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BLOCH FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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We extend the concept of Bloch functions and Bloch norm in one complex variable to holomorphic functions defined in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ with values in $\mathbb{C}^n$. It is known that in several complex variables the Bloch's Theorem will fail if we do not put some additional restrictions on functions besides the usual normalization of the derivative at the origin. We shall show that many important properties for Bloch functions in one complex variable have analogs for functions in several complex variables. In particular, we generalize Bonk's Distortion Theorem. As applications, we give lower and upper bounds of Bloch constants for various subfamilies of Bloch functions defined in $\mathbb{B}$.

1. Introduction. Let $\mathbb{C}$ be the complex plane. It is known that many results in geometric function theory of one complex variable are no longer true in several complex variables. In particular, Bloch's Theorem fails in several complex variables if there is no additional restriction on the class of functions considered (see [**2**]). K. T. Hahn [**3**] proved that the conclusion of Bloch's Theorem is true for some families of holomorphic functions defined in the unit ball of $\mathbb{C}^n$ with values in $\mathbb{C}^n$ (such as the family of bounded holomorphic functions); he also estimated the Bloch constants for these families. R. M. Timoney [**7**] studied Bloch functions defined in certain domains of $\mathbb{C}^n$ with values in $\mathbb{C}$. Sheng Gong [**8**] also did some work on the Bloch constant in several complex variables.

In this paper, we would like to discuss some properties of Bloch functions defined in the unit ball of $\mathbb{C}^n$ with values in $\mathbb{C}^n$ (some mathematicians may prefer to use the term "Bloch Mapping" instead of Bloch function). In particular, we shall prove a generalization of Bonk's Distortion Theorem for Bloch functions (see [**1**] and [**4**] for Bonk's Distortion Theorem in one complex variable). As applications, we shall give lower bounds of the related Bloch constants. The definition of Bloch functions in this paper is closely related to Timoney's definition.

Let us first introduce some notation. We write a point (or a vector)

$z \in \mathbb{C}^n$ in the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Consider $\mathbb{C}^n$ as an n-dimensional Hilbert space with the inner product and the absolute value given by

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w}_i, \qquad |z| = \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2},$$

where $z, w \in \mathbb{C}^n$. The unit ball of $\mathbb{C}^n$ is the set $\mathbb{B} = \mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$, and the unit sphere of $\mathbb{C}^n$ is denoted by $\partial\mathbb{B} = \{z \in \mathbb{C}^n : |z| = 1\}$.

A function $f : \mathbb{B} \to \mathbb{C}^n$ can also be written in the $n \times 1$ matrix form

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

where each component function f_i is a function from $\mathbb{B}$ to $\mathbb{C}$. Denote by $H(\mathbb{B})$ the family of all holomorphic functions from $\mathbb{B}$ to $\mathbb{C}^n$. The derivative of a function $f \in H(\mathbb{B})$ at a point $a \in \mathbb{B}$ is the following $n \times n$ matrix (the complex Jacobian of f)

$$f'(a) = \left(\frac{\partial f_i}{\partial z_j} \right)_{z=a}.$$

2. Biholomorphic mappings of $\mathbb{B}$ onto $\mathbb{B}$. With an $n \times n$ matrix $A = (a_{ij})$, $a_{ij} \in \mathbb{C}$, we can associate a linear mapping $L_A : \mathbb{C}^n \to \mathbb{C}^n$ by $L_A(z) = Az$, where Az denotes the matrix product of A and z. The matrix A (or the linear mapping L_A) is called unitary if

$$|Az| = |z|$$

for all $z \in \mathbb{B}$. The determinant of an $n \times n$ matrix A is denoted by $\det(A)$, and the absolute value of A (or the norm of L_A) is given by

$$|A| \ (= |L_A|) = \sup\{|Az| : z \in \mathbb{B}\} = \max\{|Az| : z \in \partial\mathbb{B}\}.$$

So a unitary matrix always has absolute value 1. For convenience, we sometimes do not distinguish between a matrix and the associated linear mapping.

Denote by $\mathrm{Aut}(\mathbb{B})$ the group of all biholomorphic mappings of $\mathbb{B}$ onto $\mathbb{B}$. An element of $\mathrm{Aut}(\mathbb{B})$ is sometimes called a biholomorphic automorphism of $\mathbb{B}$. Obviously, $L_A \in \mathrm{Aut}(\mathbb{B})$ if and only if L_A is unitary.

For $a \in \mathbb{B}$, define $\varphi_a : \mathbb{B} \to \mathbb{C}^n$ by

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z , a\rangle},$$

where

$$P_a z = \frac{\langle z , a\rangle}{|a|^2} a$$

is the projection of $\mathbb{C}^n$ onto the plane $\{z \in \mathbb{C}^n : z = ua \text{ for some } u \in \mathbb{C}\}$ (for $a = 0$ we let $P_a = 0$), and $Q_a = I - P_a$ is the projection of $\mathbb{C}^n$ onto the orthogonal complement of the above plane, where I denotes the identity mapping or the identity matrix. The results in the following lemma can be found in Rudin's book [**6**, pp. 23–30].

LEMMA 1. (1) *For any* $a \in \mathbb{B}$, $\varphi_a \in \mathrm{Aut}(\mathbb{B})$ *and* $\varphi_a \circ \varphi_a = I$.

