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ANY BLASCHKE MANIFOLD OF THE HOMOTOPY TYPE OF $\mathbf{C}P^n$ HAS THE RIGHT VOLUME

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Dedicated to Professor S. S. Chern

The aim of this paper is to prove the result stated in the title.

By a *Blaschke manifold* [**1**, p. 135], we mean a connected closed Riemannian manifold which has the property that the cut locus of each of its points, when viewed in the tangent space, is a round sphere of a constant radius. It is well known that in any Blaschke manifold, all geodesics are smoothly simply closed and have the same length. The canonical examples of a Blaschke manifold are the unit n-sphere S^n, the real, complex, quaternionic projective n-spaces $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$ and the Cayley projective plane $\mathbf{Ca}P^2$ with their standard Riemannian metric. These Blaschke manifolds will be referred to as the *canonical* Blaschke manifolds. For general informations on Blaschke manifolds, see [**1**].

The Blaschke conjecture says that *any Blaschke manifold, up to a constant factor, is isometric to a canonical Blaschke manifold*. This conjecture looks plausible, because it has been shown in [**3**, **7**] that any Blaschke manifold either is diffeomorphic to S^n or $\mathbf{R}P^n$, or is of the homotopy type of $\mathbf{C}P^n$, or is a 1-connected closed manifold having the integral cohomology ring of $\mathbf{H}P^n$ or $\mathbf{Ca}P^2$. However, so far it has been proved only for spheres and real projective spaces [**2**, **6**, **8**, **9**].

One crucial step in the proof of the Blaschke conjecture for spheres is to show that any Blaschke manifold diffeomorphic to S^n has the right volume. Hence we formulate the weak Blaschke conjecture [**10**] which says that *any Blaschke manifold has the right volume.*

Let M be a d-dimensional Blaschke manifold, UM the space of unit tangent vectors of M and CM the space of oriented closed geodesics in M. Then UM and CM are oriented connected smooth manifolds and there is a natural oriented smooth circle bundle $\pi: UM \to CM$. In [**8**], it is shown that, if e denotes the Euler class of this circle bundle, then

$$i(M) = \tfrac{1}{2}\langle e^{d-1}, [CM]\rangle$$

(i.e., one half of the value of e^{d-1} at the fundamental homology class $[CM]$ of CM) is an integer, called the *Weinstein integer* of M, and that, if ℓ denotes the length of closed geodesics in M, then

$$\operatorname{vol} M = \left(\frac{\ell}{2\pi}\right)^d i(M) \operatorname{vol} S^d.$$

Because of these results, the weak Blaschke conjecture means that any Blaschke manifold has the right Weinstein integer. Since the Weinstein integer of a Blaschke manifold depends only on the ring structure of the integral cohomology ring of its geodesic space, the weak Blaschke conjecture is essentially a topological problem rather than a geometrical problem.

The purpose of this paper is to prove the weak Blaschke conjecture for complex projective spaces. In fact, we are going to prove the following

THEOREM. *If M is a Blaschke manifold of the homotopy type of the complex projective n-space $\mathbf{C}P^n$, $n \geq 1$, then the Weinstein integer of M is equal to that of $\mathbf{C}P^n$, i.e., $\binom{2n-1}{n-1}$. In other words, if ℓ denotes the length of closed geodesics in M and S^{2n} denotes the unit $2n$-sphere, then*

$$\operatorname{vol} M = \left(\frac{\ell}{2\pi}\right)^{2n} \binom{2n-1}{n-1} \operatorname{vol} S^{2n}.$$

In particular, if closed geodesics in M are of the same length as those in $\mathbf{C}P^n$, then

$$\operatorname{vol} M = \operatorname{vol} \mathbf{C}P^n.$$

However, we are not able to prove results for complex projective spaces analogous to those for spheres as seen in [**2**, **6**]. If one succeeds in doing so, then the Blaschke conjecture for complex projective spaces follows.

Let $\mathbf{R}^k$ be the euclidean k-space of coordinates $x_1, \dots, x_k$, let D^k be the unit closed k-disk in $\mathbf{R}^k$ given by $x_1^2 + \cdots + x_k^2 \leq 1$, and let S^{k-1} be the unit $(k-1)$-sphere in $\mathbf{R}^k$ given by $x_1^2 + \cdots + x_k^2 = 1$.

For the sake of convenience, we regard $\mathbf{R}^k$ as a subspace of $\mathbf{R}^{k+1}$ by identifying every $(x_1, \dots, x_k) \in \mathbf{R}^k$ with $(x_1, \dots, x_k, 0) \in \mathbf{R}^{k+1}$. Let $\mathbf{R}^k$ be naturally oriented, let D^k have the same orientation as $\mathbf{R}^k$ and let S^{k-1} be oriented so that $\partial D^k = S^{k-1}$.

If k is even, say $k = 2n+2$, we may regard $\mathbf{R}^{2n+2}$ as the unitary $(n+1)$-space $\mathbf{C}^{n+1}$ by identifying every $(x_1, x_2, \dots, x_{2n+1}, x_{2n+2}) \in \mathbf{R}^{2n+2}$ with $(x_1 + \sqrt{-1}x_2, \dots, x_{2n+1} + \sqrt{-1}x_{2n+2}) \in \mathbf{C}^{n+1}$. Then there is a natural free orthogonal action of S^1 on S^{2n+1}. The orbit space S^{2n+1}/S^1 is the complex projective n-space which we denote by $\mathbf{C}P^n$. Since the projection of S^{2n+1} into $\mathbf{C}P^n$ is an oriented S^1 bundle, there is a natural orientation on $\mathbf{C}P^n$. Since $S^{2n+1} \subset S^{2n+3}$, $\mathbf{C}P^n \subset \mathbf{C}P^{n+1}$.

