uu'$. *Let* $a \in X \cup X^{-1}$. *Then* $wu\rho\mathscr{R}wua\rho$ *if and only if* a *is the initial letter of* u' *or* a^{-1} *is the terminal letter of* u *in the case that* u *is not the empty word, and in the case that* u *is the empty word,* a *is the initial letter of* u' *or* a^{-1} *is the terminal letter of* u'.

Proof. If u is the empty word then the statement follows immediately from Lemma 3.4, so assume that u is not the empty word.

First suppose that $wu\rho\mathscr{R}wua\rho$. Then $wu\rho = uu'u\rho\mathscr{L}u'u\rho$ since $u'u$ is a cyclic shift of w and any cyclic shift of w is an idempotent modulo ρ. Therefore, $wu\rho\mathscr{R}wua\rho$ implies that $u'u\rho\mathscr{R}u'ua\rho$ (this follows from the more general result that $t\mathscr{L}s$ implies that $t\mathscr{R}ta$ if and only if $s\mathscr{R}sa$). Since $u'u$ is a cyclic shift of w, we have by Lemma 3.4 that a is either the initial letter of u' or a^{-1} is the terminal letter of u.

For the converse, first note that if a is the initial letter of u' then ua is an initial segment of w and so, since $w\rho$ is an idempotent, $wu\rho\mathscr{R}wua\rho$. If a^{-1} is the terminal letter of u then letting $u \equiv u^*a^{-1}$ we obtain that $wua \equiv wu^*a^{-1}a \equiv u^*a^{-1}u'u^*a^{-1}a$. Since $a^{-1}u'u^*$ is a cyclic shift of w, $a^{-1}u'u^*\rho$ is an idempotent by Lemma 3.4(a) and as a result,

$$\begin{aligned} wua &\equiv wu^*a^{-1}a \equiv u^*a^{-1}u'u^*a^{-1}a\rho u^*a^{-1}aa^{-1}u'u^*\rho u^*a^{-1}u'u^* \\ &\equiv uu'u^* \equiv wu^*. \end{aligned}$$

It is now immediate that $wu\rho\mathscr{R}wu^*\rho = wua\rho$. □

LEMMA 3.6. *Let* $w = x_1x_2\cdots x_nx_1^{-1}x_2^{-1}\cdots x_n^{-1}$. *For any word* v *over* $X\cup X^{-1}$, $w\rho\mathscr{R}v\rho$ *if and only if* $v\,\rho\,wu$ *for some initial segment* u *of* w.

Proof. Suppose that $w\rho\mathscr{R}v\rho$, say $wa_1\cdots a_k\,\rho\,v$, where $a_1, \ldots, a_k \in X \cup X^{-1}$. We prove by induction on k that $wa_1\cdots a_k\rho\mathscr{R}w\rho$ implies that $wa_1\cdots a_k\,\rho\,wu$ for some initial segment u of w. If $k = 1$ then $wa_1\rho\mathscr{R}w\rho$ implies by Lemma 3.4 that $a_1 = x_1$ or x_n. If $a = x_1$ then a_1 is an initial segment of w already. If $a_1 = x_n$ then $wa_1\,\rho\,wwx_n$. Now

$$\begin{aligned} wwx_n &\equiv x_1\cdots x_nx_1^{-1}\cdots x_{n-1}^{-1}[x_n^{-1}x_1\cdots x_nx_1^{-1}\cdots x_{n-1}^{-1}]x_n^{-1}x_n \\ &\rho\, x_1\cdots x_nx_1^{-1}\cdots x_{n-1}^{-1}[x_n^{-1}x_1\cdots x_nx_1^{-1}\cdots x_{n-1}^{-1}] \end{aligned}$$

since $[x_n^{-1}x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}]$ is a cyclic shift of w and so $[x_n^{-1}x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}]\rho$ is an idempotent.

But

$$x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}[x_n^{-1}x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}] \equiv w x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}$$

and so as a consequence, $v \,\rho\, w x_1 \cdots x_n x_1^{-1} \cdots x_{n-1}^{-1}$.

Now suppose that $k > 1$. $wa_1 \cdots a_k \rho \mathscr{R} w\rho$ implies that $w\rho \mathscr{R} wa_1 \cdots a_{k-1}\rho$ and so, by the induction hypothesis, $wa_1 \cdots a_{k-1} \,\rho\, wu$ for some initial segment u of $w \equiv uu'$. If u is the empty word, then $wa_1 \cdots a_k \rho w a_k \mathscr{R} w\rho$ and this is the same as the case $k = 1$. Otherwise, by Lemma 3.5, $wu\rho \mathscr{R} wua_k\rho$ implies that a_k is the initial letter of u' or a_k^{-1} is the terminal letter of u. If a is the initial letter of u' then $v \,\rho\, wa_1 \cdots a_k \,\rho\, wua_k$ and ua_k is an initial segment of w. If a_k^{-1} is the terminal letter of u then setting $u \equiv b_1 \cdots b_m$ we obtain that $v \,\rho\, wa_1 \cdots a_k \,\rho\, wua_k$ and

$$\begin{aligned} wua_k &\equiv wb_1 \cdots b_m b_m^{-1} \\ &\equiv b_1 \cdots b_{m-1}[b_m u' b_1 \cdots b_{m-1}]b_m b_m^{-1} \\ &\;\rho\; b_1 \cdots b_{m-1}[b_m u' b_1 \cdots b_{m-1}] \end{aligned}$$

since $[b_m u' b_1 \cdots b_{m-1}]$ is a cyclic shift of w and so must be an idempotent modulo ρ. But $b_1 \cdots b_{m-1}[b_m u' b_1 \cdots b_{m-1}] \equiv wb_1 \cdots b_{m-1}$ and so $v \,\rho\, wb_1 \cdots b_{m-1}$ and $b_1 \cdots b_{m-1}$ is an initial segment of w.

Since $w\rho$ is an idempotent, the converse is immediate. □

Schützenberger graphs provide a concise, visual representation of a $\mathscr{D}$-class. Because of this, in the following theorem we describe the $\mathscr{D}$-classes of the words $\{x_1x_2 \cdots x_n x_1^{-1}x_2^{-1} : n \in \omega, n > 1\}$ relative to the variety $\mathscr{B}^1$ in this way.

Theorem 3.7. *Let $w = x_1x_2 \cdots x_n x_1^{-1}x_2^{-1} \cdots x_n^{-1}$. The following graph is isomorphic to the Schützenberger graph of w relative to $\mathscr{B}^1$, where v_1 is both the start and end vertex.*

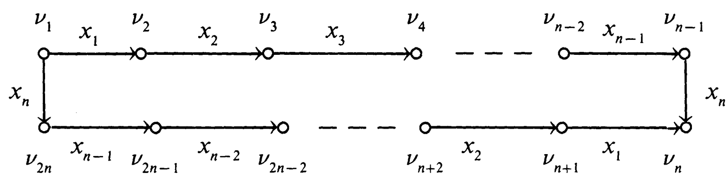

Figure 3.1
The Schützenberger graph of $w = x_1x_2 \cdots x_n x_1^{-1}x_2^{-1} \cdots x_n^{-1}$ with respect to $\mathscr{B}^1$.

