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THE EQUIVARIANT THOM ISOMORPHISM THEOREM

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In this paper we extend ordinary $RO(G)$-graded cohomology to a theory graded on virtual G-bundles over a G-space and show that a Thom Isomorphism theorem for general G-vector bundles results. Our approach uses Elmendorf's topologized spectra. We also show that the grading can be reduced from the group of virtual G-vector bundles over a space to a quotient group, using ideas from a new theory of equivariant orientations. As an application of the Thom Isomorphism theorem, we give a new calculation of the additive structure of the equivariant cohomology of complex projective spaces for $G = \mathbf{Z}/p$, partly duplicating and partly extending a recent calculation done by Lewis.

1. Introduction. Nonequivariantly, the Thom Isomorphism plays an important role in calculations of cohomology groups. Equivariantly, the Thom Isomorphism is a more elusive beast; $\mathbb{Z}$-graded ordinary cohomology is inadequate for its formulation, as can be seen by the fact that the cohomology of the one-point compactification of a representation is usually radically different from the cohomology of a point. Extending the $\mathbb{Z}$-graded theory of an $RO(G)$-graded theory [**LMM**] is still inadequate; the cohomology of the one-point compactification of a fiber of a bundle is now the suspension of that of a point, but the degree of the suspension depends on the group action on the fiber, and in general this varies as we move around in the base space.

Here we shall extend the $RO(G)$-graded theory to a theory graded on virtual G-bundles, and show that we can then prove a Thom Isomorphism theorem for general G-vector bundles. The use of virtual G-bundles for grading equivariant cohomology is not entirely new—Crabb [**C**], for example, considers stable cohomotopy, for $\mathbb{Z}/2$-actions, indexed in this way. Our approach differs in that it uses the topologized spectra that Elmendorf introduced in [**E**]. Although we could phrase our definitions in terms of parametrized spectra, we have chosen instead to take a more elementary approach. We also show that the grading can be reduced from virtual G-vector bundles to their "groupoid representations," gadgets that capture the orientability data of the bundles. This is a first step toward the construction of an ordinary theory indexed on groupoid representations; these representations are fundamental to equivariant orientation theory as presented in [**CMW**].

As an application of the equivariant Thom Isomorphism theorem, we give a new calculation of the additive structure of the equivariant cohomology of complex projective space for $G = \mathbb{Z}/p$. This partly duplicates and partly extends a recent calculation done by Lewis [**L1**] using different methods.

Along the way, we construct classifying spaces for arbitrary groupoids over the orbit category and prove a classification theorem (Theorem 3.8).

For the next section of this paper G may be any compact Lie group, but we will specialize to finite groups in the third and later sections.

2. Cohomology theories graded on $KO_G(X)$. We use Elmendorf's notion of an equivariant spectrum [**E**], which we recall here. Let $\mathscr{U}$ be a G-universe, that is, a countably infinite dimensional orthogonal G-representation which contains infinitely many copies of each finite dimensional representation of G. We recall that a G-spectrum E in the sense of Lewis, May and Steinberger [**MS**] consists of a collection of based G-spaces $E(V)$, one for each finite dimensional G-invariant subspace V of $\mathscr{U}$, together with a collection of compatible G-homeomorphisms $E(V) \cong \Omega^W(V \oplus W)$ for $V \perp W$. (see [**MS**] for details.) Elmendorf [**E**] replaces the spaces $E(V)$ with G-fiber bundles $p\colon E_n \to B_G O_n$ over the Grassmannian of n-planes in $\mathscr{U}$, with fiber over every G-invariant V a copy of $E(V)$. Each bundle is also equipped with a G-section σ giving the basepoint in each fiber. Elmendorf shows that any G-spectrum in the sense of [**LMS**] gives rise to a unique spectrum in his sense.

We want to describe how to define the E-cohomology of a space X, graded on $KO_G(X)$. We will describe here how to grade on pairs of subbundles of $X \times U$; in the Appendix we will refine this, and say in exactly what sense we can grade on $KO_G(X)$.

Let X be an unbased G-space and let ξ and ζ be finite-dimensional G-subbundles of $X \times \mathscr{U}$. Then ξ is classified by the Gauss map $\tilde{\xi}\colon X \to B_G O_n$. If E is a G-spectrum, we define $E_G^{\xi-\zeta}(X)$ as the group of G-homotopy classes of section-preserving G-maps c making the following diagram commute:

$$\begin{array}{ccc} S^{\zeta} & \xrightarrow{\ c\ } & E_n \\ \downarrow & & \downarrow p \\ X & \xrightarrow[\tilde{\xi}]{} & B_G O_n . \end{array}$$

Here, S^ζ is the fiberwise one-point compactification of ζ, and the homotopies are required to preserve sections and to cover $\tilde{\xi}$. The additive structure is a consequence of the structure of the spectrum E. This construction is functorial in X; we will give more details on this point in the Appendix. Note that we are using the notation $\xi - \zeta$ to mean simply the ordered pair (ξ, ζ).

We now generalize to pairs of G-spaces, and based G-spaces in particular. As in [**J**], a *G-ex-space* is a G-map $f: Y \to X$ with a given G-section; we also refer to this as a *based-G-space over X* if we wish to emphasize X. Many of the constructions we use here involving ex-spaces can be found in [**J**]. Given a G-ex-space $Y \to X$ as well as G-vector bundles ξ and ζ over X as above, define $\tilde{E}_G^{\xi-\zeta}(Y \to X)$ as the collection of G-homotopy classes of section-preserving G-maps c making the following diagram commute:

$$\begin{array}{ccc} \Sigma^\zeta Y & \xrightarrow{c} & E_n \\ \downarrow & & \downarrow p \\ X & \xrightarrow[\tilde{\xi}]{} & B_G O_n . \end{array}$$

Here, $\Sigma^\zeta Y$ is the fiberwise smash product of Y with S^ζ over X. This generalizes the special case described above: just take Y to be the space X^+, which is X with a disjoint section. Given a pair of G-spaces (Y, B) over X, let $C(Y, B) \to X$ denote the fiberwise cone of the inclusion $B^+ \to Y^+$ (note that the arrow is part of the notation), and define $E_G^{\xi-\zeta}((Y, B) \to X) = \tilde{E}_G^{\xi-\zeta}(C(Y, B) \to X)$. In particular, if $A \subset X$, define $E_G^{\xi-\zeta}(X, A) = \tilde{E}_G^{\xi-\zeta}(C, (X, A) \to X)$ and $\tilde{E}_G^{\xi-\zeta}(X) = E_G^{\xi-\zeta}(X, *)$ if X is based at $*$.

We have the following form of suspension isomorphism.