Throughout this paper, integers are used as coefficients in both homology and cohomology. For any oriented closed manifold Y, $[Y]$ denotes the fundamental homology class on Y. It is clear that, if g is the generator of $H^2(\mathbf{C}P^1) = H^2(\mathbf{C}P^n)$ with $g \cap [\mathbf{C}P^1] = 1$, then $g^n \cap [\mathbf{C}P^n] = 1$.

Hereafter, M always denotes a Blaschke manifold of the homotopy type of $\mathbf{C}P^n$, $n \geq 1$. Since the case $n = 1$ has been determined [**4**], we assume below that $n > 1$.

Let g be a generator of $H^2(M)$ and let M be so oriented that $g^n \cap [M] = 1$. Let UM be the closed smooth $(4n-1)$-manifold consisting of all unit tangent vectors of M, and let CM be the closed smooth $(4n-2)$-manifold consisting of all oriented closed geodesics in M. Then

(1) UM and CM are 1-connected and there is a natural oriented smooth S^{2n-1} bundle $\tau\colon UM \to M$ and a natural oriented smooth S^1 bundle $\pi\colon UM \to CM$ such that for any $u \in UM$, u is the unit tangent vector of πu at τu.

Since M is oriented, it follows from (1) that there is a natural orientation on UM and then a natural orientation on CM.

As a consequence of (1), we have

(2) The Gysin sequences of the oriented sphere bundles $\tau\colon UM \to M$ and $\pi\colon UM \to CM$, namely

$$\cdots \to H^{k-2n}(M) \xrightarrow{\smile e(\tau)} H^k(M) \xrightarrow{\tau^*} H^k(UM) \to H^{k-2n+1}(M) \to \cdots,$$
$$\cdots \to H^{k-2}(CM) \xrightarrow{\smile e} H^k(CM) \xrightarrow{\pi^*} H^k(UM) \to H^{k-1}(CM) \to \cdots$$

are exact, where $e(\tau)$ and e are the respective Euler classes of the oriented sphere bundles.

Since $e(\tau)\cap[M]$ is the Euler characteristic of M which is equal to $n+1$, it follows from (2) that

$$
\text{(3)}\quad
\begin{aligned}
H^k(UM) &= \begin{cases} \mathbf{Z} & \text{for } k = 2i \text{ or } 4n-1-2i, \\ & \qquad i = 0, \dots, n-1, \\ \mathbf{Z}_{n+1} & \text{for } k = 2n, \\ 0 & \text{otherwise}, \end{cases} \\
H^k(CM) &= \begin{cases} (i+1)\mathbf{Z} & \text{for } k = 2i \text{ or } 4n-2-2i, \\ & \qquad i = 0, \dots, n-1, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}
$$

where $\mathbf{Z}$ denotes the group of integers, $\mathbf{Z}_{n+1}$ denotes the group of integers modulo $n+1$ and $(i+1)\mathbf{Z}$ denotes the direct sum of $i+1$ copies of $\mathbf{Z}$. If a is an element of $H^2(CM)$ with $\pi^*a = \tau^*g$, then for any $i = 1, \dots, n$, $(\pi^*a)^i$ is a generator of $H^{2i}(UM)$ and for any $i = 1, \dots, n-1$, $\{a^i, a^{i-1}e, \dots, ae^{i-1}, e^i\}$ is a basis of $H^{2i}(CM)$. Moreover, $H^{2n}(CM)$ is generated by $\{a^n, a^{n-1}e, \dots, ae^{n-1}, e^n\}$ and hence the cohomology ring $H^*(CM)$ is generated by $\{a, e\}$.

Remark 1. The element $a \in H^2(CM)$ in (3) can be replaced by and only by $a + ke$ with $k \in \mathbf{Z}$. For our purpose, we shall pick a special a as specified in (5).

(4) The involution $\lambda\colon UM \to UM$ defined by $\lambda(u) = -u$, is orientation-preserving and it induces an involution $\lambda\colon CM \to CM$ such that $\lambda\pi = \pi\lambda$. Moreover, $\lambda\colon CM \to CM$ is orientation-reversing.

Proof. It is a consequence of the following facts. First, for any $x \in M$, $\lambda(\tau^{-1}x) = \tau^{-1}x$ and $\tau\colon \tau^{-1}x \to \tau^{-1}x$ is orientation-preserving. Second, for any $\gamma \in CM$, $\lambda(\pi^{-1}\gamma) = \pi^{-1}(-\gamma)$ and $\lambda\colon \pi^{-1}\gamma \to \pi^{-1}(-\gamma)$ is orientation-reversing.