Proof. By Lemma 3.6 there are at most $2n$ vertices in the Schützenberger graph Γ of w relative to $\mathscr{B}^1$ as there are $2n$ initial segments of w not identical to w. It is a simple exercise to verify, using Lemma 3.1, that if u and u' are two proper initial segments of w (that is, neither u nor u' is identical to w) then $wu\,\rho\, wu'$ implies that $u \equiv u'$. By Lemma 3.5, $(wu_1\rho, x, wu_2\rho)$ is an edge of Γ if and only if x^{-1} is the terminal letter of u_1 or x is the initial letter of u_1', where $u_1u_1' \equiv w$. If x is the initial letter of u_1', then wu_2 and wu_1x are ρ-equivalent with both u_1x and u_2 initial segments of w. Thus, $u_1x \equiv u_2$. If x^{-1} is the terminal letter of u_1 then writing $u_1 \equiv u_1^*x^{-1}$ we have $wu_1^*x^{-1}x\,\rho\, wu_2$. Since $wu_1^*\rho\mathscr{R}\, wu_1 \equiv wu_1^*x^{-1}\rho$, we have that $wu_1^*\,\rho\, wu_1^*x^{-1}x\,\rho\, wu_2$. Since both u_1^* and u_2 are initial segments of w, $wu_1^* \equiv wu_2$ and so $wu_2x^{-1} \equiv wu_1$. Finally, if u_1 is the empty word and x^{-1} is the terminal letter of w then x^{-1} is the terminal letter of $ww \equiv ww^*x^{-1}\,\rho\, w$ and $ww^*x^{-1}x\,\rho\, wu_2$. But, $ww^*x^{-1}x\,\rho\, ww^*$ and both w^* and u_2 are initial segments of w, so $wu_2 \equiv ww^*$, whence $wu_2x^{-1} \equiv ww$.

It follows from these remarks that Γ is isomorphic to the graph described above via the map which sends $wu\rho$ to $v_{|u|+1}$, for all proper initial segments u of w. □

DEFINITION 3.8. Let F be the $\mathscr{B}^1$-free inverse semigroup on $X = \{x_i : i \in \omega\}$. Let w_n be the word $x_1 \cdots x_n x_1^{-1} \cdots x_n^{-1}$ for each $n \in \omega$. Denote the ideal $\{v \in F : J_v \not\geq J_{w_n\rho}\}$ of F by $I(\tau_n)$ and let $J(\tau_n) = F/I(\tau_n)$. Now $J(\tau_n)$ is an ideal extension of $J^0_{w_n\rho}$ which is isomorphic to $B(\{1\}, 2n)$. Let $S(\tau_n)$ be the image of $J(\tau_n)$ under the canonical homomorphism into the translational hull $\Omega(J^0_{w_n\rho})$ of $J^0_{w_n\rho}$.

LEMMA 3.9. *The semigroups* $S(\tau_n)$ *and* $S(\tau_n)^1$ *belong to* $\mathscr{B}^1$, *for all* $n \in \omega$, $n \geq 2$.

Proof. The semigroup $S(\tau_n)$ is a homomorphic image of the $\mathscr{B}^1$-free inverse semigroup on X and so is an element of $\mathscr{B}^1$. The semigroup $S(\tau_n)^1 \in \mathscr{B}^1$ by Lemma 3.3. □

In the following section we will use the $S(\tau_n)$ to construct a family of inverse semigroups which belong to $\mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ but not to $\mathscr{A}_m \vee \mathscr{B}^1$, for $m \in \omega$. Before we do so, we describe the $S(\tau_n)$.

The inverse semigroup $S(\tau_n)$ is isomorphic to the Wagner representation of the $\mathscr{B}^1$-free inverse semigroup on X restricted to $R_{w_n\rho}$.

That is, if α_w is the element of $\mathscr{I}(F\mathscr{B}^1(X))$ corresponding to $w\rho$ in the Wagner representation of $F\mathscr{B}^1(X)$, then in the restricted (to $R_{w_n\rho}$) Wagner representation, α'_w corresponds to $w\rho$, where $\mathbf{d}\alpha'_w = \{u\rho \in \mathbf{d}\alpha_w : u\rho \mathscr{R} w_n\rho$ and $(u\rho)\alpha_w \mathscr{R} w_n\rho\}$ and for all $u\rho \in \mathbf{d}\alpha'_w$, $(u\rho)\alpha'_w = (u\rho)\alpha_w$.

An added advantage to using the Schützenberger graph description in Theorem 3.7 is that we can read directly from the graph the image of any word of $J(\tau_n)$ under the canonical homomorphism into $\Omega(J^0_{w_n\rho}) \cong \mathscr{I}(R_{w_n\rho})$. The inverse semigroup $S(\tau_n)$ is generated by the image of the x_i under the canonical homomorphism and, for each $i = 1, \dots, n$, the domain of the image of x_i is the set of vertices v for which there is an edge labelled by x_i starting at v and v is mapped to the terminal vertex of that edge. It is straightforward to verify that $S(\tau_n)$ is (isomorphic to) the inverse subsemigroup of $\mathscr{I}(R_{w_n\rho})$ generated by $\{\alpha_i : i = 1, \dots, n\}$ where for each i,

$$\mathbf{d}\alpha_i = \{w_n x_1 \cdots x_{i-1}\rho,\, w_n x_1 \cdots x_n x_1^{-1} \cdots x_i^{-1}\rho\}$$

and

$$w_n x_1 \cdots x_{i-1}\rho\alpha_i = w_n x_1 \cdots x_i \rho,$$

$$\begin{aligned} &w_n x_1 \cdots x_n x_1^{-1} \cdots x_i^{-1}\rho\alpha_i \\ &\quad = w_n x_1 \cdots x_n x_1^{-1} \cdots x_i^{-1} x_i \rho\, w_n x_1 \cdots x_n x_1^{-1} \cdots x_{i-1}^{-1}. \end{aligned}$$

4. Inverse semigroups in $\mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$**.** The semigroups constructed in §3 can be used to construct semigroups in $\mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ for $m \in \omega$. Since $S(\tau_n)$ is isomorphic to the Wagner representation of $F\mathscr{B}^1(X)$ restricted to $R_{w_n\rho}$, it can be represented as an inverse subsemigroup of $\mathscr{I}(R_{w_n\rho})$ for all $n \in \omega$. Thus, for any group G belonging to $\mathscr{A}_m$, $m \in \omega$, $G \,\mathrm{wr}\,(S(\tau_n), R_{w_n}) \in \mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$. The semigroups we construct in this section are inverse subsemigroups of semigroups of this form and so belong to $\mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$.