(2) $\varphi_a(0) = a$, $\varphi_a(a) = 0$, *and*

$$\varphi_a'(0) = -(1 - |a|^2) P_a - (1 - |a|^2)^{1/2} Q_a.$$

(3) $\mathrm{Aut}(\mathbb{B}) = \{L_A \circ \varphi_a : a \in \mathbb{B}$ *and* A *is a unitary matrix*$\} = \{\varphi_b \circ L_B : b \in \mathbb{B}$ *and* B *is a unitary matrix*$\}$.

(4) *If* $\varphi \in \mathrm{Aut}(\mathbb{B})$, *then*

$$|\det \varphi'(z)| = [(1 - |\varphi(z)|^2)/(1 - |z|^2)]^{(n+1)/2}.$$

Lemma 1 will be used repeatedly in the proofs of this paper.

3. Bloch functions in $\mathbb{B}$. The ball in $\mathbb{C}^n$ with center w_0 and radius r is the set

$$B(w_0 , r) = \{w \in \mathbb{C}^n : |w - w_0| < r\}.$$

For $f \in H(\mathbb{B})$, a schlicht ball of f centered at $f(a)$ is a ball with center $f(a)$ such that f maps an open subset of $\mathbb{B}$ containing a biholomorphically onto this ball. For a point $a \in \mathbb{B}$, let $r(a , f)$ be the radius of the largest schlicht ball of f centered at $f(a)$, and $r(f) = \sup\{r(a , f) : a \in \mathbb{B}\}$.

It is known that the condition $\det(f'(z_0)) \neq 0$ $(z_0 \in \mathbb{B})$ is equivalent to that f is locally schlicht (or locally injective) at the point z_0.

DEFINITION. A point z_0 is called a critical point of f if $\det(f'(z_0)) = 0$, and $f(z_0)$ is called a critical value of f (in one complex variable, such a value is called a branch point). We call a point $w_0 \in \mathbb{C}^n$ a boundary point of $f(\mathbb{B})$ if there is a sequence $\{z_n\}$ in $\mathbb{B}$ such $\{z_n\}$ has no limit point in $\mathbb{B}$ and the sequence $\{f(z_n)\}$ converges to w_0. The set of all boundary points of $f(\mathbb{B})$ is denoted by $\partial(f(\mathbb{B}))$. Note that in general $\partial(f(\mathbb{B}))$ is not the topological boundary of $f(\mathbb{B})$ in $\mathbb{C}^n$.

LEMMA 2. *Suppose $f \in H(\mathbb{B})$, G is an open subset of $\mathbb{B}$, and $a \in G$. If f maps G biholomorphically onto the schlicht ball $B(f(a), r(a, f))$, then either G and $\mathbb{B}$ have a common boundary point or there is a critical value $f(z_0)$ on the boundary of the ball $B(f(a), r(a, f))$ with the critical point z_0 on the boundary of G. Therefore, $r(a, f)$ equals the euclidean distance from $f(a)$ to a boundary point of $f(\mathbb{B})$ or to a critical value of f.*

Proof. Suppose G and $\mathbb{B}$ have no common boundary point. Then there exists a family $\{G_n\}$ of open subsets of $\mathbb{B}$, such that $G_n \supset \overline{G}$, $G_n \supset \overline{G}_{n+1}$, and

$$\bigcap\{G_n\colon n = 1, 2, \dots\} = \overline{G}.$$

By definition of $B(f(a), r(a, f))$, we can find z_n and z'_n in G_n such that $z_n \neq z'_n$ and $f(z_n) = f(z'_n)$. Without loss of generality, we may assume that $\{z_n\}$ converges to $z \in \partial G$ and $\{z'_n\}$ converges to $z' \in \partial G$; here ∂G denotes the topological boundary of G. By continuity, $f(z)$ and $f(z')$ lie on the boundary of $B(f(a), r(a, f))$, and $f(z) = f(z')$.

If $z = z'$, then it is easy to see that $f(z)$ is a critical value of f with the critical point z on the boundary of G.

Suppose $z \neq z'$. If both z and z' are not critical points of f, then the equality $f(z) = f(z')$ and the local injectivity of f at the two points z and z' will imply that f is not injective (schlicht) in G. This is impossible.

EXAMPLE (Duren and Rudin). For $n = 2$ and $\delta > 0$, set

$$f_\delta(z_1, z_2) = \begin{pmatrix} z_1 \\ z_2 + (\frac{z_1}{\delta})^2 \end{pmatrix},$$

then $f_\delta \in H(\mathbb{B})$, and $f'_\delta(0) = I$. P. Duren and W. Rudin proved that the image of f does not contain any ball of radius δ, so $r(f_\delta) \leq \delta$ (see [2] for details).

Recall that the Bloch constant in one complex variable is defined by

$$B = \inf\{r(f)\colon f \in H(\mathbb{D}) \text{ and } f'(0) = 1\},$$

where $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$. The above example shows that for $n \geq 2$, $\inf\{r(f) \colon f \in H(\mathbb{B}), \text{ and } f'(0) = I\} = 0$. Hence we should not define the Bloch constant in several complex variables to be $\inf\{r(f) \colon f \in H(\mathbb{B}) \text{ and } f'(0) = I\}$. What we will do is place some reasonable restrictions on functions in $H(\mathbb{B})$ so that the restricted family will have a positive Bloch constant.