(5) The element $a \in H^2(CM)$ in (3) can be uniquely chosen such that

$$e = a - b, \quad b = \lambda^* a.$$

Proof. Let γ be an oriented closed geodesic in M and let p and q be two points of γ which divide γ into two arcs of equal length. It is known that the union of all the closed geodesics in M which pass through p and q is a smooth 2-sphere K, and that K can be oriented so that $g\cap[K] = 1$. Let D and D' be the oriented closed 2-disks in K such that they have the same orientation as K and $\partial D = \gamma = -\partial D'$.

Since $\tau\colon UM \to M$ is an S^{2n-1} bundle with $2n-1 \geq 3$, there is a map $f\colon K \to UM$ such that for any $x \in K$, $\tau f(x) = x$, and for any $x \in \gamma$, $\pi f(x) = \gamma$. Then we have maps

$$\pi f\colon K \to CM, \quad \pi(f|D)\colon D/\partial D \to CM, \quad \pi(f|D')\colon D'/\partial D' \to CM$$

which represent three elements of $H_2(CM)$, say $\overline{e}$, $\overline{a}$, $\overline{b}$. It is not hard to see that $\overline{e}$, $\overline{a}$, $\overline{b}$ are unique and

$$\overline{e} = \overline{a} + \overline{b}.$$

Now we assert that

$$\overline{b} = \lambda_* \overline{a}.$$

Let

$$h\colon D \times [0, \pi] \to K$$

be the homotopy such that (i) for any $x \in D$, $h(x, 0) = x$, and (ii) if ξ is a geodesic segment from p to q contained in D, then for any $\theta \in [0, \pi]$, $h(\xi \times \{\theta\})$ is a geodesic segment from p to q such that ξ and $h(\xi \times \{\theta\})$ intersect at an angle θ at p and $h\colon \xi \times \{\theta\} \to h(\xi \times \{\theta\})$ is isometric. Intuitively speaking, h is the homotopy such that $h(D \times \{\theta\})$ is the closed 2-disk in K obtained by rotating D an angle θ around p and q. Therefore $h(D \times \{0\}) = D$, $h(D \times \{\pi\}) = D'$ and for any $\theta \in [0, \pi]$, $h(\partial D \times \{\theta\})$ is an oriented closed geodesic in M containing p and q such that $h(\partial D \times \{0\}) = \gamma$ and $h(\partial D \times \{\pi\}) = \lambda\gamma$. Hence we have a map

$$H'\colon \partial(D \times [0, \pi]) \to UM$$

such that (i) for any $x \in D$, $H'(x, 0) = \lambda f(x) = \lambda f h(x, 0)$ and $H'(x, \pi) = f h(x, \pi)$ and (ii) for any $(x, \theta) \in \partial D \times [0, \pi]$, $H'(x, \theta)$ is the unit tangent vector of $\lambda h(\partial D \times \{\theta\})$ at $h(x, \theta)$. Clearly for any $(x, \theta) \in \partial(D \times [0, \pi])$, $\tau H'(x, \theta) = h(x, \theta)$. Since $\pi\colon UM \to M$ is an S^{2n-1} bundle with $2n-1 \geq 3$, H' can be extended to a map

$$H\colon D \times [0, \pi] \to UM$$

such that for any $(x, \theta) \in D \times [0, \pi]$, $\tau H(x, \theta) = h(x, \theta)$. The homotopy H induces a homotopy

$$\pi H\colon D/\partial D \times [0, \pi] \to CM$$

which is a homotopy between $\lambda\pi(f|D)$ and $\pi(f|D')$. Hence $\lambda_* \overline{a} = \overline{b}$.

Let e, $a \in H^2(CM)$ be the elements as seen in (2) and (3). Then

$$e \cap \overline{e} = \pi^* e \cap \pi_*^{-1} \overline{e} = 0,$$
$$a \cap \overline{e} = \pi^* a \cap \pi_*^{-1} \overline{e} = \tau^* g \cap \tau_*^{-1}[K] = g \cap [K] = 1.$$

Moreover, we see from the Gysin homology and cohomology sequences of $\pi: UM \to CM$ that

$$e \cap \overline{a} = 1.$$

As noted in Remark 1, a can be replaced by and only by $a + ke$, where $k \in \mathbf{Z}$. Hence we can uniquely choose a such that

$$a \cap \overline{a} = 1.$$

Let

$$b = a - e.$$

It is easy to verify that

$$a \cap \overline{a} = 1, \quad a \cap \overline{b} = 0,$$
$$b \cap \overline{a} = 0, \quad b \cap \overline{b} = 1,$$

which means that $\{a, b\}$ is the basis of $H^2(CM)$ dual to the basis $\{\overline{a}, \overline{b}\}$ of $H_2(CM)$. Since $\lambda_* \overline{a} = \overline{b}$, it follows that $\lambda^* a = b$. Hence the proof is completed.