PROPOSITION 2.1. *If η is any G-subbundle of $X \times \mathscr{U}$ fiberwise perpendicular to both ξ and ζ, then*

$$\tilde{E}_G^{\xi-\zeta}(Y \to X) \cong \tilde{E}_G^{(\eta+\xi)-\zeta}(\Sigma^\eta Y \to X).$$

Proof. We recall from [**E**] the definition of the structure maps of E. Let $B_G O_{n,m}$ be the Grassmannian space of pairs of mutually orthogonal subspaces of $\mathscr{U}$ of dimension (n, m). There are maps $p_1: B_G O_{n,m} \to B_G O_n$ and $s: B_G O_{n,m} \to B_G O_{n+m}$, given by projection and sum respectively. Let γ_n denote the canonical bundle over $B_G O_n$,

and also let it denote $p_1^*\gamma_n$; similarly let γ_m denote the canonical bundle over $B_GO_{n,m}$ given by the second vector space. Then the structure maps of E are maps $\Sigma^{\gamma_m}p_1^*E_n \to s^*E_{n+m}$ over $B_GO_{n,m}$, such that the adjoint maps $p_1^*E_n \to \Omega^{\gamma_n}s^*E_{n+m}$ are homeomorphisms.

Given then a map $c\colon \Sigma^\zeta Y \to E_n$ covering $\tilde{\xi}$, we have a map $\Sigma^\eta c\colon \Sigma^\zeta\Sigma^\eta Y \to \Sigma^{\gamma_m}p_1^*E_n$ covering the map $(\tilde{\xi}, \tilde{\eta})\colon X \to B_GO_{n,m}$ induced by the classifying maps for ξ and η. The composite $\Sigma^\zeta\Sigma^\eta Y \to \Sigma^{\gamma_m}p_1^*E_n \to s^*E_{n+m} \to E_{n+m}$ covers $\widetilde{\xi+\eta}\colon X \to B_GO_{n+m}$ classifying the sum of the bundles. This describes the map $\widetilde{E}_G^{\xi-\zeta}(Y \to X) \to \widetilde{E}_G^{(\eta+\xi)-\zeta}(\Sigma^\eta Y \to X)$. That this is an isomorphism can now be checked from the fact that $p_1^*E_n \to \Omega^{\gamma_m}s^*E_{n+m}$ is a homeomorphism. □

It now follows from our definition of the cohomology of pairs, and from [J, 7.2], that we have the usual long extact sequence associated to a pair.

In order to reconcile Proposition 2.1 with the usual suspension isomorphism, involving the suspension of X, we need the following excision result. Let U be an open G-invariant subspace of X and let (Y, B) be a pair of spaces over X. We say that $(Y, B; p^{-1}(U))$ is an excisive triad over X if $\overline{U}$ has a neighborhood W in X such that $p^{-1}(W) \subset \operatorname{Int} B$.

LEMMA 2.2 (*Excision*). *Let* $(Y, B; p^{-1}(U) = V)$ *be an excisive triad over* X. *If* $\alpha = \xi - \zeta$ *is a difference of* G*-bundles over* X, *then*

$$\widetilde{E}_G^\alpha(C(Y, B) \to X) \cong \widetilde{E}_G^\alpha(C(Y - V, B - V) \to X - U).$$

In particular, if $(X, A; U)$ *is an excisive triad, then*

$$E_G^\alpha(X, A) \cong E_G^\alpha(X - U, A - U).$$

(*Here, we are using* α *to denote both* $\xi - \zeta$ *and its restriction to* $X - U$.)

Proof. If $f\colon \Sigma^\zeta C(Y, B) \to E_n$, the hypothesis allows us to see that f is fiberwise homotopic to a map that takes $\Sigma^\zeta q^{-1}(\overline{U})$ to the fiberwise basepoints, where $q\colon \Sigma^\zeta C(Y, B) \to X$ is the projection. The result follows from this. □

Note that the lemma allows one to extend the grading in $E_G^*(X, A)$ from G-bundles over X to G-bundles over $X - U$. This is needed in formulating our first version of the Thom Isomorphism theorem,

which generalizes the nonequivariant version with twisted coefficients, as well as the usual suspension isomorphism.

PROPOSITION 2.3 (*Thom Isomorphism—Twisted Form*). *If η is any G-subbundle of $X \times \mathscr{U}$ fiberwise perpendicular to both ξ and ζ, then there is a natural isomorphism*

$$E_G^{\xi-\zeta}(X) \cong \tilde{E}_G^{(\eta+\xi)-\zeta}(T\eta),$$

where $T\eta$ is the Thom space of η. (See the comment at the end of the proof for the definition of the second group.)

Proof. The result is a consequence of the following chain of natural isomorphisms in which $D(\eta)$ and $S(\eta)$ represent the unit disc and sphere bundles of η respectively, and $\alpha = \xi - \zeta$.

$$\begin{aligned} E_G^{\alpha}(X) &\cong \tilde{E}_G^{\alpha+\eta}(\Sigma^{\eta} X^{+} \to X) \\ &\cong \tilde{E}_G^{\alpha+\eta}(C(D(\eta), S(\eta)) \to X) \\ &\cong \tilde{E}_G^{\alpha+\eta}(C(D(\eta), S(\eta)) \to D(\eta)) \\ &= E_G^{\alpha+\eta}(D(\eta), S(\eta)) \\ &\cong \tilde{E}_G^{\alpha+\eta}(T\eta). \end{aligned}$$

The third isomorphism follows from the G-ex-homotopy equivalence of $C(D(\eta), S(\eta)) \to X$ and $C(D(\eta), S(\eta)) \to D(\eta)$ and Theorem A.3(iii) in the Appendix. The equality on the fourth line is by definition of the notation. Note that the last step is essentially excision and G-ex-homotopy invariance if $\alpha + \eta$ extends to a bundle over the Thom space. More precisely, let $U = T\eta - D_2\eta$, where D_2 denotes the disc bundle of radius 2. Then $(T\eta, \overline{T\eta - D\eta}; U)$ is excisive in the sense of Lemma 2.2. Hence, by Lemma 2.2 and Theorem A.3(ii), we have

$$\begin{aligned} \tilde{E}_G^{\alpha+\eta}(T\eta) &\cong E_G^{\alpha+\eta}(T\eta, \overline{T\eta - D\eta}) \\ &\cong E_G^{\alpha+\eta}(T\eta - U, \overline{T\eta - D\eta} - U) \\ &\cong E_G^{\alpha+\eta}(D(\eta), S(\eta)). \end{aligned}$$

If $\alpha + \eta$ fails to extend, we can define $\tilde{E}_G^{\alpha+\eta}(T\eta)$ to be

$$\tilde{E}_G^{\alpha+\eta}(C(D(\eta), S(\eta)),$$

following the remark after Lemma 2.2. □

The next result permits us to regard the theory as being graded on $KO_G(X)$ rather than on formal differences of representations.