For each $n \in \omega$, $n \geq 2$, let C_n denote the cyclic group of order n.

Definition 4.1. Let $m, n \in \omega$, $m, n \geq 2$. Let 1 denote the identity of C_m and let g be a generator of C_m. Let

$$A_{m,n} \subseteq C_m \,\mathrm{wr}\,(S(\tau_n), R_{w_n})$$

be defined as follows:

Let $\{\alpha_i : i = 1, \dots, n\}$ be the generators of $S(\tau_n)$ as described at the end of the previous section. For $i = 1, \dots, n-1$, define the map ϕ_i from R_{w_n} into C_m by setting

$$\mathbf{d}\phi_i = \mathbf{d}\alpha_i = \{w_n x_1 \cdots x_{i-1}\rho,\, w_n x_1 \cdots x_n x_1^{-1} \cdots x_i^{-1}\rho\}$$

and defining $(w_n x_1 \cdots x_{i-1}\rho)\phi_i = 1$, $(w_n x_1 \cdots x_n x_1^{-1} \cdots x_i^{-1}\rho)\phi_i = 1$. Define the map ϕ_n from R_{w_n} into C_m by setting $\mathbf{d}\phi_n = \mathbf{d}\alpha_n = \{w_n x_1 \cdots x_{n-1}\rho, w_n\rho\}$ and defining $(w_n x_1 \cdots x_{n-1}\rho)\phi_n = 1$, $(w_n\rho)\phi_n = g$. Then $(\phi_i, \alpha_i) \in C_m \operatorname{wr}(S(\tau_n), R_{w_n})$ for $i = 1, \ldots, n$.

Let

$$\begin{aligned} A_{m,n} = \{(\psi, \beta) \in C_m \operatorname{wr}(S(\tau_n), R_{w_n})\colon |\mathbf{d}\psi| = |\mathbf{d}\beta| \le 1\} \\ \cup \{(\phi_i, \alpha_i)\colon i = 1, \ldots, n\}. \end{aligned}$$

Define $T_{m,n}$ to be the inverse subsemigroup of $C_m \operatorname{wr}(S(\tau_n), R_{w_n})$ generated by $A_{m,n}$. Observe that $T_{m,n}$ is an ideal extension of a Brandt semigroup over the group C_m. It is not difficult to see that $T_{m,n}$ is in fact the following:

$$\begin{aligned} \{(\psi, \beta) \in C_m \operatorname{wr}(S(\tau_n), R_{w_n})\colon |\mathbf{d}\psi| = |\mathbf{d}\beta| \le 1\} \\ \cup \{(\phi_i, \alpha_i), (\phi_i, \alpha_i)^{-1}, (\phi_i, \alpha_i)(\phi_i, \alpha_i)^{-1}, \\ (\phi_i, \alpha_i)^{-1}(\phi_i, \alpha_i)\colon i = 1, \ldots, n\}. \end{aligned}$$

LEMMA 4.2. *For each* $m, n \in \omega$, $m, n \ge 2$,

(a) $T_{m,n} \in \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ *but* $T_{m,n} \notin \mathscr{B}^1$;

(b) $T_{m,n}^1 \in \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ *but* $T_{m,n}^1 \notin \mathscr{B}^1$;

(c) $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n}\rangle \subseteq \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$;

(d) $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n}^1\rangle \subseteq \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$.

Proof. $T_{m,n}^1$ is an inverse subsemigroup of $C_m \operatorname{wr}(S(\tau_n)^1, R_{w_n})$ and $S(\tau_n)^1 \in \mathscr{B}^1$ by Lemma 3.9. Thus, $T_{m,n}^1 \in \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ by the definition of the Wr operator. As a consequence, $T_{m,n} \in \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ since $T_{m,n}$ is an inverse subsemigroup of $T_{m,n}^1$. On the other hand, $T_{m,n}$ is an ideal extension of a Brandt semigroup over C_m and so contains a subgroup isomorphic to C_m. Thus, $T_{m,n} \notin \mathscr{B}^1$ since $\mathscr{B}^1$ is a combinatorial variety. Since $T_{m,n}$ is an inverse subsemigroup of $T_{m,n}^1$ we also have that $T_{m,n}^1 \notin \mathscr{B}^1$. This proves both (a) and (b).

Both $T_{m,n}^1$ and $T_{m,n}$ contain subgroups isomorphic to C_m and so $\mathscr{A}_m \subseteq \langle T_{m,n}^1\rangle$ and $\mathscr{A}_m \subseteq \langle T_{m,n}\rangle$ since $\mathscr{A}_m$ is generated by C_m. The natural homomorphism onto the second coordinate maps $T_{m,n}$ onto an inverse semigroup isomorphic to $S(\tau_n) \in \mathscr{B}^1$, and maps $T_{m,n}^1$ onto an inverse semigroup isomorphic to $S(\tau_n)^1 \in \mathscr{B}^1$. Since both $S(\tau_n)$ and $S(\tau_n)^1$ contain copies of B_2^1, it follows that $\mathscr{B}^1 \subseteq \langle T_{m,n}^1\rangle$ and $\mathscr{B}^1 \subseteq \langle T_{m,n}\rangle$. Consequently, we have that $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n}\rangle$ and $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \langle T_{m,n}^1\rangle$. It is immediate from parts (a) and (b) that $\langle T_{m,n}\rangle \subseteq \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$ and $\langle T_{m,n}^1\rangle \subseteq \operatorname{Wr}(\mathscr{A}_m, \mathscr{B}^1)$. This completes the proofs of (c) and (d). $\square$

LEMMA 4.3. *Let $m, n \in \omega$, $m, n \geq 2$. Neither $T_{m,n}$ nor $T^1_{m,n}$ satisfies the equation τ_n.*

Proof. Substitute (ϕ_i, α_i) for x_i, $i = 1, \ldots, n$. □

In the following lemma we use the term *kernel* to mean the minimum nonzero ideal of an inverse semigroup, if it exists.

LEMMA 4.4. *Let $m, n \in \omega$, $m, n \geq 2$. $T_{m,n}$ satisfies the equation τ_k for $k < n$.*

Proof. Towards a contradiction, suppose that $T_{m,n}$ does not satisfy τ_k for some $k < n$. Assume that k is the least such integer and let $(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k) \in T_{m,n}$ be such that

$$x_1 \cdots x_k x_1^{-1} \cdots x_k^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)] = (\psi, \beta)$$

is not an idempotent in $T_{m,n}$.

We first make a few observations.