Remark 2. The choice of $a \in H^2(CM)$ given in (5) is a key step of the proof of our theorem. In fact, we shall prove later that in $H^*(CM)$,

$$a^{n+1} = 0.$$

If this is shown, then our theorem can be proved as follows. Since $a^{n+1} = 0$, $b^{n+1} = \lambda^* a^{n+1} = 0$ so that

$$\begin{aligned} e^{2n-1} &= (a-b)^{2n-1} \\ &= (-1)^{n-1} \binom{2n-1}{n-1} a^n b^{n-1} + (-1)^n \binom{2n-1}{n} a^{n-1} b^n. \end{aligned}$$

By (4), $a^{n-1} b^n = -a^n b^{n-1}$ and then

$$e^{2n-1} = (-1)^{n-1} 2 \binom{2n-1}{n-1} a^n b^{n-1}.$$

By Poincaré duality, there is an element $(a^n)^* \in H^{2n-2}(CM)$ such that $a^n (a^n)^* \cap [CM] = 1$. Since $a^{n+1} = 0$, we may let $(a^n)^* = r b^{n-1}$, where $r \in \mathbf{Z}$. Therefore

$$1 = a^n (a^n)^* \cap [CM] = (a^n b^{n-1} \cap [CM])$$

so that $a^n b^{n-1} \cap [CM] = r = \pm 1$. Hence the Weinstein integer of M is

$$i(M) = \frac{1}{2} e^{2n-1} \cap [CM] = \binom{2n-1}{n-1}.$$

REMARK 3. If M is merely a Riemannian $2n$-manifold, $n > 1$, which is of the homotopy type of $\mathbf{C}P^n$ and in which all geodesics are smoothly closed and have the same length, (1), (2), (3) and (4) remain valid. Hence the stronger assumption that M is a Blaschke manifold of the homotopy type of $\mathbf{C}P^n$, $n > 1$, is used for the first time in the proof of (5).

(6) Let

$$\tau' \colon W_1 \to M, \quad \pi' \colon W_2 \to CM$$

be the smooth D^{2n} bundle and D^2 bundle associated with $\tau \colon UM \to M$ and $\pi \colon UM \to CM$ respectively. Then W_1 and W_2 are 1-connected compact smooth $4n$-manifolds with boundary UM and there is a 1-connected closed smooth $4n$-manifold W obtained by pasting together W_1 and W_2 along their common boundary UM via the identity diffeomorphism. Moreover, there is a natural involution $\lambda \colon W \to W$ such that $\lambda | UM$ and $\lambda | CM$ coincide with those given in (4) and it has M as its fixed point set.

We let W_1 be oriented so that $\partial W_1 = UM$, and let W have the same orientation as W_1.

The inclusion map of CM into W induces an isomorphism of $H^2(W)$ onto $H^2(CM)$. If we use the isomorphism to identify $H^2(W)$ with $H^2(CM)$, then

$$H^k(W) = \begin{cases} (i+1)\mathbf{Z} & \text{for } k = 2i \text{ or } 4n - 2i,\ i = 0, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

and for any $i = 1, \ldots, n$, $\{a^i, a^{i-1}e, \ldots, ae^{i-1}, e^i\}$ is a basis of $H^{2i}(W)$ and so is $\{a^i, a^{i-1}b, \cdots, ab^{i-1}, b^i\}$, where

$$b = \lambda^* a, \quad e = a - b.$$

Moreover, the cohomology ring $H^*(W)$ is generated by $\{a, e\}$ as well as by $\{a, b\}$.

Proof. The computation of $H^k(W)$ is a consequence of (3) and the Mayer-Vietoris sequence of $(W; W_1, W_2)$ and the rest is rather clear.

REMARK 4. For the special case $M = \mathbf{C}P^n$, closed geodesics in M are of length π and there is a λ-invariant homeomorphism f of W onto $\mathbf{C}P^n \times \mathbf{C}P^n$ given as follows.

Whenever $u \in UM$, there is a totally geodesic smooth 2-sphere K_u in M which is the union of the geodesic segments from τu to $\exp(\pi/2)u$, where exp is the exponential map. W_1 is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with τu. For (r, u) in W_1, we let

$$f(r, u) = (\exp(r\pi/8)u, \exp(-r\pi/8)u).$$

W_2 is obtained from $[0, 1] \times UM$ by identifying every $(0, u) \in [0, 1] \times UM$ with πu. For any $(r, u) \in [0, 1] \times UM$, there is a unique $u_r \in UM$ such that u_r is tangent to K_u at τu and the angle from u to u_r is $(1-r)\pi/2$ using the orientation on K_u. For (r, u) in W_2, we let

$$f(r, u) = (\exp(2-r)(\pi/8)u_r, \exp(-2+r)(\pi/8)u_r).$$

Notice that if πu is the equator of K_u and $f(0, u) = (x, y)$, then x is the north pole of K_u and y is the south pole of K_u.

Let us use f to identify W with $\mathbf{C}P^n \times \mathbf{C}P^n$. Then $p: W \to M$ defined by $p(x, y) = x$ is a trivial fibre bundle of fibre $\mathbf{C}P^n$ and $p: CM \to M$ is a non-trivial fibre bundle of fibre $\mathbf{C}P^{n-1}$. Hence it is preferable to consider $H^*(W)$ rather than $H^*(CM)$.

For the general case, we are not able to construct the fibration $p: W \to M$. However, we can still prove that $H^*(W)$ is isomorphic to $H^*(\mathbf{C}P^n \times \mathbf{C}P^n)$ as for the special case $M = \mathbf{C}P^n$. This is what we are going to do from now on.