DEFINITION. A function $f \in H(\mathbb{B})$ is called a Bloch function if the family

$$\mathscr{F}_f = \{g \colon g(z) = f(\varphi(z)) - f(\varphi(0)) \text{ for some } \varphi \in \operatorname{Aut}(\mathbb{B})\}$$

is a normal family.

Hence a holomorphic function f in $\mathbb{B}$ is a Bloch function if and only if each of its component functions is a Bloch function according to Timoney's definition [7]. Recall that a normal family is a family of holomorphic functions defined in a certain domain such that any sequence of functions in this family has a subsequence which converges uniformly on compact subsets of this domain. It is not difficult to see that in the case $n = 1$, the above definition is the usual definition for Bloch functions. The following theorem shows that this definition preserves some important properties of Bloch functions of one complex variable.

THEOREM 1. *Suppose* $f \in H(\mathbb{B})$. *Then*

(1) f *is a Bloch function implies that the quantity*

$$\|f\| = \sup\{|(f \circ \varphi)'(0)| \colon \varphi \in \operatorname{Aut}(\mathbb{B})\}$$

is finite.

(2) $\|f\|$ *is finite implies that* f *is a Bloch function and*

$$|f'(z)| \leq \|f\|/(1 - |z|^2)$$

for all $z \in \mathbb{B}$.

(3) *If* $0 < K < \infty$ *and* $|f'(z)| \leq K/(1 - |z|^2)^{1/2}$, *then* f *is a Bloch function and*

$$\|f\| \leq K.$$

(4) $\|f \circ \varphi\| = \|f\|$ *for all* $\varphi \in \operatorname{Aut}(\mathbb{B})$.

Proof. (1). This is obvious because $\mathscr{F}_f$ is a normal family and any function $g \in \mathscr{F}_f$ satisfies $g(0) = 0$ imply that the set

$$\{|(f \circ \varphi)'(0)| \colon \varphi \in \operatorname{Aut}(\mathbb{B})\} = \{|g'(0)| \colon g \in \mathscr{F}_f\}$$

is bounded.

(2) Let $z \in \mathbb{B}$. By Lemma 1, we have $|(\varphi_z'(0))^{-1}| \le 1/(1-|z|^2)^{-1}$. For any $\varphi \in \mathrm{Aut}(\mathbb{B})$,

$$|(f \circ \varphi)'(z)| = |[(f \circ (\varphi \circ \varphi_z))'(0)](\varphi_z'(0))^{-1}| \le \|f\|/(1-|z|^2).$$

Thus, $\{|(f \circ \varphi)'(z)| \colon \varphi \in \mathrm{Aut}(\mathbb{B})\}$ is locally uniformly bounded which implies that $\mathscr{F}_f$ is a normal family. By setting $\varphi = I$ in the above inequality we obtain

$$|f'(z)| \le \|f\|/(1-|z|^2),$$

so (2) is proved.

(3) Let $\varphi \in \mathrm{Aut}(\mathbb{B})$. From Lemma 1, $\varphi = L \circ \varphi_a$ for some $a \in \mathbb{B}$, where L is a unitary linear mapping, and

$$|\varphi'(0)| = |\varphi_a'(0)| \le (1-|a|^2)^{1/2} = (1-|\varphi(0)|^2)^{1/2}.$$

Therefore,

$$|(f \circ \varphi)'(0)| \le |f'(\varphi(0))|\,|\varphi'(0)| \le |f'(\varphi(0))|(1-|\varphi(0)|^2)^{1/2} \le K < \infty.$$

Hence f is a Bloch function with $\|f\| \le K$.

(4) This follows from the fact that $\mathrm{Aut}(\mathbb{B})$ is a group.

Remark. Theorem 1 shows that f is a Bloch function if and only if $\|f\| < \infty$. We define the quantity $\|f\|$ to be the Bloch norm of f. We should point out that the Bloch norm is only a semi-norm in the linear algebra sense.

Now we want to establish a relation between this Bloch norm we just defined and the Bloch norm in [7] defined by using the Bergman metric on $\mathbb{B}$. Let $Q_f(z)$ $(z \in \mathbb{B})$ be given by

$$Q_f(z) = \sup\{|f'(z)x| \colon x \in \mathbb{C}^n \text{ and } H_z(x, x) = 1\},$$

where H_z is the Bergman metric on $\mathbb{B}$, that is,

$$H_z(u, v) = \frac{(n+1)((1-|z|^2)\langle u, v\rangle + \langle u, z\rangle\langle z, v\rangle)}{2(1-|z|^2)^2}$$

for all $u, v \in \mathbb{C}^n$.

Theorem 2. *For any* $z \in \mathbb{B}$,

$$|(f \circ \varphi)'(0)| = \sqrt{(n+1)/2}\,Q_f(z)$$

for all $\varphi \in \mathrm{Aut}(\mathbb{B})$ *satisfying* $\varphi(0) = z$. *Therefore,*

$$\|f\| = \sqrt{(n+1)/2}\,\sup\{Q_f(z) \colon z \in \mathbb{B}\}.$$

Proof. Suppose $\varphi \in \mathrm{Aut}(\mathbb{B})$ and $z = \varphi(0)$. By Lemma 1, $\varphi = \varphi_z \circ L$, where L is a unitary linear mapping.