(i) $|\mathbf{d}\beta| = 1$: If $|\mathbf{d}\beta| = 0$ then we immediately have that (ψ, β) is an idempotent. If $|\mathbf{d}\beta| = 2$ then the (ψ_i, β_i) all belong to the same $\mathscr{D}$-class, namely, the $\mathscr{D}$-class D of (ψ, β). [This is because $T_{m,n}$ is completely semisimple and so $\mathscr{D} = \mathscr{J}$. Thus, the $\mathscr{D}$-class of (ψ, β) is contained in the $\mathscr{D}$-class of (ψ_i, β_i) for all i. But if $|\mathbf{d}\beta| = 2$, then the $\mathscr{D}$-class of (ψ, β) is a maximal $\mathscr{D}$-class in $T_{m,n}$ and so (ψ, β) is $\mathscr{D}$-related to (ψ_i, β_i) for all i.] But D^0 is a Brandt semigroup and as such satisfies τ_k. Since $x_1 \cdots x_k x_1^{-1} \cdots x_k^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)] = (\psi, \beta)$ in D^0 and $(\psi, \beta) \neq 0$, we conclude that, in this case, (ψ, β) is an idempotent. The only remaining possibility is that $|\mathbf{d}\beta| = 1$.

(ii) If $\mathbf{d}\beta = \{v\}$ then $v\beta = v$ and $v\psi$ is not an idempotent. We know that β is an idempotent of $(S(\tau_n), R_{w_n})$ since the natural homomorphism of $T_{m,n}$ onto its second coordinate has image $S(\tau_n)$ which, by Lemma 3.9, is a member of $\mathscr{B}^1$ and $\mathscr{B}^1$ satisfies the equation τ_k. Thus, $v\beta = v$. Also, $v\psi$ is not an idempotent lest $(\psi, \beta) = (\psi, \beta)^2$.

(iii) If (ψ, β) is not an idempotent then for any cyclic shift $y_1 \cdots y_n y_1^{-1} \cdots y_n^{-1}$ of $x_1 \cdots x_k x_1^{-1} \cdots x_k^{-1}$ we have that $y_1 \cdots y_n y_1^{-1} \cdots y_n^{-1}[(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)]$ is not an idempotent. To see this note that if $y_1 \cdots y_n y_1^{-1} \cdots y_n^{-1}$ is a cyclic shift of $x_1 \cdots x_k x_1^{-1} \cdots x_k^{-1}$

then $y_1 \cdots y_n y_1^{-1} \cdots y_n^{-1}[(\psi_1, \beta_1), \dots, (\psi_k, \beta_k)] = (\psi', \beta')$ can be expressed as $(\varphi_1, \gamma_1)(\varphi_2, \gamma_2)$ where $(\psi, \beta) = (\varphi_2, \gamma_2)(\varphi_1, \gamma_1)$. If $\{v\} = \mathbf{d}\beta$ then $v\gamma_2 \in \mathbf{d}\beta'$ and $v\gamma_2\beta' = v\gamma_2$ because $v\gamma_2\gamma_1\gamma_2 = v\gamma_2$ since $v\gamma_2\gamma_1 = v\beta = v$. Then

$$v\gamma_2\psi' = (v\gamma_2\varphi_1)(v\gamma_2\gamma_1\varphi_2) = (v\gamma_2\varphi_1)(v\varphi_2) = (v\varphi_2)(v\gamma_2\varphi_1)$$

since C_m is abelian. But $(v\varphi_2)(v\gamma_2\varphi_1) = v\psi$ which is not an idempotent and so, as a result, (ψ', β') is not an idempotent.

(iv) For some $i \in \{1, \dots, k\}$, $(\psi_i, \beta_i) = (\varphi_n, \alpha_n)$ or $(\varphi_n, \alpha_n)^{-1}$. By (ii), if $\mathbf{d}\beta = \{v\}$ then $v\beta = v$. Therefore, if (ψ, β) is not an idempotent then $v\psi$ is not the identity of C_m. The only elements of $T_{m,n}$ which can contribute non-identity elements to $v\psi$ are those (ψ, β) for which $|\mathbf{d}\beta| = 1$, (ϕ_n, α_n) and $(\phi_n^{-1}, \alpha_n^{-1})$. Now

$$\begin{aligned} v\psi = (v\psi_1)(v\beta_1\psi_2)\cdots(v\beta_1\cdots\beta_{k-1}\psi_k)(v\beta_1\cdots\beta_k\psi_1^{-1}) \\ (v\beta_1\cdots\beta_k\beta_1^{-1}\psi_2^{-1})\cdots(v\beta_1\cdots\beta_k\beta_1^{-1}\cdots\beta_{k-1}^{-1}\psi_k^{-1}). \end{aligned}$$

If (ψ_i, β_i) is such that $|\mathbf{d}\beta_i| = 1$, then in this factorization of $v\psi$, ψ_i contributes $v\beta_1 \cdots \beta_{i-1}\psi_i = g$, say, and $v\beta_1 \cdots \beta_k\beta_1^{-1} \cdots \beta_{i-1}^{-1}\psi_i^{-1} = g^{-1}$, since g^{-1} is the only element of $\mathbf{r}\psi_i^{-1}$. Thus, the contributions to this factorization of $v\psi$ by ψ_i cancel and so, if (ψ, β) is not an idempotent, one of the (ψ_i, β_i) must be (ϕ_n, α_n) or $(\phi_n, \alpha_n)^{-1}$.

(v) None of the (ψ_i, β_i) is an idempotent. This follows from the general observation that if $e = e^2$ and $aebec$ is not an idempotent then $aebec = aea^{-1}(abc)c^{-1}ec$ and so abc cannot be an idempotent. Thus, (ψ_i, β_i) an idempotent contradicts the minimality of k.

As a consequence of the aforementioned observations, the following assumptions concerning the (ψ_i, β_i) can be made. First of all, by (iii) and (iv) we may assume that $(\psi_1, \beta_1) = (\phi_n, \alpha_n)$. Secondly, assume that the k-tuple $\langle(\psi_1, \beta_1), \dots, (\psi_k, \beta_k)\rangle$ contains a maximal number of elements from the kernel of $T_{m,n}$ among the collection of k-tuples from $T_{m,n}$ whose first element is (ϕ_n, α_n) and which witness that $T_{m,n}$ does not satisfy τ_k.

There are two stages to the remainder of the proof. The first stage is showing that exactly one of the (ψ_i, β_i) is a member of the kernel of $T_{m,n}$. We do this in four parts.

(1) For any $i \in \{1, \dots, k\}$, both (ψ_i, β_i) and $(\psi_{i+1}, \beta_{i+1})$ do not belong to the kernel of $T_{m,n}$.