(7) The fixed point set M of $\lambda: W \to W$ is a closed smooth $2n$-manifold such that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

Moreover, there is a smooth imbedding

$$\phi: \mathbf{C}P^n \to W$$

such that

(i) $a^n \cap \phi_*[\mathbf{C}P^n] = 1$, $b \cap \phi_*[\mathbf{C}P^n] = 0$,

(ii) M and $\phi(\mathbf{C}P^n)$ intersect transversally at a single point and

(iii) $\phi(\mathbf{C}P^n)$ and $\lambda\phi(\mathbf{C}P^n)$ intersect transversally at an odd number of points.

Proof. Since the homomorphism of $H^2(W)$ into $H^2(M)$ induced by the inclusion map of M into W maps a into g, we infer that $a^n \cap [M] = g^n \cap [M] = 1$. Since M is the fixed point set of $\lambda: W \to W$ and λ is orientation-preserving, it follows that

$$b \cap [M] = \lambda^* a \cap [M] = a \cap \lambda_*[M] = a \cap [M].$$

Hence $e \cap [M] = (a - b) \cap [M] = 0$.

Let $\phi': \mathbf{C}P^1 \to CM$ be a smooth imbedding homotopic to the imbedding of $\pi(f|D)$ of $D/\partial D$ $(= \mathbf{C}P^1)$ into CM given in the proof of (5). Then

$$a \cap \phi'_*[\mathbf{C}P^1] = 1, \quad b \cap \phi'_*[\mathbf{C}P^1] = 0.$$

Since for any $k = 3, \ldots, 2n-2$, $\pi_k(CM) = \pi_k(UM) = \pi_k(M) = 0$ and since $\dim CM > 2 \dim \mathbf{C}P^{n-1}$, ϕ' can be extended to a smooth imbedding $\phi'': \mathbf{C}P^{n-1} \to CM$.

Let T be a closed tubular neighborhood of $\mathbf{C}P^{n-1}$ in $\mathbf{C}P^n$ and let $\pi' : W_2 \to CM$ be the D^2 bundle we had earlier. Then ϕ'' can be extended to a smooth imbedding $\phi''': T \to W_2$ such that

$$\phi'''(T) = \pi'^{-1}\phi''(\mathbf{C}P^{n-1}).$$

Clearly $\phi'''(\partial T)$ is a smooth $(2n-1)$-sphere in UM at which $\phi'''(T)$ intersects UM transversally. Since $\pi_{2n-1}(W_1) = \pi_{2n-1}(M) = 0$ and $\dim W = 2 \dim \mathbf{C}P^n > 4$, we infer that ϕ''' can be extended to a smooth imbedding $\phi: \mathbf{C}P^n \to W$ such that $\phi(\mathbf{C}P^n - T) \subset W_1$. From the construction of ϕ, we see that

$$a \cap \phi_*[\mathbf{C}P^1] = 1, \quad b \cap \phi_*[\mathbf{C}P^1] = 0.$$

Therefore for any $i = 2, \ldots, n$,

$$a \cap \phi_*[\mathbf{C}P^i] = \phi_*[\mathbf{C}P^{i-1}], \quad b \cap \phi_*[\mathbf{C}P^i] = 0.$$

Hence

$$a^n \cap \phi_*[\mathbf{C}P^n] = 1, \quad b \cap \phi_*[\mathbf{C}P^n] = 0.$$

Let $p : \widetilde{W} \to W$ be the smooth S^1 bundle of Euler class e. From its Gysin sequence, we see that

$$H^k(\widetilde{W}) = \begin{cases} \mathbf{Z} & \text{for } k = 2i \text{ or } 4n+1-2i, \; i = 0, \ldots, n; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any $i = 0, \ldots, n$, $(p^*a)^i$ is a generator of $H^{2i}(\widetilde{W})$. Since $e \cap [M] = 0$, $p^{-1}M$ is diffeomorphic to $S^1 \times M$ so that there is an oriented closed smooth submanifold M' of $p^{-1}M$ such that

$p: M' \to M$ is an orientation-preserving diffeomorphism. Now

$$(p^*a)^n \cap [M'] = a^n \cap p_*[M'] = a^n \cap [M] = 1.$$

Hence $[M']$ is a generator of $H_{2n}(\widetilde{W})$.

Since $e^n \cap \phi_*[\mathbf{C}P^n] = a^n \cap \phi_*[\mathbf{C}P^n] = 1$, $p^{-1}\phi(\mathbf{C}P^n)$ is a $(2n+1)$-sphere. From the Gysin sequence of $p: \widetilde{W} \to W$, we see that $[p^{-1}\phi(\mathbf{C}P^n)]$ is a generator of $H_{2n+1}(\widetilde{W})$. Therefore, by Poincaré duality, $[M'] \cap [p^{-1}\phi(\mathbf{C}P^n)] = \pm 1$. Hence $[M] \cap \phi_*[\mathbf{C}P^n] = \pm 1$. That $[M] \cap \phi_*[\mathbf{C}P^n] = 1$ is a consequence of the choice of the orientation of W. In fact, ϕ may be so chosen that the closed $2n$-disk $\phi(\mathbf{C}P^n) \cap W_1$ intersects M transversally at exactly one point.

Altering ϕ by a homotopy if it is necessary, we may assume that $\phi(\mathbf{C}P^n)$ and $\lambda\phi(\mathbf{C}P^n)$ intersect transversally at finitely many points. Besides the point $M \cap \phi(\mathbf{C}P^n)$, other points in $\phi(\mathbf{C}P^n) \cap \lambda\phi(\mathbf{C}P^n)$ are in pairs. Hence $\phi_*[\mathbf{C}P^n] \cap (\lambda\phi)_*[\mathbf{C}P^n] = \text{odd integer}$.