PROPOSITION 2.4. *Let* (ξ, ζ) *and* (ξ', ζ') *be pairs of representations with* $[\xi - \zeta] = [\xi' - \zeta']$ *in* $KO_G(X)$. *Then* $\widetilde{E}_G^{\xi-\zeta}(f) \cong \widetilde{E}_G^{\xi'-\zeta'}(f)$ *for any G-space* $f: Y \to X$ *over* X.

Proof. Since $[\xi - \zeta] = [\xi' - \zeta']$ in $KO_G(X)$, there is a G-bundle η with $\xi \oplus \zeta' \oplus \eta \cong \xi' \oplus \zeta \oplus \eta$. One has

$$\begin{aligned}\widetilde{E}_G^{\xi-\zeta}(Y \to X) &\cong \widetilde{E}_G^{\xi}(\Sigma^{\zeta} Y \to X)\\ &\cong \widetilde{E}_G^{\xi\oplus\zeta'\oplus\eta}(\Sigma^{\zeta\oplus\zeta'\oplus\eta} Y \to X)\\ &\cong \widetilde{E}_G^{\xi'\oplus\zeta\oplus\eta}(\Sigma^{\zeta\oplus\zeta'\oplus\eta} Y \to X)\\ &\cong \widetilde{E}_G^{\xi'-\zeta'}(Y \to X).\end{aligned}$$

□

Although we have not specified choices of isomorphisms in the proof above, it is possible to make $\widetilde{E}_G^*(-)$ into a functor on the category of pairs (f, α) where f is a G-space over X (X allowed to vary) and where α is a "virtual bundle" over X. This is deliberately vague, but the Appendix gives the details. It does not appear that we can create a well-defined functor on the category of pairs (f, α), where $\alpha \in KO_G(X)$; some bundle maps must be specified. However, we will write $\widetilde{E}_G^{\alpha}(Y \to X)$ when $\alpha \in KO_G(X)$, with the understanding that this is natural in f and α only when we specify a particular difference of subbundles of $X \times \mathscr{U}$ representing α and particular pairs of bundle maps.

3. Classifying spaces and ordinary cohomology graded on $KO_G(X)$. We shall see in this section that, when the theory is ordinary cohomology, the grading introduced in the previous section reduces to a quotient of $KO_G(X)$, and that this gives a Thom Isomorphism where the shift in the grading is by the "dimension" of the bundle, rather than the bundle itself. We restrict attention here to G finite, the details in the compact Lie case being somewhat messy. Let HT be the G-spectrum representing ordinary $RO(G)$-graded equivariant cohomology with coefficients in the Mackey functor T. (See, for example, **[LMM]** or **[CW]**.)

In order to describe the reduction of the grading, we first recall the following definitions from **[CMW]**, which are fundamental to the theory of equivariant orientations presented there. If X is a

G-space, the *fundamental groupoid* $\pi(X; G)$ *of* X is the category whose objects are the G-maps $x: G/H \to X$, where H ranges over the subgroups of G; equivalently, x is a point in X^H. A morphism $x \to y$, $y: G/K \to X$, is the equivalence class of a pair (σ, ω), where $\sigma: G/H \to G/K$ is a G-map, and where $\omega: G/H \times I \to X$ is a G-homotopy from x to $y \circ \sigma$. Two such maps are equivalent if there is a G-homotopy $k: \omega \simeq \omega'$ such that $k(\alpha, 0, t) = x(\alpha)$ and $k(\alpha, 1, t) = y \circ \sigma(\alpha)$ for $\alpha \in G/H$ and $t \in I$.

Let $\mathscr{G}$ be the category of G-orbits and G-maps between them. There is a functor $\phi: \pi(X; G) \to \mathscr{G}$, given by $\phi(x: G/H \to X) = G/H$ on objects, and by $\phi(\sigma, \omega) = \sigma$ on morphisms. This turns $\pi(X; G)$ into a groupoid over $\mathscr{G}$ in the sense of **[CMW]**. Precisely:

DEFINITION 3.1. A *groupoid* over a small category $\mathscr{B}$ consists of a small category $\mathscr{C}$ and a functor $\phi: \mathscr{C} \to \mathscr{B}$ satisfying the following properties:

(a) For each object b of $\mathscr{B}$, the fiber $\phi^{-1}(\mathrm{id}_b)$ is a groupoid in the classical sense (i.e., all morphisms in C covering an identity map in B are isomorphisms).

(b) (Source-lifting) For each object y of $\mathscr{C}$ and each morphism $\alpha: a \to \phi(y)$ in $\mathscr{B}$, there exists a morphism $\beta: x \to y$ in $\mathscr{C}$ such that $\phi(\beta) = \alpha$.

(c) (Divisibility) For each pair of morphisms $\alpha: x \to y$ and $\alpha': x' \to y$ in $\mathscr{C}$ and each morphism $\beta: \phi(x) \to \phi(x')$ such that $\phi(\alpha) = \phi(\alpha')\beta$, there exists a morphism $\gamma: x \to x'$ in $\mathscr{C}$ such that $\alpha = \alpha'\gamma$ and $\phi(\gamma) = \beta$.

A groupoid $\mathscr{C}$ has *unique divisibility* if the map γ asserted to exist in (c) is unique. All the groupoids we consider will have this property. Notice in particular that $\pi(X; G)$ is a groupoid over $\mathscr{G}$ with unique divisibility, and that the fiber $\phi^{-1}(G/H)$ is the usual fundamental groupoid $\pi(X^H)$.

DEFINITION 3.2. If $\phi: \mathscr{C} \to \mathscr{B}$ is a continuous functor of topological categories and α and b are objects in $\mathscr{C}$, let $h\mathscr{C}(a, b)$ be the space obtained from the map $\phi: \mathscr{C}(a, b) \to \mathscr{B}(\phi(a), \phi(b))$ by replacing each preimage $\phi^{-1}(f)$ with $\pi_0\phi^{-1}(f)$, its space of components (using the quotient topology). This gives a new category $h\mathscr{C}$ having the same objects as $\mathscr{C}$, and a functor $h\phi: h\mathscr{C} \to \mathscr{B}$. Say that $\mathscr{C}$ is a *groupoid-like category over* $\mathscr{B}$ if $h\phi: h\mathscr{C} \to \mathscr{B}$ is a groupoid over $\mathscr{B}$.

