PACIFIC JOURNAL OF MATHEMATICS
Vol. 151, No. 2, 1991

A NOTE ON HOMOTOPY COMPLEX SURFACES WITH NEGATIVE TANGENT BUNDLES

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We propose some problems concerning a weak rigidity phenomenon on compact complex manifolds with negative tangent bundles. Some observations have been made in the two dimensional case as an easy consequence of classification theory, and Yau's theorem on the rigidity of $\mathbb{C}P_2$. We point out that among the class of complex surfaces of general type with $c_1^2 - c_2 > 0$ the cotangent dimension is a homotopy invariant possibly except in the case of $S^2 \times S^2$.

1. Results and problems. A compact complex manifold M with a negative tangent bundle $T(M)$ in the sense of Grauert if $T(M)$ is a strongly pseudoconvex manifold with the zero section as its only exceptional variety. This is the same thing as to say its cotangent bundle $T^*(M)$ is ample. If X is a compact Kähler manifold with negative bisectional curvature, then $T(X)$ is negative by the formula of bicurvature [**6**]. This class of manifolds in general admit nontrivial local moduli [**13**, **14**] unlike those Kähler manifolds with Nakano-negative curvatures which are locally rigid in the classical sense of Kodaira-Spencer [**4**, **10**]. The aim of this paper is to propose a problem on a weak rigidity phenomenon of complex structures on compact complex manifolds with negative tangent bundles and to point out some observations in the complex two dimensional case.

Problem. Let M be a compact Kähler manifold with negative bisectional curvature. Does it admit another complex structure, with the fixed topological (homotopic, or homeomorphic, or diffeomorphic) type, in which there exists a Kähler metric with positive definite Ricci tensor? A closely related problem is to ask whether the complex structures of negative tangent bundle and rational type are exclusive to each other.

An analogous open problem in differential topology and algebraic geometry is to ask whether general type and rational type complex structures can coexist on a compact four dimensional manifold with a fixed differentiable structure. In [**1**] R. Barlow proved that on a homeomorphic $\mathbb{C}P_2$ with eight points blown up, there exist both rational type and general type complex structures. Barlow's surface cannot carry a negative tangent bundle as a consequence of a simple computation of Chern numbers. On the other hand, a theorem of S. T. Yau determined that the homotopy $\mathbb{C}P_2$ admits only the standard complex structure [17]. Our problem is probably more natural from the angle of a curvature formulation of Schwarz's lemma [18] than the one stated above. We do not have any analytic or geometric technique to settle down our major problem at this point. Nevertheless, there is some hope that one can resolve the problem in the case of complex surfaces through an analysis of cotangent sheaf. We make several observations along this line based on a trick due to F. Bogomolov [3].

THEOREM 1. *Let X be a compact complex two-fold with a negative tangent bundle and M be a minimal compact complex two-fold which is not diffeomorphic to the standard $S^2 \times S^2$. Suppose X and M satisfy one of the following conditions:*

(a) *X and M have finite fundamental groups and they are homotopic to each other.*

(b) *X and M are homeomorphic.*

Then the cotangent dimension of M is equal to two.

The definition of cotangent dimension can be found in §2. It can be proved that a complex surface of cotangent dimension two must be of general type.

The following assertion follows from standard topological fact.

THEOREM 2. *Let X be a compact complex two-fold with a negative tangent bundle and M be a minimal non-spin compact complex two-fold. Suppose they satisfy one of the following two conditions:*

(a) *X and M are homotopic and their fundamental groups are finite.*

(b) *X and M are homeomorphic.*

Then the cotangent dimension of M is equal to two.

We recall here that a manifold has a spin structure iff its second Steifel-Whitney class $w_2 = 0$. To end our introduction, we would like to mention some open problems which should be clarified on our road.

Problem 1. Let X be a compact complex two-fold with negative tangent bundle. Suppose M is a compact complex two-fold homotopic (homeomorphic, diffeomorphic) to X. Is $T(M)$ negative?