Let N be an integer $> 4n$, let

$$\lambda: \mathbf{C}P^N \times \mathbf{C}P^N \to \mathbf{C}P^N \times \mathbf{C}P^N$$

be the involution defined by $\lambda(x, y) = (y, x)$ and let $\{a, b\}$ be the basis of $H^2(\mathbf{C}P^N \times \mathbf{C}P^N)$ such that

$$a \cap [\mathbf{C}P^N \times \mathbf{C}P^N] = [\mathbf{C}P^{N-1} \times \mathbf{C}P^N],$$
$$b \cap [\mathbf{C}P^N \times \mathbf{C}P^N] = [\mathbf{C}P^N \times \mathbf{C}P^{N-1}].$$

(8) There is a smooth imbedding

$$f: W \to \mathbf{C}P^N \times \mathbf{C}P^N$$

such that $f\lambda = \lambda f$, $f^*a = a$ and $f^*b = b$. Moreover, there is a natural isomorphism

$$H^{2n}(\mathbf{C}P^N \times \mathbf{C}P^N) \cong H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

which maps every $x \in H^{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$ into $x \cap f_*[W] \in H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$.

Proof. There is a smooth map $f': W \to \mathbf{C}P^N$ such that f'^* maps the generator g of $H^2(\mathbf{C}P^N)$ into a. Since $\dim \mathbf{C}P^N > 2\dim W$, f' can be approximated by a smooth imbedding homotopic to f'. (See [**5**].) Therefore we may assume that f' is a smooth imbedding. Hence $f: W \to \mathbf{C}P^N \times \mathbf{C}P^n$ defined by $f(x) = (f'x, \lambda f'x)$ is as desired.

By Poincaré duality, there is an isomorphism $H^{2n}(W) \cong H_{2n}(W)$ which maps every $x \in H^{2n}(W)$ into $x \cap [W] \in H_{2n}(W)$. Since

$$f^*\colon H^{2n}(\mathbf{C}P^N \times \mathbf{C}P^N) \to H^{2n}(W)$$

and

$$f_*\colon H_{2n}(W) \to H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

are isomorphisms, the second part of (8) follows.

Now we consider an oriented λ-invariant connected closed smooth $4n$-submanifold X of $\mathbf{C}P^N \times \mathbf{C}P^N$, $n \geq 1$, which has the following properties of W (or rather of fW).

(a) Let $f\colon X \to \mathbf{C}P^N \times \mathbf{C}P^N$ be the inclusion map. Then for any $i = 0, \dots, n$,

$$f_*\colon H_{2i}(X) \to H_{2i}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

is surjective. Moreover, there is an isomorphism

$$H^{2n}(\mathbf{C}P^N \times \mathbf{C}P^N) \cong H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

which maps every $x \in H^{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$ into $x \cap [X] \in H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$.

(b) The fixed point set M of $\lambda\colon X \to X$ is a closed smooth $2n$-manifold which can be so oriented that

$$a^n \cap [M] = 1, \quad e \cap [M] = 0.$$

(c) There is a smooth imbedding $\phi\colon \mathbf{C}P^n \to X$ such that

$$a^n \cap \phi_*[\mathbf{C}P^n] = 1, \quad b \cap \phi_*[\mathbf{C}P^n] = 0.$$

(d) $[M] \cap \phi_*[\mathbf{C}P^n] = 1$,

$$\phi_*[\mathbf{C}P^n] \cap (\lambda\phi)_*[\mathbf{C}P^n] = \text{odd integer}.$$

For any $k = 0, \dots, 2n$, we let $P_k(a, b)$ be the group of homogeneous polynomials in variables a and b of degree k with integral coefficients. Then for any $i = 0, \dots, 2n$,

$$H^{2i}(\mathbf{C}P^N \times \mathbf{C}P^N) = P_i(a, b).$$

As a consequence of (a), (b), (c), (d) above, we have

(9) There are unique $p(a, b)$, $q(a, b) \in P_n(a, b)$ such that

$$p(a, b) \cap [X] = [M], \quad q(a, b) \cap [X] = \phi_*[\mathbf{C}P^n].$$

Moreover,

$$a^n p(a, b) \cap [X] = 1, \quad e p(a, b) \cap [X] = 0;$$
$$a^n q(a, b) \cap [X] = 1, \quad b q(a, b) \cap [X] = 0.$$

Furthermore,

$$p(a, b)q(a, b) \cap [X] = 1,$$
$$q(a, b)q(b, a) \cap [X] = \text{odd integers}.$$

(10) (i) For any $i = 0, \ldots, n$, $a^i b^{n-i} p(a, b) \cap [X] = 1$.
(ii) $p(a, b) = p(b, a)$.
(iii) $p(1, 0) = p(0, 1) = q(1, 1) = 1$.
(iv) Let K be the subgroup of $H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N) = P_{n+1}(a, b)$ consisting of the elements x with $x \cap [X] = 0$ and let L be the subgroup of $P_{n+1}(a, b)$ generated by $\{a^n b, a^{n-1}b^2, \ldots, a^2 b^{n-1}, ab^n\}$. Then

$$P_{n+1}(a, b) = K \oplus L,$$

$q(0, 1) = \pm 1$ and $\{aq(b, a), bq(a, b)\}$ is a basis of K.
(v) $aq(b, a) - bq(a, b) = q(0, 1)ep(a, b)$.