Suppose that both (ψ_i, β_i) and $(\psi_{i+1}, \beta_{i+1})$ belong to the kernel of $T_{m,n}$. If $\mathbf{d}\beta_i = \{v_i\}$ and $\mathbf{d}\beta_{i+1} = \{v_{i+1}\}$ then $v_i\beta_i = v_{i+1}$ since

$\beta_i\beta_{i+1} \neq 0$ and $v_{i+1}\beta_{i+1} = v_i$ since $\beta_i^{-1}\beta_{i+1}^{-1} \neq 0$. It follows that

$$v_i\beta_i\beta_{i+1} = v_i \quad \text{and} \quad v_{i+1}\beta_{i+1}\beta_i = v_{i+1}$$

and

$$\begin{aligned}(v_{i+1}\psi_i^{-1})(v_{i+1}\beta_i^{-1}\psi_{i+1}^{-1}) &= (v_i\beta_i\psi_i^{-1})(v_i\psi_{i+1}^{-1}) \\ &= (v_i\psi_i)^{-1}(v_i\beta_{i+1}^{-1}\psi_{i+1})^{-1} \\ &= (v_i\psi_i)^{-1}(v_{i+1}\psi_{i+1})^{-1} \\ &= (v_{i+1}\psi_{i+1})^{-1}(v_i\psi_i)^{-1} \quad \text{(since } C_m \text{ is abelian)} \\ &= [(v_i\psi_i)(v_{i+1}\psi_{i+1})]^{-1}.\end{aligned}$$

As a consequence of this we have that

$$x_1 \cdots x_{i-1}x_{i+2} \cdots x_k x_1^{-1} \cdots x_{i-1}^{-1}x_{i+2}^{-1} \cdots x_k^{-1} \\ [(\psi_1, \beta_1), \ldots, (\psi_{i-1}, \beta_{i-1}), (\psi_{i+2}, \beta_{i+2}), \ldots, (\psi_k, \beta_k)]$$

is equal to (ψ, β), which is not an idempotent by assumption. Thus, $T_{m,n}$ does not satisfy the equation τ_{k-2}, contrary to our choice of k. Note that under these conditions, $k \geq 3$, by observation (iv). In the case $k = 3$, the conclusion is that $T_{m,n}$ does not satisfy τ_1 which is absurd since all inverse semigroups satisfy the equation $xx^{-1} \in E$.

(2) If (ψ_i, β_i) is an element of the kernel then

(i) if $\mathbf{d}\beta_i = \{wx_1 \cdots x_j\rho\}$, then $wx_1 \cdots x_j\rho\beta_i = wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho$;

(ii) if $\mathbf{d}\beta_i = \{wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho\}$, then $wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho\beta_i = wx_1 \cdots x_j\rho$.

(i) We have assumed that $(\psi_1, \beta_1) = (\phi_n, \beta_n)$ and so $i \neq 1$. Let $\mathbf{d}\beta_{i-1} = \{v_1, v_2\}$ (by (1) $|\mathbf{d}\beta_{i-1}| = 2$), and suppose that $v_1\beta_{i-1} = u_1$ and $v_2\beta_{i-1} = u_2$. Now, $\beta_{i-1}\beta_i \neq 0$ so one of u_1 and u_2 must be $wx_1 \cdots x_j\rho$, say $u_1 = wx_1 \cdots x_j\rho$. Also, $\beta_{i-1}^{-1}\beta_i^{-1} \neq 0$ so one of v_1 and v_2 must be $wx_1 \cdots x_j\rho\beta_i$. If $v_1 = wx_1 \cdots x_j\rho\beta_i$ then $(\psi_{i-1}, \beta_{i-1})$ can be replaced by $(\hat{\psi}, \hat{\beta})$ where $\mathbf{d}\hat{\beta} = \{v_1\}$ and $v_1\hat{\beta} = u_1$ and $v_1\hat{\psi} = v_1\psi_{i-1}$. This new substitution witnesses that $T_{m,n}$ does not satisfy τ_k. Following the argument in (1) above, we obtain that $T_{m,n}$ does not satisfy τ_{k-2}, contradicting the minimality of k. Thus, $v_2 = wx_1 \cdots x_j\rho\beta_i$. By observation (v), β_{i-1} is α_p or α_p^{-1} for some $p \in \{1, \ldots, n\}$.

If $\beta_{i-1} = \alpha_p$ then $v_1\beta_{i-1} = wx_1 \cdots x_j\rho$ implies that $v_1x_p\rho = wx_1 \cdots x_j\rho$ and hence that either $p = j$ and $v_1\,\rho\,wx_1 \cdots x_{j-1}$ or $j = n, p = 1$ and $v_1\,\rho\,wx_1 \cdots x_n x_1^{-1}$. Thus, $wx_1 \cdots x_j\rho\beta_i = v_2 =$

$wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho$, by the definition of α_p or $wx_1 \cdots x_n \rho\beta_i = v_2 = w\rho$, which is what we want to prove.

If $\beta_{i-1} = \alpha_p^{-1}$ then $v_1\beta_{i-1} = wx_1 \cdots x_j\rho$ implies that $v_1 x_p^{-1}\rho = wx_1 \cdots x_j\rho$ and hence that $v_1 \, \rho \, wx_1 \cdots x_p$ and $p = j+1$. Note that in this case $j \neq n$ since if u is an initial segment of w, then $wux_p^{-1}\, \rho \, wx_1 \cdots x_n$ is impossible by Lemma 3.5. Therefore, $wx_1 \cdots x_j\rho\beta_i = v_2 = wx_1 \cdots x_n x_1^{-1} \cdots x_{p-1}^{-1}\rho wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}$, by the definition of α_p^{-1}.

(ii) As in (i) we can assume that $\mathbf{d}\beta_{i-1} = \{v_1, wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho\beta_i\}$ and that $v_1\beta_{i-1} = wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho$. Again, by observation (v), we may assume that $\beta_{i-1} = \alpha_p$ or α_p^{-1}.

If $\beta_{i-1} = \alpha_p$ then $v_1 x_p \rho = wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho$ and hence $p = j+1$ and $v_1 \, \rho \, wx_1 \cdots x_n x_1^{-1} \cdots x_{j+1}^{-1}$. Note that if $j = n$, $wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1} \, \rho \, w$ and so for any initial segment u of w, $wux_p \, \rho \, w$ is impossible, by Lemma 3.5. Therefore, by the definition of α_p, $wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho\beta_i = wx_1 \cdots x_j\rho$.

If $\beta_{i-1} = \alpha_p^{-1}$ then $v_1 x_p^{-1}\rho = wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho$ and so $p = j$ and $v_1 \, \rho \, wx_1 \cdots x_n x_1^{-1} \cdots x_n x_1^{-1} \cdots x_{j-1}^{-1}$ or $j = n$, $p = 1$, $v_1 \, \rho \, wx_1$. By the definition of α_p^{-1}, $wx_1 \cdots x_n x_1^{-1} \cdots x_j^{-1}\rho\beta_i = wx_1 \cdots x_j\rho$ and if $j = n$, $p = 1$, $w\rho\beta_i = v_2 = wx_1 \cdots x_n\rho$.