Problem 2. Does $S^2 \times S^2$ admit a complex structure of general type (or even a complex structure with negative tangent bundle) upon the fixed underlying topological (diffeomorphic, homeomorphic, or homotopic) type?

Problem 3. Let M be the compact four dimensional differentiable manifold obtained by $\mathbb{C}P_2$ blowing up one or two points (i.e., $\mathbb{C}P_2\#\overline{\mathbb{C}P_2}$ or $\mathbb{C}P_2\#2\overline{\mathbb{C}P_2}$, where $\overline{\mathbb{C}P_2} = \mathbb{C}P_2$ with opposite orientation, "#" means connected sum). Does M admit a complex structure of general type (or even a complex structure with negative tangent bundle)?

Problem 4. Let X be a compact complex manifold covered by a bounded domain in $\mathbb{C}^2$. Is the cotangent dimension of X equal to two? Suppose M is a compact complex two-fold homotopic (or homeomorphic, or diffeomorphic) to X; is the cotangent dimension equal to two?

Problem 5. Is the property that the cotangent dimension is equal to two for complex surfaces a homotopic (homeomorphic, diffeomorphic) invariant?

Problem 6. Explore all our problems to higher dimensional situations.

2. Background materials and related results. Let V be a compact complex manifold and D be a divisor, the D-dimension according to Iitaka **[11]** is the number

$$K(D, V) = \begin{cases} \max \dim \Phi_{mD}(V) & \text{if } N(D, V) \neq \phi, \\ -\infty & \text{if } N(D, V) = \phi, \end{cases}$$

where $N(D, V) = \{m > 0 \mid \dim_{\mathbb{C}} H^0(V, mD) \geq 1\}$ and Φ_{mD} is the canonical meromorphic mapping defined by a basis $\{\Phi_0, \Phi_1, \dots, \Phi_n\}$ of $H^0(V, mD)$, namely

$$\begin{aligned} \Phi_{mD} : V &\to \mathbb{C}P^n \\ z &\to (\Phi_0(z) : \Phi_1(z) : \cdots : \Phi_n(z)) \end{aligned}$$

(in homogeneous coordinates).

The canonical dimension (or Kodaira dimension) $K(V)$ of V is defined to be $K(D, V)$ when D is chosen as the canonical divisor K_V of V. This is an invariant intrinsically attached to V. A compact complex manifold V is of general type iff $K(V) = \dim_{\mathbb{C}} V$. Another equivalent characterization is that

$$\overline{\lim_{k\to\infty}} \frac{P_k(V)}{k^m} > 0,$$

where $m = \dim_{\mathbb{C}} V$, $P_k(V) = \dim_{\mathbb{C}} H^0(V, kK_V)$.

Let E be a holomorphic vector bundle of rank r over a compact complex manifold X, $\dim_{\mathbb{C}} X = m$. We denote by $S^n(E)$ the nth symmetric power of E. Let $P(E)$ be the projective bundle of the hyperplane through the origin in the fibers of E. We denote by $\pi\colon P(E) \to X$ the natural projection and $O_{P(E)}(1)$ the tautological line bundle on $P(E)$. There are canonical isomorphisms

$$\pi_*\colon O_{P(E)}(n) \to S^n(E)$$

and

$$H^0(P(E), O_{P(E)}(n)) \cong H^0(X, S^n E), \quad \text{for } n \geq 0.$$

E is ample iff $O_{P(E)}(1)$ is an ample line bundle over $P(E)$ in the usual sense of algebraic geometry. From the viewpoint of pseudoconvexity in several complex variables, it is equivalent to E^* being (Grauert) negative (i.e., E^* is a strongly pseudoconvex manifold with the zero section as its only exceptional variety). If E is ample over X, then a standard argument in algebraic geometry and several complex variables implies that $H^0(X, S^n E)$ has enough sections to yield an embedding of X into a Grassmannian, for sufficiently large n.