If $u \in \partial\mathbb{B}$, then $v = Lu \in \partial\mathbb{B}$. Write v in the form $v = tz + w$, where $t \in \mathbb{C}$, $w \in \mathbb{C}^n$, and $\langle z, w\rangle = 0$. Let $x = \varphi_z'(0)v = \varphi'(0)u$. By Lemma 1 (2),

$$\varphi_z'(0)z = -(1-|z|^2)z, \qquad \varphi_z'(0)w = -(1-|z|^2)^{1/2}w.$$

By calculation,

$$x = -(1-|z|^2)tz - (1-|z|^2)^{1/2}w,$$

and

$$(f\circ\varphi)'(0)u = f'(z)x, \qquad H_z(x, x) = (n+1)/2.$$

Conversely, for any $x \in \mathbb{C}^n$ with $H_z(x, x) = (n+1)/2$, we can verify that

$$u = (\varphi'(0))^{-1}x \in \partial\mathbb{B}.$$

Hence

$$\begin{aligned} |(f\circ\varphi)'(0)| &= \sup\{|(f\circ\varphi)'(0)u| : |u| = 1\} \\ &= \sup\{|f'(z)x| : H_z(x, x) = (n+1)/2\} \\ &= \sqrt{(n+1)/2}\,Q_f(z). \end{aligned}$$

The quantity $\sup\{Q_f(z) : z \in \mathbb{B}\}$ is the Bloch norm defined in [7] except that the gradient of f has been replaced by $f'(z)$ in the definition of $Q_f(z)$. By using Theorem 2 we can calculate (or estimate) the Bloch norm of a holomorphic function in $H(\mathbb{B})$ explicitly.

It is easy to see that when $n = 1$, $\|f\| = \sup\{Q_f(z) : z \in \mathbb{B}\}$ reduces to the usual Bloch norm. Recall that a holomorphic function f defined in the unit disk $\mathbb{D}$ of the complex plane is a Bloch function if and only if

$$\sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty.$$

One interesting question is: Can we use the above inequality to define Bloch functions when $n \geq 2$? The next theorem will show that the answer is affirmative.

LEMMA 3. (1) *Suppose $A = (a_{ij})$, $a_{ij} \in \mathbb{C}$, is an $n \times n$ matrix. If there is a point $x \in \mathbb{C}^n$ and a constant $K > 0$, such that*

$$|Ax| \leq K|x| \quad \text{and} \quad |Ay| \leq K|y|$$

for all $y \in \mathbb{C}^n$ satisfying $\langle x, y\rangle = 0$, then $|A| \leq \sqrt{2}K$.

(2) *Suppose* $f \in H(\mathbb{B})$. *Let*

$$m(f) = \sup\{(1-|z|^2)|f'(z)|: z \in \mathbb{B}\}.$$

Then there is an absolute constant $c \geq 1$, *such that for any* $z \in \mathbb{B}$,

$$|f'(z)z| \leq m(f)|z|/(1-|z|^2),$$

and

$$|f'(z)w| \leq cm(f)|w|/(1-|z|^2)^{1/2}$$

for any $w \in \mathbb{C}^n$ *satisfying* $\langle z, w\rangle = 0$.

Proof. (1) For any $\zeta \in \mathbb{C}^n$, we can write ζ as

$$\zeta = ux + y,$$

where $u \in \mathbb{C}$, $y \in \mathbb{C}^n$, and $\langle x, y\rangle = 0$. Hence,

$$|A\zeta| = |uAx + Ay| \leq |u|\,|Ax| + |Ay| \leq K(|u|\,|x| + |y|) \leq \sqrt{2}K|\zeta|,$$

therefore $|A| \leq \sqrt{2}K$.

(2) The proof of (2) is similar to the proofs of Theorem 4.7 and Lemma 4.8 in Timoney's paper [7]. The inequality

$$|f'(z)z| \leq m(f)|z|/(1-|z|^2)$$

follows from the definition of $m(f)$. We need only to show that there is an absolute constant $c \geq 1$ such that

$$|f'(z)w| \leq cm(f)/(1-|z|^2)^{1/2}$$

for all $w \in \mathbb{C}^n$ satisfying $\langle z, w\rangle = 0$ and $|w| = 1$.

Define $g: \mathbb{C}^2 \to \mathbb{C}^n$ by

$$g(u, v) = f(uz + vw).$$

Then

$$\frac{\partial g}{\partial u}(u, v) = f'(uz + vw)z.$$

By Cauchy's Integral Formula,

$$\frac{\partial^2 g}{\partial u \partial v}(u, 0) = \frac{1}{2\pi i}\int_{|v|=\alpha} \frac{f'(uz + vw)z}{v^2}\,dv.$$

By setting

$$\alpha = \frac{1}{\sqrt{3}}(1-|uz|^2)^{1/2}$$

in the above equality, we can get

$$\left|\frac{\partial^2 g}{\partial u \partial v}(u, 0)\right| \leq \frac{3\sqrt{3}|z|m(f)}{2(1-|uz|^2)^{3/2}}.$$

Here we have used the fact

$$(1 - |uz|^2 - |v|^2)|f'(uz + vw)z|$$
$$= (1 - |uz + vw|^2)|f'(uz + vw)z| \le m(f)|z|.$$