Proof.
(i) Since, by (9), $(a - b)p(a, b) \cap [X] = 0$, we have

$$ap(a, b) \cap [X] = bp(a, b) \cap [X].$$

Hence for any $i = 0, \ldots, n$,

$$a^i b^{n-i} p(a, b) \cap [X] = a^n p(a, b) \cap [X]$$

which is equal to 1 by (9).
(ii) Since $\lambda^* a = b$, $\lambda^* b = a$ and $\lambda_*[X] = [X]$, it follows from (i) and (9) that

$$a^n p(b, a) \cap [X] = b^n p(a, b) \cap [X] = 1,$$
$$ep(b, a) \cap [X] = -ep(a, b) \cap [X] = 0.$$

Hence, by (9), $p(b, a) = p(a, b)$.
(iii) By (9) and (ii),

$$1 = p(a, b)q(a, b) \cap [X] = p(1, 0)a^n q(a, b) \cap [X]$$
$$= p(1, 0) = p(0, 1).$$

Let $q(a, b) = \sum_{i=0}^n \beta_i a^i b^{n-i}$. Then, by (9) and (i),

$$1 = q(a, b)p(a, b) \cap [X] = \sum_{i=0}^n \beta_i a^i b^{n-i} p(a, b) \cap [X]$$
$$= \sum_{i=0}^n \beta_i = q(1, 1).$$

(iv) By (a),

$$a^n \cap [X],\ a^{n-1}b \cap [X],\ \dots,\ ab^{n-1} \cap [X],\ b^n \cap [X]$$

are linearly independent elements of $H_{2n}(\mathbf{C}P^N \times \mathbf{C}P^N)$. Therefore

$$a^{n-1} \cap [X],\ a^{n-2}b \cap [X],\ \dots,\ ab^{n-2} \cap [X],\ b^{n-1} \cap [X]$$

are linearly independent elements of $H_{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ and hence K does not have more than two linearly independent elements.

By (9),

$$\begin{aligned} q(0,1) &= q(0,1)a^n q(a,b) \cap [X] \\ &= q(a,b)q(b,a) \cap [X] = \text{odd integers.} \end{aligned}$$

We infer that in $P_{n+1}(a, b)$,

$$aq(b,a),\ a^n b,\ a^{n-1}b^2,\ \dots,\ a^2 b^{n-1},\ ab^n,\ bq(a,b)$$

are linearly independent. Therefore $\{aq(b,a), bq(a,b)\}$ generates a subgroup of K of finite index.

Let $\{r(a,b), s(a,b)\}$ be a basis of K. Then

$$\{r(a,b),\ a^n b,\ a^{n-1}b^2,\ \dots,\ a^2 b^{n-1},\ ab^n,\ s(a,b)\}$$

is a basis of $P_{n+1}(a, b)$ so that we may assume that

$$r(1,0) = 1, \quad r(0,1) = 0, \quad s(1,0) = 0, \quad s(0,1) = 1.$$

Therefore there are $r_1(a,b),\ s_1(a,b) \in P_n(a,b)$ such that

$$r(a,b) = ar_1(a,b), \quad s(a,b) = bs_1(a,b).$$

From this result, it follows that

$$aq(b,a) = q(0,1)r(a,b) = q(0,1)ar_1(a,b)$$

so that

$$q(b,a) = q(0,1)r_1(a,b).$$

Since, by (iii), $q(1,1) = 1$, we infer that

$$q(0,1) = \pm 1.$$

Hence

$$aq(b,a) = \pm r(a,b), \qquad bq(a,b) = \pm s(a,b)$$

and consequently $\{aq(b,a), bq(a,b)\}$ is a basis of K.

(v) By (9), $ep(a, b)$ is in K and by (iv), $\{aq(b, a), bq(a, b)\}$ is a basis of K. Then for some integers s and t,

$$ep(a, b) = saq(b, a) + tbq(a, b).$$

By setting $a = 1$ and $b = 0$, we obtain $sq(0, 1) = 1$ by (iii). Therefore $s = q(0, 1)$. Similarly, $t = -q(0, 1)$. Hence our assertion follows.

$$p(a, b) = \sum_{i=0}^{n} a^{n-i} b^i \quad \text{and} \quad q(a, b) = b^n. \tag{11}$$

Proof. Assume first that $n = 1$. By **[4]**, we may set

$$M = \mathbf{C}P^1.$$

As seen in Remark 4, which is valid for $n = 1$, we may let W be $\mathbf{C}P^1 \times \mathbf{C}P^1$ and let M be the diagonal set in $\mathbf{C}P^1 \times \mathbf{C}P^1$. As we have done earlier, we let $\{a, b\}$ be the basis of $H^2(\mathbf{C}P^1 \times \mathbf{C}P^1)$ such that

$$a \cap [\mathbf{C}P^1 \times \mathbf{C}P^1] = [\mathbf{C}P^0 \times \mathbf{C}P^1],$$
$$b \cap [\mathbf{C}P^1 \times \mathbf{C}P^1] = [\mathbf{C}P^1 \times \mathbf{C}P^0],$$

and let $p(a, b)$ and $q(a, b)$ be the elements of $H^2(\mathbf{C}P^1 \times \mathbf{C}P^1)$ such that

$$p(a, b) \cap [W] = [M], \quad q(a, b) \cap [W] = [\mathbf{C}P^1 \times \mathbf{C}P^0].$$

It is not hard to see that

$$p(a, b) = a + b, \quad q(a, b) = b.$$

Hence (11) holds for $n = 1$.