Let us now assume E is a rank two negative holomorphic vector bundle over a complex surface V. Following from a result of Kleiman [7], the Chern numbers satisfy $C_1^2(E) > 0$, $C_2(E) > 0$ and $C_1^2(E) - C_2(E) > 0$, where $C_1(E)$ and $C_2(E)$ are the first and second Chern classes of E. In particular, if the tangent bundle of V is negative, the Chern numbers of V will satisfy $C_1^2 > 0$, $C_2 > 0$, and $C_1^2 - C_2 > 0$; here C_1 and C_2 are the Chern classes of V. The following lemma, due to Bogomolov [3], will play an important role in our discussions. For the convenience of the readers and for the completion of this paper, we include the proof here.

LEMMA 2.1 ([3]). *Let E be an r-dimensional holomorphic vector bundle over a compact complex two-fold V s.t. $C_1^2(E) - C_2(E) > 0$.*

Then either $\dim_{\mathbb{C}} H^0(V, S^nE)$ *or* $\dim_{\mathbb{C}} H^0(V, S^nE^*)$ *grows faster than* cn^{r+1}, *for some constant* $c > 0$.

In particular, if E *is the tangent bundle* $T(V)$, *then either* $\dim_{\mathbb{C}} H^0(V, S^nT)$ *or* $\dim_{\mathbb{C}} H^0(V, S^nT^*)$ *grows faster than* cn^3.

Proof. By the Hirzebruch-Riemann-Roch theorem and a computation due to Bogomolov [3], one has the following identity:

$$\begin{aligned}\chi(V, S^nE) &= \dim_{\mathbb{C}} H^0(V, S^nE) - \dim_{\mathbb{C}} H^1(V, S^nE) \\ &\quad + \dim_{\mathbb{C}} H^2(V, S^nE) \\ &= -p_{r+1}(n)C_2 + \frac{r+2n-1}{2(r+1)}p_r(n)C_1^2 \\ &\quad + p_{r-1}(n+1)\frac{K^2+\chi}{12} - \frac{1}{2}p_r(n)C_1K,\end{aligned}$$

where $p_{n_1}(n_2) = n_2(n_2+1)\cdots(n_2+n_1-1)/n_1!$, $K =$ canonical class of V, $\chi =$ Euler class of V, $C_1 =$ first Chern class of E, and $C_2 =$ second Chern class of E.

We observe that as n tends to infinity, the terms $p_{r+1}(n)$ and $\frac{r+2n-1}{2(r+1)}p_r(n)$ will dominate. Both of these two terms are polynomials in n with degree $r+1$. As n grows to infinity, we can write

$$\begin{aligned}&\dim_{\mathbb{C}} H^0(V, S^nE) - \dim_{\mathbb{C}} H^1(V, S^nE) + \dim_{\mathbb{C}} H^2(V, S^nE) \\ &\quad = \frac{2n+r-1}{2(r+1)}p_r(n)C_1^2 - p_{r+1}(n)C_2 \\ &\qquad + \{\text{terms of polynomials in } n \text{ of degree } < r+1\}.\end{aligned}$$

We further observe that

$$\lim_{n\to\infty} \frac{p_{r+1}(n)}{\frac{2n+r-1}{2(r+1)}p_r(n)} = \lim_{n\to\infty} \frac{2(n+r)}{(2n+r-1)} = 1.$$

With all the above information we can write

$$\chi(S^nE) = p_{r+1}(n)(C_1^2 - C_2) + o(n^{r+1}),$$

where $\lim_{r\to\infty}(o(n^{r+1})/n^{r+1}) = 0$. Thus we have the inequality

$$\dim_{\mathbb{C}} H^0(V, S^nE) + \dim_{\mathbb{C}} H^2(V, S^nE) > a \cdot n^{r+1} + o(n^{r+1}),$$

where $a = (C_1^2 - C_2)(1-\varepsilon)/(r+1)!$, $C_1^2 - C_2 > 0$, $0 < \varepsilon < 1$. By Serre duality $H^2(V, S^nE) \cong H^0(V, K \otimes S^nE)$, we have

$$\dim_{\mathbb{C}} H^0(V, S^nE) + \dim_{\mathbb{C}} H^0(V, K \otimes S^nE^*) > a \cdot n^{r+1} + o(n^{r+1}).$$