By calculation,

$$\left|\frac{\partial g}{\partial v}(1, 0) - \frac{\partial g}{\partial v}(0, 0)\right| = \left|\int_0^1 \frac{\partial^2 g}{\partial u \partial v}(t, 0)\, dt\right|$$
$$\le \int_0^1 \frac{3\sqrt{3}|z|m(f)}{2(1 - |tz|^2)^{3/2}}\, dt = \frac{3\sqrt{3}|z|m(f)}{2(1 - |z|^2)^{1/2}}.$$

Therefore,

$$|f'(z)w| = \left|\frac{\partial g}{\partial v}(1, 0)\right| \le \frac{3\sqrt{3}|z|m(f)}{2(1 - |z|^2)^{1/2}} + \left|\frac{\partial g}{\partial v}(0, 0)\right|$$
$$= \frac{3\sqrt{3}|z|m(f)}{2(1 - |z|^2)^{1/2}} + |f'(0)w|$$
$$\le \left(\frac{3\sqrt{3}|z|}{2(1 - |z|^2)^{1/2}} + 1\right) m(f) \le \frac{\sqrt{31}}{2} m(f)/(1 - |z|^2)^{1/2}.$$

Hence we can choose $c = \sqrt{31}/2$ and (2) is proved.

THEOREM 3. *There is an absolute constant* $C > 0$ *such that*

$$m(f) \le \|f\| \le Cm(f)$$

for all $f \in H(\mathbb{B})$. *Therefore,* f *is a Bloch function if and only if*

$$m(f) = \sup\{(1 - |z|^2)|f'(z)|\colon z \in \mathbb{B}\} < \infty.$$

Proof. The inequality $m(f) \le \|f\|$ follows from Theorem 1(2).

Given $\varphi \in \mathrm{Aut}(\mathbb{B})$, by Lemma 1, $\varphi = \varphi_z \circ L$, where $z \in \mathbb{B}$ and L is a unitary linear mapping. Further, $\varphi(0) = \varphi_z(0) = z$ and $\varphi_z'(0) = -(1 - |z|^2)P_z - (1 - |z|^2)^{1/2}Q_z$.

Let $x = L^{-1}(z)$. Then by Lemma 3(2),

$$|(f \circ \varphi)'(0)x| = |f'(z)(\varphi_z'(0)z)|$$
$$= (1 - |z|^2)|f'(z)z| \le m(f)|z| = m(f)|x|.$$

For any $y \in \mathbb{C}^n$ satisfying $\langle x, y\rangle = 0$, let $w = L(y)$, then $\langle z, w\rangle = \langle x, y\rangle = 0$. Lemma 3(2) shows that there is an absolute constant $c \ge 1$, such that

$$|(f \circ \varphi)'(0)y| = |f'(z)(\varphi_z'(0)w)|$$
$$= (1 - |z|^2)^{1/2}|f'(z)w| \le cm(f)|w| = cm(f)|y|.$$

By Lemma 3(1), $|(f\circ\varphi)'(0)| \le \sqrt{2}cm(f)$, and hence $\|f\| \le Cm(f)$ with $C = \sqrt{2}$. In fact, the proof of Lemma 2 gives $C = \sqrt{2}c = \sqrt{31/2} < 4$.

4. Bloch constant for functions defined in $\mathbb{B}$. Now we are ready to define the Bloch constant for various subfamilies of Bloch functions in $\mathbb{B}$.

DEFINITION. Let $\mathscr{B}(K) = \{f \in H(\mathbb{B}): \|f\| \le K\}$, $1 \le K < \infty$. Define

$$B(K) = \inf\{r(f): f \in \mathscr{B}(K) \text{ and } \det(f'(0)) = 1\}.$$

We call $B(K)$ the Bloch constant of order K.

We have added the restriction $\|f\| \le K$ in the above definition of the Bloch constant. Later we shall see that $B(K)$ is a positive constant which depends on K when $n \ge 2$. Therefore, the Bloch norm $\|\ \|$ plays an essential role in the estimation of $B(K)$. It turns out that there is another quantity which is also important. This quantity is given by

$$\begin{aligned}\|f\|_0 &= \sup\{|\det(g'(0))|^{1/n}: g \in \mathscr{F}_f\}\\ &= \sup\{(1-|z|^2)^{(n+1)/(2n)}|\det(f'(z))|^{1/n}: z \in \mathbb{B}\}.\end{aligned}$$

The second equality follows from the identity

$$|\det(\varphi'(0))| = (1-|\varphi(0)|^2)^{(n+1)/2}, \qquad \varphi \in \operatorname{Aut}(\mathbb{B}),$$

which is a special case of Lemma 1(4). We call $\|f\|_0$ the prenorm of f. I would like to thank the referee for pointing out that the quantity $\|\ \|_0$ does satisfy the triangle inequality.

THEOREM 4. *If $f \in H(\mathbb{B})$, then*

(1) $\|f\|_0 \le \|f\|$, *and equality holds when* $n = 1$.

(2) $\|f\circ\varphi\|_0 = \|f\|_0$ *for all* $\varphi \in \operatorname{Aut}(\mathbb{B})$.

(3) $\|f\|_0 = 1$ *and* $\det(f'(0)) = 1$ *implies that* $|\det(f'(z))| = 1 + o(|z|)$.

(4) *For the case $n \ge 2$, there is a constant $c = c_n > 0$ such that $r(f) \le c\|f\|_0$ for all $f \in H(\mathbb{B})$.*

Proof. (1) is true because the inequality $|A|^n \ge |\det(A)|$ holds for any $n\times n$ matrix A.