(3) At most one of the (ψ_i, β_i) belongs to the kernel of $T_{m,n}$.

Suppose that (ψ_j, β_j) and $(\psi_{j+p}, \beta_{j+p})$ are two members of the kernel of $T_{m,n}$ and they are the first two such elements appearing in the sequence $\{(\psi_1, \beta_1), \ldots, (\psi_k, \beta_k)\}$. Let $\mathbf{d}\beta_j = \{v_1\}$, $\mathbf{d}\beta_{j+p} = \{u_1\}$, $v_1\beta_j = v_2$ and $v_1\psi_j = g_1$, and $u_1\beta_{j+p} = u_2$ and $u_1\psi_{j+p} = g_2$. The claim is that if (ψ, β) is not an idempotent then neither is the following:

$$x_1 \cdots x_{j-1} x_{j+1}^{-1} \cdots x_{j+p-1}^{-1} x_{j+p+1} \cdots x_k x_1^{-1} \cdots x_{j-1}^{-1} x_{j+1} \cdots x_{j+p-1} x_{j+p+1}^{-1} \cdots x_k^{-1}$$

when (ψ_i, β_i) is substituted for x_i for all x_i appearing in the expression. Call this element (ψ', β'). If the claim is correct then $T_{m,n}$ does not satisfy τ_{k-2}, contrary to our assumptions. We first show that $\mathbf{d}\beta' \supseteq \mathbf{d}\beta$ and β' equals β on $\mathbf{d}\beta$. Now, with $\mathbf{d}\beta = \{v\}$,

$v\beta_1 \cdots \beta_{j-1} = v_1$;
$v_1 \in \mathbf{d}x_{j+1}^{-1} \cdots x_{j+p-1}^{-1}[(\psi_{j+1}, \beta_{j+1}), \ldots, (\psi_{j+p-1}, \beta_{j+p-1})]$ and
$v_1\beta_{j+1}^{-1} \cdots \beta_{j+p-1}^{-1} = u_2$;

$$u_2 \in \mathbf{d}x_{j+p+1} \cdots x_k x_1^{-1} \cdots x_{j-1}^{-1}[(\psi_{j+p+1}, \beta_{j+p+1}), \dots, (\psi_k, \beta_k), (\psi_1, \beta_1), \dots, (\psi_{j-1}, \beta_{j-1})]$$
$$u_2\beta_{j+p+1} \cdots \beta_k \beta_1^{-1} \cdots \beta_{j-1}^{-1} = v_2;$$
$$v_2 \in \mathbf{d}x_{j+1} \cdots x_{j+p-1}[(\psi_{j+1}, \beta_{j+1}), \dots, (\psi_{j+p-1}, \beta_{j+p-1})] \text{ and}$$
$$v_2\beta_{j+1} \cdots \beta_{j+p-1} = u_1;$$
$$u_1 \in \mathbf{d}x_{j+p+1}^{-1} \cdots x_k^{-1}[(\psi_{j+p+1}, \beta_{j+p+1}), \dots, (\psi_k, \beta_k)] \text{ and}$$
$$u_1\beta_{j+p+1}^{-1} \cdots \beta_k^{-1} = v\beta = v.$$

Thus, $v \in \mathbf{d}\beta'$ and $v\beta' = v\beta = v$. By calculation one sees that $v\psi$ must be equal to $v\psi' g_1 g_2 g_1^{-1} g_2^{-1}$, since C_m is abelian, and thus, $v\psi = v\psi'$. Therefore, if (ψ, β) is not an idempotent, then neither is (ψ', β'). It now follows that at most one of the (ψ_i, β_i) belongs to the kernel of $T_{m,n}$.

(4) Exactly one of the (ψ_i, β_i) is a member of the kernel of $T_{m,n}$.

First of all, observe that if none of the (ψ_i, β_i) belongs to the kernel then each (ψ_i, β_i) is (ϕ_p, α_p) or $(\phi_p, \alpha_p)^{-1}$ for some p. By the definition of the α_p, if $v\beta_1 \cdots \beta_k \in \mathbf{d}\beta_1^{-1}$ then $v\beta_1 \cdots \beta_k\beta_1^{-1} = v$. This is because if $v = wu\rho$ for some initial segment u of w then $v\beta_1 \cdots \beta_k = wu'\rho$ for some initial segment u' of w and the difference between the lengths of u and u' is not greater than k and hence strictly less than n. It follows that $v\beta_1 \cdots \beta_k$ must be $v\beta_1$. By the same reasoning we can conclude that, for all $1 \le i \le k$, $v\beta_1 \cdots \beta_k\beta_1^{-1} \cdots \beta_i^{-1} = v\beta_1 \cdots \beta_{i-1}$. Since $\mathbf{d}\beta = \{v\}$, we can replace each (ψ_i, β_i) with an element of the kernel and conclude that if (ψ, β) is not an idempotent then neither is the result of this new substitution. But this cannot be since the kernel of $T_{m,n}$ is a Brandt semigroup over an abelian group and so satisfies the equation τ_k. Therefore, exactly one of the (ψ_i, β_i) belongs to the kernel of $T_{m,n}$. This completes the first stage of the proof.

Let (ψ_j, β_j) be the only member of $\{(\psi_1, \beta_1), \dots, (\psi_k, \beta_k)\}$ which belongs to the kernel of $T_{m,n}$. Let $\mathbf{d}\beta_j = \{v_1\}$, $v_1\beta_j = v_2$ and $v_1\psi_j = g_1$. We consider the following two cases: (i) $v_1 \,\rho\, wx_1 \cdots x_p$; and (ii) $v_1 \,\rho\, wx_1 \cdots x_n x_1^{-1} \cdots x_p^{-1}$.