We observe that the minimal model $\widetilde{V}$ of V must also satisfy the inequality $C_1^2 - C_2 > 0$. By [2, Table 10, pp. 188], we see that $\widetilde{V}$ has to be algebraic. Hence V is also algebraic. We can therefore choose a divisor D on V so that both D and $D - K$ are effective. This gives an exact sequence which arises from the restriction of $S^n E^* \otimes K$ to D,

$$0 \to S^n E^* \otimes K \otimes (-D) \to S^n E^* \otimes K \to S^n E^* \otimes K|_D \to 0.$$

This associates to an exact sequence of cohomologies,

$$\begin{aligned} 0 &\to H^0(V, S^n E^* \otimes K \otimes (-D)) \to H^0(V, S^n E^* \otimes K) \\ &\to (D, S^n E^* \otimes K|_D) \to H^1(V, S^n E^* \otimes K \otimes (-D)) \to \cdots. \end{aligned}$$

This implies that

$$\begin{aligned} \dim_{\mathbb{C}} H^0(V, S^n E^* \otimes K) - \dim_{\mathbb{C}} H^0(V, S^n E^* \otimes K \otimes (-D)) \\ \leq \dim_{\mathbb{C}} H^0(D, S^n E^* \otimes K|_D). \end{aligned}$$

By the Riemann-Roch theorem over curves, one obtains

$$\dim_{\mathbb{C}} H^0(D, S^n E^* \otimes K|_D) \leq \alpha \cdot n^r + \beta,$$

where α and β are positive constants. Nevertheless, if $D - K$ is effective, one has the inequality

$$\dim_{\mathbb{C}} H^0(V, S^n E^* \otimes K \otimes (-D)) \leq \dim_{\mathbb{C}} H^0(V, S^n E^*).$$

We observe that

$$\dim_{\mathbb{C}} H^0(V, S^n E^* \otimes K) - \dim_{\mathbb{C}} H^0(V, S^n E^*) \leq \alpha \cdot n^r + \beta.$$

Combining the arguments above for sufficiently large n and for $c = C_1^2 - C_2(1-\varepsilon)/(r+1)!$, one finally concludes with the following desired inequality

$$\dim_{\mathbb{C}} H^0(V, S^n E) + \dim_{\mathbb{C}} H^0(V, S^n E^*) > c \cdot n^{r+1}.$$

Definition. Let E be a rank r holomorphic vector bundle over a compact complex manifold X. The E-dimension of E over X is the number

$$e(E, X) = \begin{cases} d - r + 1 & \text{if } d \neq -\infty, \\ -r & \text{if } d = -\infty, \end{cases}$$

where $d =$ the D-dimension of $O_{P(E)}(1)$ over $P(E)$.

Definition. The cotangent dimension of X, namely $\mathrm{Cod}(X)$, is the number $e(E, X)$ when E is the cotangent bundle of X.

We state some remarks concerning cotangent dimension and refer to **[9]** for their proofs.

REMARK 2.1. It is immediate from the definition that $\mathrm{Cod}(X)$ takes one of the numbers from $\{-\dim_{\mathbb{C}} X, \dots, 0, \dots, \dim_{\mathbb{C}} X\}$.

REMARK 2.2 **[9]**. The cotangent dimension is a bimeromorphic invariant.

REMARK 2.3 **[9]**. It follows basically from the classification of complex surfaces that for an algebraic surface X, the inequality $\mathrm{Cod}(X) \leq K(X)$ holds if $K(X) \geq 0$. In particular, for a compact complex two-fold M, if $\mathrm{Cod}(M) = 2$, then it is of general type. Here we should notice that for any compact complex two-fold M, if $\mathrm{Cod}(M) = 2$, then it admits two independent meromorphic functions. Hence M must be algebraic by an old theorem of Chow and Kodaira **[5]**.