(2) follows because $\operatorname{Aut}(\mathbb{B})$ is a group.

(3) is an immediate consequence of the inequality

$$|\det(f'(z))| \le (1-|z|^2)^{-(n+1)/2}.$$

To prove (4), let $B(a)$ be the ball centered at $f(a)$ with radius $r(a, f)$, V_n be the volume of $\mathbb{B}$, and

$$d_n = \int_{\mathbb{B}} (1 - |z|^2)^{-(n+1)} \, dv(z),$$

where the integral is taken with respect to volume. Then $0 < c = (d_n/V_n)^{1/(2n)} < \infty$, and

$$\begin{aligned} V_n r(a, f)^{2n} &= \text{the volume of } B(a) \\ &\leq \int_{\mathbb{B}} |\det(f'(z))|^2 \, dv(z) \leq \|f\|_0^{2n} d_n, \end{aligned}$$

so $r(a, f) \leq c\|f\|_0$ for all $a \in \mathbb{B}$ and (4) is proved.

REMARK. Theorem 4(4) implies that $\|f\|_0 = \|f\| = \infty$ when $r(f) = \infty$ and $n \geq 2$. But unlike the case $n = 1$, the converse is not true. For example, let $F \in H(\mathbb{B})$ be the function satisfying

$$F(0) = 0, \qquad F'(z) = \begin{pmatrix} 1/(1 - z_1)^{n+1} & 0 \\ 0 & I_{n-1} \end{pmatrix},$$

where I_{n-1} is the identity matrix of rank $n - 1$. Then $\|F\|_0 = \|F\| = \infty$ and $r(F) \leq 1 < \infty$.

The next theorem is a $\mathbb{C}^n$ version of Bonk's Distortion Theorem (see [**1**] and [**4**]).

THEOREM 5. *If* $f \in H(\mathbb{B})$, $\|f\|_0 = 1$, *and* $\det(f'(0)) = 1$, *then*

$$|\det(f'(z))| \geq \operatorname{Re}\{\det(f'(z))\} \geq (1 - \sqrt{n+2}|z|)/(1 - |z|/\sqrt{n+2})^{n+2}$$

for all z *satisfying* $|z| \leq 2\sqrt{n+2}/(n+3)$. *The above inequality is best possible.*

Proof. For any $\zeta \in \partial\mathbb{B}$, define a holomorphic function $g\colon \mathbb{D} \to \mathbb{C}$ by

$$g(u) = (1 - a_n T(u))^{(n+1)} \det(f'(T(u)\zeta)),$$

where

$$T(u) = a_n(1 - u)/(1 - a_n^2 u), \qquad a_n = 1/(n+2)^{1/2}.$$

Note that T maps $\mathbb{D}$ onto the disk $D = \{w : |1 - a_n w|^2 < 1 - |w|^2\} \subset \mathbb{D}$. By calculation, $g(1) = 1$. From Theorem 4(3), $g'(1) = 1$. Since $\|f\|_0 = 1$ and $T(\mathbb{D}) = D$, we have

$$\begin{aligned} |g(u)| &= |1 - a_n T(u)|^{(n+1)} |\det(f'(T(u)\zeta))| \\ &\leq (1 - |T(u)|^2)^{(n+1)/2} |\det(f'(T(u)\zeta))| \leq 1 \end{aligned}$$

for all $u \in \mathbb{D}$. So $g(\mathbb{D}) \subset \mathbb{D}$. The classical Julia Lemma (see **[4]**) shows that g maps the horodisk

$$\Delta(r) = \{u \in \mathbb{C}: (|1-u|^2/(1-|u|^2)) < r\} \qquad (r > 0)$$

of $\mathbb{D}$ into itself. In particular,

$$\operatorname{Re} g(u) \geq u \quad \text{for all } u \in [-1, 1],$$

which is equivalent to

$$\operatorname{Re}\{\det(f'(v\zeta))\} \geq (1 - \sqrt{n+2}v)/(1 - v/\sqrt{n+2})^{n+2}$$

for all v satisfying $0 \leq v \leq 2\sqrt{n+2}/(n+3)$. This inequality is best possible because there is a function $F \in H(\mathbb{B})$ satisfying $F(0) = 0$ and

$$F'(z) = \begin{pmatrix} \frac{1-\sqrt{n+2}z_1}{(1-z_1/\sqrt{n+1})^{n+2}} & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

It is not difficult to verify that $\|F\|_0 = 1$, and $\det(F'(0)) = 1$.