Let $hG\mathscr{B}$ be the category of orthogonal G-bundles over G-orbits and G-homotopy classes of linear maps, so there is again a functor $\phi\colon hG\mathscr{B} \to \mathscr{G}$, giving the base-space. A *representation of* $\pi(X; G)$ is a functor $\rho\colon \pi(X; G) \to hG\mathscr{B}$ such that $\phi\rho = \phi$; that is, it is a functor over $\mathscr{G}$. We add representations in the obvious way, by taking direct sums of bundles. Taking isomorphism classes of representations gives a monoid; applying the Grothendieck construction gives $RO(\pi(X; G))$, the group of virtual representations of the fundamental groupoid of X. If V is a representation of G, we will write V also for the representation of $\pi(X; G)$ that sends every object over G/H to the bundle $G/H \times V$, and every morphism over $\sigma\colon G/H \to G/K$ to $\sigma \times 1$. In this way we can think of $RO(G) \subset RO(\pi(X; G))$.

If ξ is a G-bundle over X, then ξ determines a representation $\rho(\xi)$ of $\pi(X; G)$ given by $\rho(\xi)(x\colon G/H \to X) = x^*(\xi)$ on objects. $\rho(\xi)$ is defined on maps using the lifting property for G-bundles. Since ρ preserves addition, it extends to a homomorphism $\rho\colon KO_G(X) \to RO(\pi(X; G))$. Let $DO_G(X)$ denote the image of ρ. Roughly speaking, $DO_G(X)$ is the group of equivalence classes of virtual bundles over X where two virtual G-bundles are equivalent if they have the same local representations and the same action of the fundamental groupoid on fibers. We think of $\rho(\alpha)$ as the *dimension* of α. The main theorem of this section is:

THEOREM 3.3. *If* α *and* $\beta \in KO_G(X)$ *and* $\rho(\alpha) = \rho(\beta)$, *then* $H_G^\alpha(Y \to X; T) \cong H_G^\beta(Y \to X; T)$.

An immediate consequence of Theorem 3.3 is that we can grade ordinary cohomology on $DO_G(-)$ instead of $KO_G(-)$. In fact, the ultimate result in this direction should be that we can grade ordinary cohomology on the group of stable spherical representations of $\pi(X; G)$; this we may show in a future paper. Another consequence of the theorem, together with Proposition 2.3, is the following.

THEOREM 3.4 (*Thom Isomorphism*). *If* η *is any* G*-vector bundle over* X *and if* $\alpha \in DO_G(X)$, *then there is a natural isomorphism*

$$H_G^\alpha(X; T) \cong \widetilde{H}_G^{\alpha + g\rho(\eta)}(T\eta; T)$$

for any coefficient system T. *In particular, if* $\rho(\eta) = V$, *then*

$$H_G^\alpha(X; T) \cong \widetilde{H}_G^{\alpha+V}(T\eta; T). \qquad \square$$

As in Proposition 2.3, we define $\tilde{H}_G^{\alpha+\rho(\eta)}(T\eta; T)$ to be $H_G^{\alpha+\rho(\eta)}(D(\eta), S(\eta); T)$ if α and η do not extend to the Thom space. The second conclusion of Theorem 3.4 has been previously noticed for $RO(G)$-graded cohomology; the condition $\rho(\eta) = V$ implies that η is an "orientable V-bundle" in the sense of **[W2]**, but is slightly stronger.

The proof of Theorem 3.3 requires several preliminary results on classifying spaces of groupoids over $\mathscr{G}$. Several such constructions are known; see for example **[CMW]** for one case. If the groupoids in question have discrete topologies, then there are simpler constructions than the one that follows **[M2]**, but we have need of the topological case. We shall work with a set $G\mathscr{U}_n$ of n-dimensional G-vector bundles over orbits. These G-bundles have the form $v(H, V)\colon G \times_H V \to G/H$, where $H \subset G$ and $V \subset \mathscr{U}$ is an H-invariant n-dimensional subspace. A *category over* $G\mathscr{U}_n$ is a category whose underlying set of objects is $G\mathscr{U}_n$ and which comes equipped with a functor $\phi\colon C \to \mathscr{G}$ taking the object $v(H, V)$ to G/H. If $\mathscr{C}$ is a topological category over $G\mathscr{U}_n$, we define the *classifying G-space of* $\mathscr{C}$ by the formula

$$B_G\mathscr{C} = B(*, \mathscr{C}, \phi).$$

Here, $B(-, -, -)$ denotes the two-sided geometric bar construction **[M1]**. On the simplicial level $B_k(*, \mathscr{C}, \phi)$, for $k > 0$, consists of tuples $(*[f_1, \ldots, f_k]gH_k)$ where $f_i\colon v(H_i, V_i) \to v(H_{i-1}, V_{i-1})$ is a morphism in $\mathscr{C}$ and $gH_k \in G/H_k$. We take $B_0(*, \mathscr{C}, \phi)$ to be the space of cosets $(*[\ ]_gH_0)_v$ indexed on the objects $v = v(H_0, V_0)$ in $\mathscr{C}$. G acts on $B_G\mathscr{C}$ through its action on the last coordinate.

Following are the examples we shall need.

EXAMPLES 3.5. (i) Take $\mathscr{C} = O_n$, the category of pairs (H, V), where V is an H-invariant n-dimensional subspace of $\mathscr{U}$, and where a morphism $(H, V) \to (J, W)$ is an orthogonal G-bundle map $G \times_H V \to G \times_J W$. The resulting space $B_G\mathscr{C}$ is the classifying space B_GO_n for n-dimensional orthogonal G-bundles **[W1]**, and is G-homotopy equivalent to the Grassmannian of n-planes in $\mathscr{U}$.

(ii) Take $\mathscr{C} = hO_n$, the homotopy category over $\mathscr{G}$ associated with O_n by Definition 3.2. If $[\mathscr{D}, hO_n]_{\mathscr{G}}$ denotes the set of natural isomorphism classes of maps over $\mathscr{G}$, then $[\pi(X; G), hO_n]_{\mathscr{G}}$ is the set of isomorphism classes of n-dimensional representations of $\pi(X; G)$. Note that there is a discretization map $B_GO_n \to B_GhO_n$. In Theorem 3.8 we will show that B_GhO_n classifies representations of the fundamental groupoids of G-spaces, so that the discretization map represents the passage from a G-vector bundle to the associated groupoid representation.