Now we proceed by induction on n and assume that our assertion holds when n is replaced by $n - 1$, $n > 1$. Since

$$X \subset \mathbf{C}P^N \times \mathbf{C}P^N \subset \mathbf{C}P^{N+1} \times \mathbf{C}P^{N+1},$$

we can use a λ-equivariant isotopy to alter X so that the following hold.

(1) $\phi(\mathbf{C}P^n)$ is contained in $\mathbf{C}P^{N+1} \times \mathbf{C}P^N$ and intersects $\mathbf{C}P^N \times \mathbf{C}P^{N+1}$ transversally at $\phi(\mathbf{C}P^{n-1})$.

(2) M and X are transversal to $\mathbf{C}P^N \times \mathbf{C}P^{N+1}$.

(3) $X' = X \cap (\mathbf{C}P^N \times \mathbf{C}P^N)$ is a connected closed smooth $(4n-4)$-manifold invariant under λ.

Let X' be oriented so that

$$[X'] = ab \cap [X].$$

We claim that X' satisfies (a), (b), (c), (d) with $n-1$ in place of n.

For any $i = 0, \dots, n-2$,

$$\begin{aligned} f_*H_{2i}(X') &= ab \cap f_*H_{2i+4}(X) \\ &= ab \cap H_{2i+4}(\mathbf{C}P^N \times \mathbf{C}P^N) = H_{2i}(\mathbf{C}P^N \times \mathbf{C}P^N). \end{aligned}$$

By (10), (iv),

$$ab \cup f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) = f^*H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N).$$

Then

$$ab \cap f_*H_{2n+2}(X) = f_*H_{2n-2}(X) = H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$$

and hence

$$f_*H_{2n-2}(X') = f_*(ab \cap H_{2n+2}(X)) = H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N).$$

Since

$$\begin{aligned} &f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap [X'] \\ &\quad = f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap (ab \cap [X]) \\ &\quad = (ab \cup f^*H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)) \cap [X] \\ &\quad = f^*H^{2n+2}(\mathbf{C}P^N \times \mathbf{C}P^N) \cap [X] \\ &\quad \cong f_*H_{2n-2}(X) = f_*H_{2n-2}(X'), \end{aligned}$$

it follows that there is an isomorphism of $H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ onto $H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ which maps every $x \in H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ into $x \cap f_*[X'] \in H_{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$. The rest is rather obvious.

By the induction hypothesis, $q'(a, b) = b^{n-1}$ is the unique element of $H^{2n-2}(\mathbf{C}P^N \times \mathbf{C}P^N)$ such that

$$q'(a, b) \cap [X'] = \phi_*[\mathbf{C}P^{n-1}]$$

so that

$$ab^n \cap [X] = b^{n-1} \cap (ab \cap [X]) = \phi_*[\mathbf{C}P^{n-1}].$$

Then

$$a(b^n - q(a, b)) \cap [X] = \phi_*[\mathbf{C}P^{n-1}] - a \cap \phi_*[\mathbf{C}P^n] = 0.$$

Therefore, by (10), (iv),

$$b^n - q(a, b) = kq(b, a)$$

for some integer k. Since, by (10), (iii), $q(1, 1) = 1$, it follows that

$k = 0$ and hence

$$q(a, b) = b^n.$$

From this result and (10), (v), it is clear that

$$p(a, b) = \sum_{i=0}^{n} a^{n-i} b^i$$

follows.

Proof of our theorem. In $H^*(W)$,

$$a^{n+1} = aq(b, a) = 0$$

and then in $H^*(CM)$,

$$a^{n+1} = 0.$$

Hence our assertion follows as seen in Remark 2.

References

[1] A. Besse, *Manifolds All of Whose Geodesics are Closed*, Ergebnisse der Mathematik, No. 93, Springer-Verlag, Berlin, 1978.
[2] A. Berger, *Blaschke's Conjecture for Spheres*, Appendix D in the book above by A. Besse.
[3] R. Bott, *On manifolds all of whose geodesics are closed*, Ann. Math., **60** (1954), 375–382.
[4] L. W. Green, *Aug Wiedersehensflachen*, Ann. math., **78** (1963), 289–299.
[5] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv., **36** (1961), 47–81.
[6] J. L. Kazdan, *An Inequality Arising in Geometry*, Appendix E in the book above by A. Besse.
[7] H. Samelson, *On manifolds with many closed geodesics*, Portugal. Math., **22** (1963), 193–196.
[8] A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geom., **9** (1974), 513–517.
[9] C. T. Yang, *Odd-dimensional Wiedersehen manifolds are spheres*, J. Differential Geom., **15** (1980), 91–96.
[10] ——, *On the Blaschke conjecture*, Yau, Seminar on Differential Geometry, pp. 159–171.

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