LEMMA 4. *Suppose $A = (a_{ij})$ is an $n \times n$ matrix. If $|A| > 0$, then for any unit vector $\zeta \in \partial\mathbb{B}$, the following inequality holds:*

$$|A\zeta| \geq |\det(A)|/|A|^{n-1}.$$

Proof. If $A^* = (\overline{a_{ji}})$, then the product A^*A is a positive semi-definite matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ $(0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n)$ be the n eigenvalues of the matrix A^*A. Then

$$(\lambda_n)^{1/2} = \max\{|Av|: v \in \partial\mathbb{B}\} = |A|, \quad (\lambda_1)^{1/2} = \min\{|Av|: v \in \partial\mathbb{B}\},$$

and

$$|\det(A)| = (\lambda_1\lambda_2\cdots\lambda_n)^{1/2}.$$

Hence,

$$|A\zeta| \geq (\lambda_1)^{1/2} \geq |\det(A)|/|A|^{n-1}.$$

THEOREM 6. *If $f \in \mathscr{B}(K)$, $\|f\|_0 = 1$, and $\det(f'(0)) = 1$, then*

$$\begin{aligned} r(0, f) &\geq C(K, n) \\ &= K^{1-n}\int_0^{1/\sqrt{n+2}} [(1-t^2)^{n-1}(1-\sqrt{n+2}t)/(1-t/\sqrt{n+2})^{n+2}]\,dt \\ &\geq K^{1-n}\frac{1}{e}\frac{\sqrt{n+2}}{n}\left(\left(1+\frac{1}{n+1}\right)^{n+1} - 2\right). \end{aligned}$$

Proof. We have seen in Lemma 2 that $r(0, f)$ equals either the euclidean distance from $f(0)$ to a boundary point of $f(\mathbb{B})$ or the euclidean distance from $f(0)$ to a critical value of f. Hence there exists a line segment Γ of euclidean length $r(0, f)$ from $f(0)$ to a point in $\partial(f(\mathbb{B}))$ or to a critical value of f. Let γ be the inverse image of Γ under the mapping f. Then by Theorem 5, γ is a smooth curve from 0 to $\partial\mathbb{B}$ or to a point $z_0 \in \mathbb{B}$ with $|z_0| \geq 1/\sqrt{n+2}$ and

$$\det(f'(z_0)) = 0.$$

By Lemma 4,

$$\begin{aligned} r(0, f) &= \left|\int_\Gamma dw\right| = \int_\Gamma |dw| = \int_\gamma |f'(z)\,dz| = \int_\gamma \left|f'(z)\frac{dz}{|dz|}\right|\,|dz| \\ &\geq \int_\gamma \frac{|\det(f'(z))|}{|f'(z)|^{n-1}}\,d|z| \geq \int_0^{1/\sqrt{n+2}} \frac{|\det(f'(z)|}{|f'(z)|^{n-1}}\,d|z|. \end{aligned}$$

Theorem 5 and the inequality $|f'(z)| \leq K/(1-|z|^2)$ (see Theorem 1(2)) imply that the last quantity is not less than $C(K, n)$.

Example. Let $F_K \in H(\mathbb{B})$ be the function satisfying $F_K(0) = 0$ and

$$F_K'(z) = \begin{pmatrix} \frac{(1-\sqrt{n+2}z_1)}{K^{n-1}(1-z_1/\sqrt{n+2})^{n+2}} & 0 \\ 0 & KI_{n-1} \end{pmatrix}.$$

For K sufficiently large, we have

$$|F_K'(z)| \leq K/(1-|z|^2)^{1/2},$$

thus $F_K \in \mathscr{B}(K)$ by Theorem 1(3). By direct calculations, we obtain

$$\|F_K\|_0 = 1, \qquad \det(F_K'(0)) = 1,$$

and

$$r(0, F_K) \leq eC(K, n).$$

Therefore, the lower bound on $r(0, f)$ in Theorem 6 is a reasonable estimate; it is not off by more than a factor of e.

Corollary. $K^{1-n} \geq B(K) \geq C(K, n)$.

Proof. By definition, there are functions $g_m \in \mathscr{B}(K)$ $(m = 1, 2, \dots)$ such that

$$\det(g_m'(0)) = 1, \qquad r(g_m) \to B(K).$$

Note that $1 \leq \|g_m\|_0 \leq K$. Let $h_m = g_m/\|g_m\|_0$, then

$$h_m \in \mathscr{B}(K), \quad \|h_m\|_0 = 1.$$

A standard normal family argument and Theorem 4(2) imply that

$$B(K) = \inf\{r(f)\colon f \in \mathscr{B}(K) \text{ and } \|f\|_0 = 1\}.$$

Therefore,

$$r(g_m) \geq r(h_m) \to B(K).$$

For each m, let

$$\mathscr{F}_m = \{f\colon f(z) = h_m(\varphi(z)) - h_m(\varphi(0)) \text{ for some } \varphi \in \operatorname{Aut}(\mathbb{B})\}.$$

By Theorem 1(4) and Theorem 4(2), all functions in $\mathscr{F}_m$ have the same prenorm $\| \ \|_0$ and the same Bloch norm $\| \ \|$ as h_m does. Since h_m is a Bloch function, $\mathscr{F}_m$ is a normal family. Hence we can find a function $f_m \in \mathscr{B}(K)$, such that

$$r(f_m) = r(h_m) \to B(K), \qquad \|f_m\|_0 = \|h_m\|_0 = 1,$$

and

$$\det(f_m'(0)) = 1.$$

Theorem 6 gives

$$B(K) = \lim r(f_m) \geq \lim r(0, f_m) \geq C(K, n).$$

The upper bound is obtained from the function

$$F(z) = \begin{pmatrix} K^{1-n} z_1 \\ K z_2 \\ \vdots \\ K z_n \end{pmatrix}.$$

REMARK. (1) Note that $C(K, 1) = \sqrt{3}/4$; this is the well-known lower bound of the classical Bloch constant obtained by L. V. Ahlfors in 1937 (M. Bonk [**1**] has recently improved the lower bound of the classical Bloch constant to $\sqrt{3}/4 + 10^{-14}$).