REMARK 2.4 **[9]**. Let M_1 and M_2 be two compact complex manifolds such that M_1 is an unramified holomorphic cover of M_2. Then $\mathrm{Cod}(M_1) = \mathrm{Cod}(M_2)$.

The following observation is an easy consequence of the classification theory of complex surfaces.

THEOREM 2.2. *Let M be a compact complex two-fold with $C_1^2 - C_2 > 0$. Then M is either of general type with cotangent dimension two or of rational type.*

For the proof we observe that one can read from **[3**, Table 10, pp. 188**]** that a *minimal* compact complex two-fold with $C_1^2 - C_2 > 0$ can only be rational or of general type. On the other hand if M satisfies $C_1^2 - C_2 > 0$, its minimal model must also satisfy the same inequality. The cotangent dimension is equal to two is a consequence of Lemma 2.1.

3. Proofs of Theorems 1 and 2.

(A) *Proof of Theorem* 1. *Case* (a). Let $\widetilde{M}$ and $\widetilde{X}$ be the universal coverings of M and X respectively. Since $\pi_1(M)$ and $\pi_1(X)$ are finite, $\widetilde{M}$ and $\widetilde{X}$ are compact complex manifolds. It follows from assumption that $\widetilde{M}$ and $\widetilde{X}$ must be homotopic to each other. It is a consequence of a theorem in **[15]**, **[8]**, **[16]** that the Hirzebruch index of $\widetilde{M}$ and $\widetilde{X}$ must be equal. The index of a compact complex two-fold is equal to $\frac{1}{3}(C_1^2 - 2C_2)$. Moreover, it is elementary to prove that the Euler number for an oriented simply connected four-fold is a homotopy invariant. We have therefore proved that C_1^2 and C_2 for $\widetilde{M}$ and $\widetilde{X}$ are identical. The tangent bundle $T(\widetilde{X})$ is negative because X is assumed to be so. By a theorem of Kleiman [7], $C_2(\widetilde{X}) > 0$ and $C_1^2(\widetilde{X}) - C_2(\widetilde{X}) > 0$. It follows that $C_1(\widetilde{M}) > 0$, $C_2(\widetilde{M}) > 0$, and $C_1^2(\widetilde{M}) - C_2(\widetilde{M}) > 0$. Moreover, one has immediately $C_1^2(M) > 0$, $C_2(M) > 0$ and $C_1^2(M) - C_2(M) > 0$ because $\widetilde{M}$ is a finite unramified cover of M. From Lemma 2.1, Remark 2.3, and Theorem 2.2, M is either of general type with cotangent dimension two or of rational type. (This also follows from the classification theory of complex surfaces, namely [**2**, Table 10, p. 188]. This is a fact that a minimal compact complex two-fold with $C_1^2 > 0$ and $C_2 > 0$ must be of general type or of rational type. But it should be noticed the "minimal assumption" has been used here if you apply this alternate argument.) Finally, a minimal rational two-fold is either diffeomorphic to $\mathbb{C}P_2$ or to $S^2 \times S^2$. However, by Yau's theorem [**17**] X cannot be homotopic to $\mathbb{C}P_2$ because X has a negative tangent bundle. The possibility for being $S^2 \times S^2$ is also dropped by the assumption in Theorem 1.

Case (b). It is a known fact that the Chern numbers for compact complex two-folds are homeomorphic invariants (for a proof, we refer to [**2**, Theorem 2.6, p. 116]). The other part of the proof is then a line to line copy of part (a).

(B) *Proof of Theorem* 2. The proof is parallel to (A). The only new ingredient is the fact that $S^2 \times S^2$ is a spin manifold.

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Received July 7, 1990. Research partially supported by NSF Grant.

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