(iii) If $V \subset \mathscr{U}$ is an H-invariant n-dimensional subspace of $\mathscr{U}$, let $K(T|H, V)$ be an H-equivariant Eilenberg-Mac Lane space of type $(T|H, V)$ [**LMM, CW**]. Take $\mathscr{C} = \mathscr{K}T_n$, the category whose morphisms $v(H, V) \to v(J, W)$ covering $f: G/H \to G/J$ consist of the fiberwise G-homotopy equivalences $G \times_H K(T|H, V) \to G \times_J K(T|J, W)$ covering f. The resulting space $B_G\mathscr{K}T_n$ classifies G-fibrations whose fibers over H-fixed points are H-equivariantly equivalent to $K(T|H, V)$ for some V, by [**W1**; 2.3.6]. In particular, in the spectrum HT, the fibration $HT_n \to B_GO_n$ is classified by a G-map $\chi: B_GO_n \to B_G\mathscr{K}T_n$.

(iv) Take $\mathscr{C} = h\mathscr{K}T_n$, the homotopy category over $\mathscr{G}$ associated with $\mathscr{K}T_n$. The morphism spaces $\mathscr{K}T_n(a, b)$ have weakly contractible components. To see this, it suffices to show that

$$\pi_n \mathrm{map}(K(T, V), K(T, V))^K = 0 \quad \text{for } n > 0,$$

but one has that

$$\pi_n \mathrm{map}(K(T, V), K(T, V))^K \cong \tilde{H}_K^{V-n}(K(T, V); T) = 0$$

for $n > 0$. This in turn is true since $\pi_m(K(T, V)^J) \cong \tilde{H}_J^V(S^m; T) \cong \tilde{H}_m^J(S^V; T) = 0$ for $m < |V^J|$, which, by [**L2**, 7.1] is enough to show the vanishing of $H_K^{V-n}(K(T, V); T)$ for $n > 0$. It follows that the discretization functor $h: \mathscr{K}T_n \to h\mathscr{K}T_n$ is a weak equivalence on morphism spaces. We shall see below that h induces a weak G-equivalence of classifying spaces.

If α is an object of $\mathscr{C}$, let $\mathscr{C}_\alpha$ be the category of maps out of α, or the category of objects under α; i.e., the objects are maps $\alpha \to \beta$ in $\mathscr{C}$ and morphisms are commutative triangles. Also define $\mathscr{C}/\phi$ to be the category whose objects are those of $\mathscr{C}$ and whose morphisms are equivalence classes of morphisms of $\mathscr{C}$ under the identification $f \approx g$ if $\phi(f) = \phi(g)$.

LEMMA 3.6. *Let $\mathscr{C}$ be a groupoid-like category over $\mathscr{G}$. Then the space $(B_G\mathscr{C})^H$ is a disjoint union of copies of the connected spaces $B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha)$, one for each isomorphism class $[\alpha]$ of objects in $h\mathscr{C}$ with $\phi(\alpha) = G/H$.*

Proof. Let α be an object in $\mathscr{C}$ with $\phi(\alpha) = G/H$. It is clear that $B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha) \subset (B_G\mathscr{C})^H$. It follows from the source-lifting property that $(B_G\mathscr{C})^H$ is the union of subspaces of this form. Inclusion of the first simplicial filtration is a π_0-equivalence, and we can use this, together with the divisibility property, to see that if α and β

are non-isomorphic objects of $h\mathscr{C}$ mapping to G/H under ϕ, then $B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha)$ and $B(*, \mathscr{C}, (\mathscr{C}/\phi)_\beta)$ lie in distinct components of $(B_G\mathscr{C})^H$. Further, each subspace $B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha)$ may be seen to be connected by again considering π_0. The result now follows by combining this collection of observations. □

LEMMA 3.7. *The map* $B_Gh\colon B_G\mathscr{K}T_n \to B_Gh\mathscr{K}T_n$ *is a weak* G*-equivalence.*

Proof. Let $H \subset G$; we show that $(B_Gh)^H$ is a (nonequivariant) weak equivalence. In view of the previous lemma, it suffices to show that, for each $\alpha \in G\mathscr{U}_n$, the map

$$B(*, \mathscr{K}T_n, (\mathscr{K}T_n/\phi)_\alpha) \to B(*, h\mathscr{K}T_n, (h\mathscr{K}T_n/\phi)_\alpha)$$

is a weak equivalence. For this, consider the diagram

$$\begin{array}{ccccc}
\mathscr{K}T_n(\alpha, \alpha)_1 & \longrightarrow & B(*, \mathscr{K}T_n, (\mathscr{K}T_n)_\alpha) & \longrightarrow & B(*, \mathscr{K}T_n, (\mathscr{K}T_n/\phi)_\alpha) \\
\downarrow & & \downarrow & & \downarrow \\
h\mathscr{K}T_n(\alpha, \alpha)_1 & \longrightarrow & B(*, h\mathscr{K}T_n, (h\mathscr{K}T_n)_\alpha) & \longrightarrow & B(*, h\mathscr{K}T_n, (h\mathscr{K}T_n/\phi)_\alpha),
\end{array}$$

where $\mathscr{C}(\alpha, \alpha)_1$ is the space of morphisms $\alpha \to \alpha$ in $\mathscr{C}$ over the identity on $\phi(\alpha)$. Each row is a quasifibration, using the Dold-Thom criterion; for details, see [**M1**, §7.6] (our groupoid-like property replaces his group-like assumption). As mentioned in 3.5(iv), the vertical map on the left is a weak equivalence. The spaces in the middle of each line are contractible, so the middle map is also an equivalence. Therefore, the map on the right is a weak equivalence. □

Finally, we need the following important result.

THEOREM 3.8. *Let* $\phi\colon \mathscr{C} \to \mathscr{G}$ *be a groupoid with unique divisibility such that the topology on* $\mathscr{C}$ *is discrete. If* X *has the* G*-homotopy type of a* $G-CW$ *complex, then*

$$[X, B_G\mathscr{C}]_G \cong [\pi(X; G), \mathscr{C}]_{\mathscr{G}},$$

where $[-, -]_G$ *denotes* G*-homotopy classes of* G*-maps, and* $[-, -]_{\mathscr{G}}$ *denotes natural isomorphism classes of maps over* $\mathscr{G}$. *In other words,* B_G *is right adjoint to* π.