(i) If $v_1 \,\rho\, wx_1 \cdots x_p$ then $v_2 = wx_1 \cdots x_n x_1^{-1} \cdots x_p^{-1}\rho$ by the first stage, part (2). Since $(\psi_1, \beta_1) = (\phi_n, \alpha_n)$ and $k < n$, by the constraints on the (ψ_i, β_i) discussed thus far, for some $1 < q < j$, $(\psi_q, \beta_q) = (\phi_n, \alpha_n)^{-1}$. [That is, because for $i = 1, \dots, j-1$,

(ψ_i, β_i) is either (ϕ_h, α_h) or $(\phi_h, \alpha_h)^{-1}$, for some h, and the projection map of $T_{m,n}$ onto its second coordinate has image $S(\tau_n)$, we have that $v\beta_1\beta_2\cdots\beta_{j-1} = vx_{i_1}x_{i_2}\cdots x_{i_{j-1}}\rho$, for some $x_{i_1}, x_{i_2}, \ldots, x_{i_{j-1}} \in X \cup X^{-1}$, and that $x_{i_1}x_{i_2}\cdots x_{i_{j-1}}$ labels a path in the Schützenberger graph of $x_1\cdots x_n x_1^{-1}\cdots x_n^{-1}\rho$ from v to $wx_1\cdots x_p\rho$. Since $j-1 < k < n$, this path must traverse the edge labelled x_n^{-1} with terminal vertex v. Thus, for some $1 < q < j$, $(\psi_q, \beta_q) = (\phi_n, \alpha_n)^{-1}$.] Assume that q is the least such integer. Because $k < n$ and each of the (ψ_i, β_i) is either (ϕ_h, α_h) or $(\phi_h, \alpha_h)^{-1}$, for some h, for $1 < i \le q$, as a consequence of the definitions of the (ϕ_h, α_h), we have that $v\beta_1\cdots\beta_q = v$ and $(v\psi_1)(v\beta_1\psi_2)\cdots(v\beta_1\cdots\beta_{q-1}\psi_q) = 1$. In a likewise manner we obtain that

$$(v\beta_1\cdots\beta_k)\beta_1^{-1}\cdots\beta_q^{-1} = v\beta_1\cdots\beta_k$$

and

$$\begin{aligned}&[(v\beta_1\cdots\beta_k)\psi_1^{-1}][(v\beta_1\cdots\beta_k)\beta_1^{-1}\psi_2^{-1}]\\ &\qquad\cdots[(v\beta_1\cdots\beta_k)\beta_1^{-1}\cdots\beta_{q-1}^{-1}\psi_q^{-1}] = 1.\end{aligned}$$

As a result, $x_{q+1}\cdots x_k x_{q+1}^{-1}\cdots x_k^{-1}[(\psi_{q+1}, \beta_{q+1}), \ldots, (\psi_k, \beta_k)]$ is not an idempotent if (ψ, β) is not an idempotent, contrary to our choice of k.

(ii) If $v_1\,\rho\, wx_1\cdots x_n x_1^{-1}\cdots x_p^{-1}$ then $v_2\,\rho\, wx_1\cdots x_p$. Using a similar argument to that used in (i) above, we can assume that (ψ_1, β_1) is the only (ψ_i, β_i) equal to (ϕ_n, α_n) for $i < j$. Moreover, the same argument can be used to show that at most one of the (ψ_i, β_i) is equal to (ϕ_n, α_n) for $j < i \le k$. In this case, by the constraints on the (ψ_i, β_i) and the definitions of the (ϕ_i, α_i) and their inverses, (ψ_k, β_k) is equal to (ϕ_n, α_n). Thus, the only (ψ_i, β_i) equal to (ϕ_n, α_n) are (ψ_1, β_1) and (ψ_k, β_k). But for any inverse semigroup, $axaa^{-1}ya^{-1}$ is not an idempotent implies that xy is not an idempotent. It would then follow that $T_{m,n}$ does not satisfy the equation τ_{k-2}, a contradiction.

Since every inverse semigroup satisfies τ_1, the proof is complete if we can show that, for $n > 2$, $T_{m,n}$ satisfies τ_2. This is not difficult to verify directly: Suppose that $(\psi, \beta) \in T_{m,n}$ is such that $(\phi_n, \alpha_n)(\psi, \beta)(\phi_n, \alpha_n)^{-1}(\psi, \beta)^{-1}$ is not an idempotent. Since $\mathscr{B}^1$ does satisfy τ_2, we have that $\alpha_n\beta\alpha_n^{-1}\beta^{-1}$ is an idempotent. Thus, for all $v \in \mathbf{d}\alpha_n\beta\alpha_n^{-1}\beta^{-1} \subseteq \mathbf{d}\alpha_n$, $v\alpha_n\beta\alpha_n^{-1}\beta^{-1} = v$. Therefore, both

v and $v\alpha_n$ (which are not equal) are in the domain of β. For either v in the domain of α_n, there is no pair (ψ, β) in $T_{m,n}$ such that $\mathbf{d}\beta = \{v, v\alpha_n\}$. It follows that $T_{m,n}$ must satisfy τ_2. □

LEMMA 4.5. *Let $m, n \in \omega$, $m, n \geq 2$. $T^1_{m,n}$ satisfies the equation τ_k for $k < n$, but $T^1_{m,n}$ does not satisfy the equation τ_k for $k \geq n$.*

Proof. This is an immediate consequence of Lemmas 4.4 and 4.3. □

REMARK. The only property of the varieties $\mathscr{A}_m$ that we used in the construction of the $T_{m,n}$'s was that they each satisfied the equations τ_n, $n \in \omega$. This is also true of the variety $\mathscr{AG}$, the variety of abelian groups. Thus, in a similar way, we can construct a family of inverse semigroups $\{T^1_n\}$ such that, for each n, T^1_n satisfies the equations τ_k, for $k < n$, but T^1_n does not satisfy the equations τ_k, for $k \geq n$. Moreover, for each $n \in \omega$, $\mathscr{AG} \vee \mathscr{B}^1 \subseteq \langle T^1_n \rangle \subseteq \mathscr{AG} \circ \mathscr{B}^1$.

5. A class of varieties in the interval $[\mathscr{A}_m, \mathscr{B}^1]$. The inverse semigroups defined in the previous section can be used to define an infinite collection of varieties in the interval $[\mathscr{A}_m, \mathscr{B}^1]$. Once it is established that the interval $[\mathscr{A}_m, \mathscr{B}^1]$ is infinite, it can then be shown that other intervals which coincide with ν-classes are infinite.

NOTATION 5.1. Let $m \in \omega$. For each $n \in \omega$, define the variety $\mathscr{V}_{m,n}$ to be the variety of inverse semigroups generated by $\{T^1_{m,k} : k \geq n\}$.

PROPOSITION 5.2. *Let $m, n \in \omega$, with $m, n > 1$.*
(a) *$\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$;*
(b) *$\mathscr{V}_{m,n}$ does not satisfy τ_j for $j \geq n$;*
(c) *$\mathscr{V}_{m,n} \supset \mathscr{V}_{m,n+1}$ (the containment is proper).*

Proof. (a) By Lemma 4.5, $T^1_{m,k}$ satisfies τ_j for $j < k$. Therefore, each generator of $\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$, and hence $\mathscr{V}_{m,n}$ satisfies τ_j for $j < n$.

(b) By Lemma 4.3, $T^1_{m,j}$ does not satisfy τ_j. Since $T^1_{m,j}$, $j \geq n$, is a generator of $\mathscr{V}_{m,n}$, the equation τ_j is not satisfied by $\mathscr{V}_{m,n}$, for all $j \geq n$.