(2) Let B be the classical Bloch constant (for holomorphic functions of one complex variable). When $K \geq [2\sqrt{3}/3]^{1/n}$, we can improve the upper bound in the above corollary to

$$K^{1-n} B \geq B(K).$$

To demonstrate this, let $H(\mathbb{D})$ be the family of all holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$, then we have the following well-known identity which was proved by E. Landau in 1929:

$$B = \inf\{r(f) \colon f \in H(\mathbb{D}),\ \|f\| = 1,\ \text{ and } f'(0) = 1\}.$$

Note that (since $n = 1$) in the above identity,

$$r(f) = \sup\{\text{radii of schlicht disks of } f\},$$

and $\|f\|$ is the usual Bloch norm of f.

Suppose $f \in H(\mathbb{D})$, $\|f\| = 1$, and $f'(0) = 1$. We need only to show that

$$K^{1-n} r(f) \geq B(K).$$

Define $F_f \in H(\mathbb{B})$ by

$$F_f(z) = \begin{pmatrix} K^{1-n} f(z_1) \\ K z_2 \\ \vdots \\ K z_n \end{pmatrix}.$$

It is elementary but lengthy to verify that $\|F_f\| = K$ when $K \geq [2\sqrt{3}/3]^{1/n}$ (see **[5]** for details). Also, $\det(F_f'(0)) = 1$. Hence $F_f \in \mathscr{B}(K)$, and

$$K^{1-n} r(f) \geq r(F_f) \geq B(K).$$

In fact, if $\|F_f\| \leq K$ is true for all $K \geq 1$, then by the same argument, $B(K) \leq K^{1-n} B$ will be true for all $K \geq 1$.

5. Corresponding results for locally schlicht functions. We know that a function $f \in H(\mathbb{B})$ is locally schlicht in $\mathbb{B}$ if and only if $\det(f'(z))$ does not vanish in $\mathbb{B}$. Recall that in the geometric function theory of one complex variable, the locally schlicht Bloch constant is defined to be

$$B_0 = \inf\{r(f) \colon f \in H(\mathbb{D}),\ f'(0) = 1,\ \text{ and } f'(z) \neq 0 \text{ for all } z \in \mathbb{D}\}.$$

Liu and Minda **[4]** established a sharp distortion theorem (an analog of Bonk's Distortion Theorem) for locally schlicht Bloch functions of one complex variable. For several complex variables, the analogs of Theorem 5 and Theorem 6 (and its corollary) hold. We will omit the proofs because the ideas are the same (see **[5]** for details).

For $1 \leq K < \infty$, define

$$\mathscr{B}_0(K) = \{f \in \mathscr{B}(K) \colon \det(f'(z)) \neq 0 \text{ for all } z \in \mathbb{B}\},$$
$$B_0(K) = \inf\{r(f) \colon f \in \mathscr{B}_0(K),\ \text{ and } \det(f'(0)) = 1\}.$$

THEOREM 7. *Suppose* $f \in H(\mathbb{B})$, $\|f\|_0 = 1$, *and* $\det(f'(0)) = 1$. *If* $\det(f'(z)) \neq 0$ *for all* $z \in \mathbb{B}$, *then*

$$|\det(f'(z))| \geq (1 - |z|)^{-(n+1)} \exp\{-(n+1)|z|/(1-|z|)\}$$

for all $z \in \mathbb{B}$. *This inequality is best possible.*

The following function is an extremal function for Theorem 7.

$$F(z) = \begin{pmatrix} \frac{1}{(1-z_1)^{n+1}} \exp\{-(n+1)z_1/(1-z_1)\} \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

THEOREM 8. *If* $f \in \mathscr{B}_0(K)$ *satisfies the conditions of Theorem* 7, *then*

$$r(0, f) \geq C_0(K, n) = K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{n+1}} \exp\{-(n+1)t/(1-t)\}\, dt.$$

COROLLARY. $K^{1-n} \geq B_0(K) \geq C_0(K, n)$.

REMARK. (1) $C_0(K, 1) = 1/2$ is currently the largest known lower bound for the locally schlicht Bloch constant of one complex variable.

(2) By the same argument as in §4, $B_0(K) \leq K^{1-n} B_0$ is true when $K \geq [2\sqrt{3}/3]^{1/n}$.

(3) Finally, we would like to point out that all of the results in this paper have analogs for holomorphic functions defined in the unit polydisc of $\mathbb{C}^n$, see **[5]**.

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REFERENCES

[1] M. Bonk, *On Bloch's constant*, Proc. Amer. Math. Soc., **110** (1990), 889–894.

[2] P. Duren and W. Rudin, *Distortion in several variables*, Complex Variables, **5** (1986), 323–326.2.

[3] K. T. Hahn, *Higher diimensional generalizations of the Bloch constant and their lower bounds*, Trans. Amer. Math. Soc., **179** (1973), 263–274.

[4] X. Liu and D. Minda, *Distortion theorems for Bloch functions*, Transl. Amer. Math. Soc., to appear.

[5] X. Liu, *Topics in geometric function theory*, Ph.D. Dissertation, University of Cincinnati, in preparation.

[6] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[7] R. M. Timoney, *Bloch functions in several complex variables* I, Bull. London Math. Soc., **12** (1980), 241–267.

[8] Sheng Gong, private communication.

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