Proof. We construct maps in both directions. Start with the map

$$\Phi\colon [X, B_G\mathscr{C}]_G \to [\pi(X; G), \mathscr{C}]_{\mathscr{G}}.$$

This will be induced by an equivalence of groupoids $\mu\colon \mathscr{C} \to \pi(B_G\mathscr{C}; G)$ over $\mathscr{G}$ (i.e., this will be the inverse of the counit of the adjunction we are constructing). On objects, μ is specified by taking α to the vertex $((*[\ ]eH)_\alpha; 1)$ in $(B_G\mathscr{C})^H$, where $\phi(\alpha) = G/H$. (Here we are following May's notation from [**M1**, §11].) If $\omega\colon \alpha \to \beta$ is a morphism in $\mathscr{C}$, then define $\mu(\omega)$ to be the homotopy class of the path $t \mapsto (*[\omega]eH; (t, 1-t))$. This is a path from $\mu(\alpha)$ to $g\mu(\beta)$, where $\phi(\omega)(eH) = gK$, and thus specifies a morphism in $\pi(B_G\mathscr{C}; G)$ from $\mu(\alpha)$ to $\mu(\beta)$ covering $\phi(\omega)$. Next, we show that μ is an equivalence by showing that it restricts to an isomorphism of sketeta. (A skeleton of a groupoid over $\mathscr{G}$ has only one object from each isomorphism class over the identity on each orbit in $\mathscr{G}$.) By Lemma 3.6, the image of a skeleton of $\mathscr{C}$ is a skeleton of $\pi(B_G\mathscr{C}; G)$, and μ is a one-to-one correspondence on objects. It remains to show that μ is bijective on morphisms, and we do this in two steps. First, we show that μ is bijective on automorphisms over the identity. One has the principal fibration $B(*, \mathscr{C}, \mathscr{C}_\alpha) \to B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha)$ with fiber $\mathrm{Aut}(\alpha)$ and contractible total space, where $\mathrm{Aut}(\alpha)$ is the group of automorphisms over $\mathrm{id}_{\phi(\alpha)}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Aut}(\alpha) & \longrightarrow & B(*, \mathscr{C}, \mathscr{C}_\alpha) & \longrightarrow & B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha) \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \nu} & & \| \\
\Omega B(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha) & \longrightarrow & PB(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha) & \longrightarrow & \mathscr{B}(*, \mathscr{C}, (\mathscr{C}/\phi)_\alpha).
\end{array}$$

Here the bottom row is the path-space fibration,

$$\nu(*[f_1, \ldots, f_n]f; x)(t) = (*[f_1, \ldots, f_n, f]1_\alpha; (xt, 1-t)),$$

$\zeta(f) = (*[f]1_\alpha; (t, 1-t))$, and $\mu = \pi_0(\zeta)$. Each row is fibration sequence with contractible total space. It now follows that ζ is a homotopy equivalence on the fibers, and hence that μ is an isomorphism on maps over the identity. The unique divisibility property now implies that μ is an isomorphism on sets of morphisms over maps other than the identity. The map Φ is now specified by $\Phi[f] = \mu^{-1} \circ \pi(f; G)$.

We now construct the inverse, Ψ, of Φ. Start with a map of groupoids $\theta\colon \pi(X; G) \to \mathscr{C} \cong \pi(B_G\mathscr{C}; G)$. θ specifies a well-defined homotopy class of G-maps from the 1-skeleton of X to $B_G\mathscr{C}$. Further, we can extend such a map over the 2-skeleton, since θ is a map of groupoids, so that inessential loops go to inessential loops. Since, by the discussion above, each component Y of each fixed set of $B_G\mathscr{C}$ has $\pi_i(Y) = 0$ if $i \geq 2$, it follows by elementary obstruction theory

that we can extend the map to all of X. We define $\Psi(\theta)$ to be the homotopy class of this extension $\overline{\theta}$.

From the constructions, it is clear that $\Phi\Psi = 1$. On the other hand, since it is easy to see that $\overline{\theta}$ is unique up to G-homotopy, it follows that Φ is injective, and we are done. □

We can now prove Theorem 3.3.

Proof of Theorem 3.3. If α and β are two elements of $KO_G(X)$ agreeing in $DO_G(X)$, we can assume that $\alpha = [\xi - \eta]$ and $\beta = [\zeta - \eta]$ with ξ and ζ giving the same representation of $\pi(X; G)$. Consider the following diagram:

$$\begin{array}{ccccc} X & \overset{\xi}{\underset{\zeta}{\rightrightarrows}} & B_G O_n & \xrightarrow{\chi} & B_G \mathscr{K} T_n \\ & \searrow & \downarrow \delta & & \downarrow \simeq \\ & & B_G h O_n & \xrightarrow{h\chi} & B_G h\mathscr{K} T_n . \end{array}$$

Since ξ and ζ give the same representation, and since $B_G h O_n$ classifies representations of $\pi(X; G)$ by the theorem just proved, the composites $\delta \circ \xi$ and $\delta \circ \zeta$ are G-homotopic. It now follows by a diagram chase that $\chi \circ \xi$ and $\chi \circ \zeta$ are G-homotopic. Since now the pullbacks along ξ and ζ of the bundle $HT_n \to B_G O_n$ are fiber homotopy equivalent, it follows that $H_G^{\alpha}(Y \to X; T) \cong H_G^{\beta}(Y \to X; T)$, as claimed. □

4. Cohomology of projective spaces. Let $G = \mathbb{Z}/p$. In [**L1**], Lewis gives an explicit equivariant cell structure for the complex projective space $\mathbb{C}P(V) = S(V)/U(1)$, where $S(V)$ is the unit sphere of the unitary representation V. He then uses this decomposition to compute its ordinary $RO(G)$-graded cohomology with Burnside coefficients. Here we show how to obtain the same result using the Thom isomorphism theorem, and also extend the calculation to a larger grading.

Before stating the result, we need some notation. Since $G = \mathbb{Z}/p$, we can decompose V as a sum of one-dimensional unitary representations: $V = L_0 \oplus L_1 \oplus \cdots$. Let $\lambda_0 = 0$, and for each $i > 0$, let

$$\lambda_i = \sum_{j=0}^{i-1} \overline{L}_i \otimes_{\mathbb{C}} L_j .$$

For $i \geq 0$ let

$$n_i = \dim_{\mathbb{R}} \lambda_i^G .$$

Thus, n_i is twice the number of times L_i appears up to isomorphism in $\sum_{j=0}^{i-1} L_j$. If $\alpha \in KO_G(\mathbb{C}P(V))$, let α_i be the restriction of α to the point $\mathbb{C}P(L_i)$, thought of as a representation of G, i.e. $\alpha_i \in RO(G)$. We shall denote ordinary cohomology with Burnside coefficients by $H_G^*(-)$.

THEOREM 4.1. *Assume that* $\alpha \in KO_G(\mathbb{C}P(V))$ *and that*

$$\dim_{\mathbb{R}}(\alpha_i)^G - \dim_{\mathbb{R}}(\alpha_j)^G \leq n_i - n_j \text{ whenever } i > j.$$

Then

$$H_G^\alpha(\mathbb{C}P(V)) \cong \sum_{i \geq 0} H_G^{\alpha_i - \lambda_i}(*).$$

Further, if we grade on the coset $\alpha + RO(G)$, *then this isomorphism is one of modules over the* $RO(G)$*-graded cohomology of a point.*

The following is now an immediate consequence of the theorem.