(c) $\{T^1_{m,k} : k \geq n\} \supset \{T^1_{m,k} : k \geq n+1\}$ and so $\mathscr{V}_{m,n} = \langle T^1_{m,k} : k \geq n\rangle \supset \langle T^1_{m,k} : k \geq n+1\rangle = \mathscr{V}_{m,n+1}$. That the containment is proper follows from parts (a) and (b). □

As a consequence of Proposition 5.2, the collection of varieties of inverse semigroups $\{\mathscr{V}_{m,n}: n > 1\}$ forms an infinite chain in the lattice of varieties of inverse semigroups. Furthermore, by Lemma 4.2, $\mathscr{A}_m \vee \mathscr{B}^1 \subseteq \mathscr{V}_{m,n} \subseteq \mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1)$. Since $\mathrm{Wr}(\mathscr{A}_m, \mathscr{B}^1) = \mathscr{A}_m \circ \mathscr{B}^1$, and the ν-class of $\mathscr{A}_m \vee \mathscr{B}^1$ is the interval $[\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m \circ \mathscr{B}^1]$, we have the following result.

THEOREM 5.3. *The ν-class of the variety $\mathscr{A}_m \vee \mathscr{B}^1$ possesses an infinite descending chain of varieties.*

Using Theorem 5.3, we can show that other intervals in $\mathscr{L}(\mathscr{I})$ are infinite.

LEMMA 5.4. *Let $\mathscr{V} \in [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_n \circ \mathscr{B}^1]$, where $\mathscr{A}_m$ is the variety of abelian groups of exponent m, and let $\mathscr{U} \in [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m^{\max}]$. Then*

$$\ker \rho(\mathscr{U} \vee \mathscr{V}) = \ker \rho(\mathscr{V}) \quad \textit{and} \quad \mathrm{tr}\, \rho(\mathscr{U} \vee \mathscr{V}) = \mathrm{tr}\, \rho(\mathscr{U}).$$

Proof. $\mathscr{A}_m \subseteq \mathscr{V}$ and so $\mathscr{A}_m^{\max} \subseteq \mathscr{V}^{\max}$. Therefore,

$$\mathscr{V} \subseteq \mathscr{U} \vee \mathscr{V} \subseteq \mathscr{A}_m^{\max} \vee \mathscr{V} \subseteq \mathscr{V}^{\max} \vee \mathscr{V} = \mathscr{V}^{\max}.$$

Since $\ker \rho(\mathscr{V}) = \ker \rho(\mathscr{V}^{\max})$, it follows that $\ker \rho(\mathscr{U} \vee \mathscr{V}) = \ker \rho(\mathscr{V})$.

Also,

$$\mathscr{U} \subseteq \mathscr{U} \vee \mathscr{V} \subseteq \mathscr{U} \vee \mathscr{V} \vee \mathscr{G} = \mathscr{U} \vee (\mathscr{A}_m \vee \mathscr{B}^1) \vee \mathscr{G} = \mathscr{U} \vee \mathscr{G}.$$

Since $\mathrm{tr}\, \rho(\mathscr{U}) = \mathrm{tr}\, \rho(\mathscr{U} \vee \mathscr{G})$, we have that $\mathrm{tr}\, \rho(\mathscr{U} \vee \mathscr{V}) = \mathrm{tr}\, \rho(\mathscr{U})$. □

THEOREM 5.5. *Let $\mathscr{U} \in [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m^{\max}]$. Then the interval $[\mathscr{U}, (\mathscr{A}_m \circ \mathscr{B}^1) \vee \mathscr{U}]$ contains an infinite descending chain.*

Proof. The function $\theta: [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m \circ \mathscr{B}^1] \to [\mathscr{U}, (\mathscr{A}_m \circ \mathscr{B}^1) \vee \mathscr{U}]$ defined by $\mathscr{V}\theta = \mathscr{V} \vee \mathscr{U}$ is one-to-one on $[\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m \circ \mathscr{B}^1]$ by Lemma 5.4 and the fact that all varieties $\mathscr{V}$ in this interval are such that $\mathrm{tr}\, \rho(\mathscr{V}) = \mathrm{tr}\, \rho(\mathscr{A}_m \vee \mathscr{B}^1)$. Clearly θ is order-preserving, and the result follows from Theorem 5.3. □

COROLLARY 5.6. *Let $\mathscr{U}$ be a combinatorial variety contained in $\mathscr{A}_m^{\max}$ and containing $\mathscr{B}^1$. Then the ν-class of $\mathscr{U} \vee \mathscr{A}_m$, that is, $[\mathscr{U} \vee \mathscr{A}_m, \mathscr{A}_m \circ \mathscr{U}]$, contains an infinite descending chain.*

Proof. By Theorem 5.5, since $\mathscr{U} \vee \mathscr{A}_m \in [\mathscr{A}_m \vee \mathscr{B}^1, \mathscr{A}_m^{\max}]$ and $(\mathscr{A}_m \circ \mathscr{B}^1) \vee \mathscr{U} \subseteq \mathscr{A}_m \circ \mathscr{U}$. □

REMARK. The results of this section are true for the variety $\mathscr{AG}$ as well. That is, if $\mathscr{V}_n$ denotes the variety of inverse semigroups generated by $\{T_n^1 : k \geq n\}$, the analogous results to Proposition 5.2 hold and the remaining results of this section are true when we replace $\mathscr{A}_m$ by $\mathscr{AG}$.

REFERENCES

[B] J. L. Bales, *On product varieties of inverse semigroups*, J. Austral. Math. Soc., **28** (1979), 107–119.

[BS] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, GTM 78, Springer-Verlag, New York (1981).

[C] D. F. Cowan, *A class of varieties of inverse semigroups*, to appear in J. Algebra.

[H] C. H. Houghton, *Embedding inverse semigroups in wreath products*, Glasgow Math. J., **17** (1976), 77–82.

[K1] E. I. Kleiman, *On the basis of identities of Brandt semigroups*, Semigroup Forum, **13** (1977), 209–218.

[K2] ——, *Bases of identities of varieties of inverse semigroups*, Sibirsk. Matem. Zh., **20** (1979), 760–777 (Russian).

[P] M. Petrich, *Inverse Semigroups*, Wiley, New York (1984).

[PR] M. Petrich and N. R. Reilly, *E-unitary covers and varieties of inverse semigroups*, Acta Sci. Math., Szeged, **46** (1983), 59–72.

[R1] N. R. Reilly, *Modular sublattices of the lattice of varieties of inverse semigroups*, Pacific J. Math., **89** (1980), 405–417.

[R2] ——, *Varieties of completely semisimple inverse semigroups*, J. Algebra, **65** (1980), 427–444.

[S] J. B. Stephen, *Presentations of inverse monoids*, J. Pure Appl. Algebra, **63** (1990), 81–112.

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