COROLLARY 4.2 (*L. G. Lewis*). *If* (n_i) *is a non-decreasing sequence, then the conclusion of Theorem* 4.1 *holds for any* $\alpha \in RO(G)$.

Lewis also computes the multiplicative structure, and this is far more complicated. For details, see [**L1**]. Both our result and Lewis' hinge on explicit information about the $RO(G)$-graded cohomology of a point as first calculated by Stong (unpublished) and we state what little we shall need in the following lemma. (A complete description of the cohomology of a point appears in [**L1**].)

LEMMA 4.3. *Let* $G = \mathbb{Z}/p$ *and* $\sigma \in RO(G)$. *Then* $H_G^\sigma(*) = 0$ *when* $\dim_{\mathbb{R}}(\sigma)$ *and* $\dim_{\mathbb{R}}(\sigma^G)$ *are odd,* $\dim_{\mathbb{R}}(\sigma) < 0$, *and* $\dim_{\mathbb{R}}(\sigma^G) \leq 1$. □

Proof of Theorem 4.1. We first assume that V is finite dimensional, and induct on $m = \dim_{\mathbb{C}} V$. The beginning of the induction, $m = 1$, is trivial. For the inductive step, we consider the inclusion

$$\iota\colon \mathbb{C}P(L_1 \oplus \cdots \oplus L_n) \hookrightarrow \mathbb{C}P(L_0 \oplus L_1 \oplus \cdots \oplus L_n).$$

By a standard geometric argument, the normal bundle of ι is $\nu = \mathrm{Hom}(\xi, L_0) = \overline{\xi} \otimes L_0$, where ξ is the canonical line bundle over the subspace $\mathbb{C}P(L_1 \oplus \cdots \oplus L_n)$, and the associated Thom space is $\mathbb{C}P(L_0 \oplus L_1 \oplus \cdots \oplus L_n)$ with basepoint $\mathbb{C}P(L_0)$. By the Thom isomorphism,

$$\widetilde{H}_G^\alpha(\mathbb{C}P(L_0 \oplus L_1 \oplus \cdots \oplus L_n)) \cong H_G^{\alpha - \nu}(\mathbb{C}P(L_1 \oplus \cdots \oplus L_n)).$$

(Note that α and ν are defined on all of $\mathbb{C}P(L_0 \oplus L_1 \oplus \cdots \oplus L_n)$.) By induction, $H_G^{\alpha-\nu}(\mathbb{C}P(L_1 \oplus \cdots \oplus L_n)) \cong \sum_{i>0} H_G^{\alpha_i-\lambda_i}(*)$, where $\lambda_i = \sum_{0\le j\le i-1} L_i \otimes_{\mathbb{C}} L_j$ and where the hypothesis on $\alpha-\nu$ and the sequence $L_1, \ldots, L_n$ may be checked to hold. The long exact sequence of the pair $(\mathbb{C}P(V), \mathbb{C}P(L_0))$ now becomes

$$\cdots \to \sum_{i>0} H_G^{\alpha_i-\lambda_i}(*) \to H_G^{\alpha}(\mathbb{C}P(V)) \to H_G^{\alpha_0}(*) \xrightarrow{\partial} \sum_{i>0} H_G^{\alpha_i-\lambda_i+1}(*) \to \cdots.$$

We claim that the connecting homomorphism ∂ is zero. To see this, it suffices to show that ∂ is zero with α replaced by $\alpha-\alpha_0$, since ∂ is a map of $RO(G)$-graded $H_G^*(*)$-modules and the generator of $H_G^*(*)$ lies in degree 0. However, the summands $H_G^{\alpha_i-\alpha_0-\lambda_i+1}(*)$ are all zero, by Lemma 4.3. Since $H_G^*(*)$ is projective as a module over itself, the resulting short exact sequence splits, completing the inductive step, and hence the proof for the case V finite.

If V is countably infinite dimensional, the result follows by passage to inverse limits—the $\lim^1$-terms vanishing since the maps of the system are epimorphisms. □

Appendix. Virtual bundles. We construct, carefully, a category of virtual bundles, in order to explain how to reduce the grading of cohomology theories given in §2 from pairs of bundles to a category whose isomorphism classes give $KO_G(X)$, and obtain a well-defined functor of X. This construction is essentially the one that appears in **[CMW]**.

DEFINITION A1. (a) Let $\mathscr{U}$ be a G-universe. Define $G\mathscr{V}$ to be the category whose objects are pairs (X, ξ), where X is a G-space and ξ is a finite-dimensional G-subbundle of $X \times \mathscr{U}$. The morphisms are the G-bundle maps. We shall usually write ξ for (X, ξ).

(b) Let $\mathscr{S}G\mathscr{V}$ be the category whose objects are pairs of bundles over the same base space, and whose maps $(\xi, \zeta) \to (\xi', \zeta')$ are equivalence classes of pairs of G-bundle maps (f, g): $(\xi+\eta, \zeta+\eta) \to (\xi'+\eta', \zeta'+\eta')$ covering the same map of base spaces. The equivalence relation is the one generated by three elementary equivalences. First, $(f, g) \approx (f+\mathrm{id}_\mu, g+\mathrm{id}_\mu)$, where, if ξ' and ζ' are bundles over X and f and g cover the map F on base spaces, then μ is any subbundle of $X \times \mathscr{U}$ perpendicular to $\xi'+\eta'$ and $\zeta'+\eta'$, with $F^*\mu$ perpendicular to $\xi+\eta$ and $\zeta+\eta$, and by id_μ we mean the canonical map $F^*\mu \to \mu$.

Secondly, let

$$(f, g)\colon (\xi+\eta, \zeta+\eta) \to (\xi'+\eta', \zeta'+\eta')$$
$$\approx (h, k)\colon (\xi+\mu, \zeta+\mu) \to (\xi'+\mu', \zeta'+\mu')$$

if there are bundle maps $\eta \to \mu$ and $\eta' \to \mu'$ making the following diagrams both fiber homotopy commute:

$$\begin{array}{ccc} \xi+\eta & \longrightarrow & \xi'+\eta' \\ \downarrow & & \downarrow \\ \xi+\mu & \longrightarrow & \xi'+\mu' \end{array} \quad \text{and} \quad \begin{array}{ccc} \zeta+\eta & \longrightarrow & \zeta'+\eta' \\ \downarrow & & \downarrow \\ \zeta+\mu & \longrightarrow & \zeta'+\mu'. \end{array}$$

The last relation is fiber-homotopy of maps (over a fixed base map). We call the objects of $\mathscr{S}G\mathscr{V}$ *virtual bundles*, and the morphisms *virtual bundle maps*.

(c) If X is a G-space, let $\mathscr{S}G\mathscr{V}(X)$ be the subcategory of $\mathscr{S}G\mathscr{V}$ consisting of virtual bundles over X and maps over the identity map of X.

We are interested mainly in the isomorphism classes of these categories, for we have:

PROPOSITION A.2. *There is a natural one-to-one correspondence between the isomorphism classes of objects of* $\mathscr{S}G\mathscr{V}(X)$ *and the elements of* $KO_G(X)$.

Proof. The correspondence is given by taking the isomorphism class of the pair (ξ, ζ) to the element $[\xi - \zeta] \in KO_G(X)$, and it is easy to check that this gives a well-defined bijection. In particular, notice that the elementary equivalences of A.1(b) above add no new isomorphisms. □

Now let GEx be the category of G-ex-spaces. Both GEx and $\mathscr{S}G\mathscr{V}$ map to the category $G\mathscr{U}$ of G-spaces by taking base-spaces, and we can form the pullback category $GEx \times_{G\mathscr{U}} \mathscr{S}G\mathscr{V}$, the category of G-ex-spaces and virtual bundles on the base-spaces.

The following result shows the sense in which we can grade cohomology theories of $KO_G(X)$. It also shows in what sense it is homotopy invariant.

THEOREM A3. (i) *If* E *is a* G*-spectrum and* $(W \to X, \xi, \zeta)$ *is an object of* $GEx \times_{G\mathscr{U}} h\mathscr{S}G\mathscr{V}$, *then the assignment* $(W \to X, \xi, \zeta) \mapsto \tilde{E}_G^{\xi-\zeta}(W \to X)$ *defines a contravariant functor on* $GEx \times_{G\mathscr{U}} h\mathscr{S}G\mathscr{V}$.

(ii) *The functor of part* (i) *factors through the homotopy category of* $GEx \times_{G\mathscr{U}} h\mathscr{S}G\mathscr{V}$.

Proof. (i) Recall that an element of $\tilde{E}_G^{\xi'-\zeta'}(W \to X)$ is a homotopy class of maps making the diagram

$$\begin{array}{ccc} \Sigma^{\zeta'} W & \dashrightarrow & E_n \\ \downarrow & & \downarrow \\ X & \xrightarrow[\tilde{\xi}']{} & B_G O_n \end{array}$$

commute, where $\tilde{\xi}'$ is the Gauss map. Now suppose given a G-ex-map $(\widetilde{F}, F): (Y \to Z) \to (W \to X)$, with $\widetilde{F}: Y \to W$ and $F: Z \to X$. Suppose also given a virtual bundle map $(\xi, \zeta) \to (\xi', \zeta')$ over F, represented by the pair of bundle maps $(f, g): (\xi + \eta, \zeta + \eta) \to (\xi' + \eta', \zeta' + \eta')$. Starting with $\alpha \in \tilde{E}_G^{\xi'-\zeta'}(W \to X)$ represented by the diagram above, we suspend the diagram by η', obtaining

$$\begin{array}{ccc} \Sigma^{\zeta'+\eta'} W & \dashrightarrow & E_{n+k} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\tilde{\xi}'+\tilde{\eta}']{} & B_G O_{n+k}. \end{array}$$

Now, the bundle map $\xi + \eta \to \xi' + \eta'$ specifies a homotopy class of homotopies from $\tilde{\xi} + \tilde{\eta}$ to $(\tilde{\xi}' + \tilde{\eta}') \circ F$; using the covering homotopy property of fiber bundles, and the G-map $\Sigma^{\zeta+\eta} Z \to \Sigma^{\zeta'+\eta'} W$ specified by $\widetilde{F}$ and the bundle map $\zeta + \eta \to \zeta' + \eta'$, gives us a commutative diagram

$$\begin{array}{ccc} \Sigma^{\zeta+\eta} Z & \dashrightarrow & E_{n+k} \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\tilde{\xi}+\tilde{\eta}]{} & B_G O_{n+k}. \end{array}$$

Finally, we can take adjoints to obtain the diagram

$$\begin{array}{ccc} \Sigma^{\zeta} Z & \dashrightarrow & E_m \\ \downarrow & & \downarrow \\ Y & \xrightarrow[\tilde{\xi}]{} & B_G O_m \end{array}$$

representing an element of $\widetilde{E}_G^{\xi-\zeta}(Y \to X)$ (for details about the suspension and adjunction used, see [E]). This defines the map

$$\widetilde{E}_G^{\xi'-\zeta'}(W \to X) \to E_G^{\xi-\zeta}(Z \to Y).$$

We can write what we just did as defining a sequence of homomorphisms

$$\begin{aligned}\widetilde{E}_G^{\xi'-\zeta'}(W \to X) &\to \widetilde{E}_G^{(\xi'+\eta')-(\zeta'+\eta')}(W \to X)\\ &\to \widetilde{E}_G^{(\xi+\eta)-(\zeta+\eta)}(Z \to Y) \to \widetilde{E}_G^{\xi-\zeta}(Z \to Y).\end{aligned}$$

We now need to show that this construction respects the elementary equivalences given in A.1.(b) above. That it respects the first is easy to see. For the second, suppose given maps $\eta \to \mu$ and $\eta' \to \mu'$ making the diagram in A.1.(b) commute, we can form the following commutative diagram:

$$\begin{array}{ccccc} & & \widetilde{E}_G^{(\xi'+\eta')-(\zeta'+\eta')}(W \to X) & \longrightarrow & \widetilde{E}_G^{(\xi+\eta)-(\zeta+\eta)}(Z \to Y) & & \\ & \nearrow & \downarrow & & \downarrow & \searrow & \\ \widetilde{E}_G^{\xi'-\zeta'}(W \to X) & & & & & & \widetilde{E}_G^{\xi-\zeta}(Z \to Y) \\ & \searrow & & & & \nearrow & \\ & & \widetilde{E}_G^{(\xi'+\mu')-(\zeta'+\mu')}(W \to X) & \longrightarrow & \widetilde{E}_G^{(\xi+\mu)-(\zeta+\mu)}(Z \to Y). & & \end{array}$$

The vertical arrow are defined in the same way that the middle arrow was defined in the step above. One can now check that the outer triangles and middle square of this diagram commute, showing that the paths around the top and the bottom give the same homomorphism.

Finally, to see that this construction respects fiber-homotopy is straightforward, since we are only concerned with homotopy classes in the defining diagram.

For (ii), again we are only concerned with homotopy classes, so this is straightforward. □

One should not expect to make cohomology into a functor on the obvious category of pairs $(Z \to X, \alpha \in KO_G(X))$, as the virtual bundle maps are crucial to the functoriality and there is no canonical way of choosing such maps in this situation.

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