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SOME MORE STATES MODELS FOR LINK INVARIANTS

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In this paper I present two new states models for a specialization of the Kauffman link invariant $F(a, z)$ and show that these models occur naturally together. The particular specialization we are concerned with has been investigated by Lickorish and Millett and identified with a probability generating function. Kauffman has also found a description (different from the one given here) of this evaluation of $F(a, z)$.

1. The Jones polynomial $V(t)$.

We begin by briefly recalling Kauffman's states model for the Jones polynomial $V(t)$, upon which the results of this paper are modelled (see [1], [3] for more details). It is interesting to compare this with the models obtained below, especially in view of the fact that $V(t)$ may be obtained from the Kauffman link invariant $F(a, z)$ of [3] by making the right substitutions for a and z. In fact, $V(t) = F(t^{-3/4}, -(t^{-1/4} + t^{1/4}))$ (see [4]).

We start by defining the bracket polynomial $\langle\mathscr{D}\rangle$ of an unoriented link diagram $\mathscr{D}$. By a state ν of $\mathscr{D}$ we mean (for the moment) an assignation to each crossing c in $\mathscr{D}$ of one of the two alternatives shown in Figure 1, the value $\nu(c)$ of the state at that crossing being x or x^{-1} accordingly.

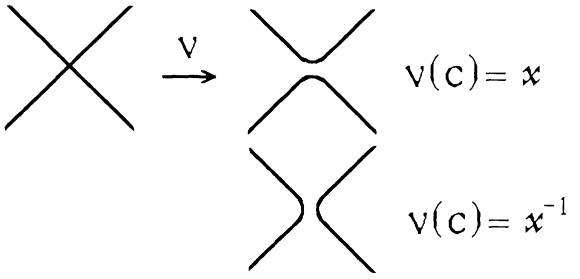


FIGURE 1

For a given state ν, we may 'undo' each crossing in $\mathscr{D}$ in the indicated manner and then count the number of connected components of the resulting diagram $\nu(\mathscr{D})$. Write $|\nu|$ for this number. We then

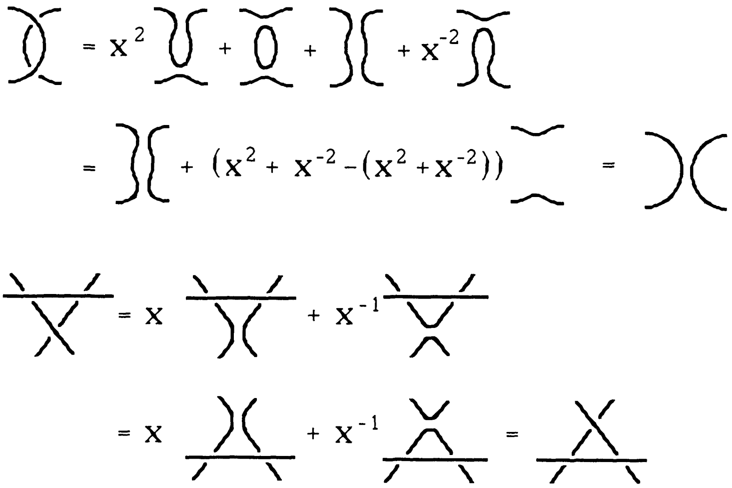


FIGURE 2

define $\langle\mathscr{D}\rangle$ by

$$\langle\mathscr{D}\rangle = \sum_{\text{states } \nu} (-x^2 - x^{-2})^{|\nu|-1} \prod_{\text{crossings } c} \nu(c).$$

It is then easy to show that $\langle\mathscr{D}\rangle$ is invariant under the action of Reidemeister moves of types II and III (see [**8**]) on $\mathscr{D}$ (see Figure 2), and the Reidemeister move I just multiplies $\langle\mathscr{D}\rangle$ by a power of $(-x)^3$.

Now giving $\mathscr{D}$ an orientation, recall that the writhe w of $\mathscr{D}$ is defined as the number of positive crossings minus the number of negative crossings (where these are defined as in Figure 3). It rapidly follows that $J(\mathscr{D}) = \langle\mathscr{D}\rangle(-x)^{-3w}$ is invariant under all three Reidemeister moves and is therefore a link invariant. Now an examination of the behaviour of $J(x)$ on skein triplets (i.e. triplets of link diagrams which are identical everywhere except in the neighbourhood of a particular crossing where they are as in Figure 3) leads to the discovery that it satisfies a linear relation and in fact that $J(x) = V(x^4)$.

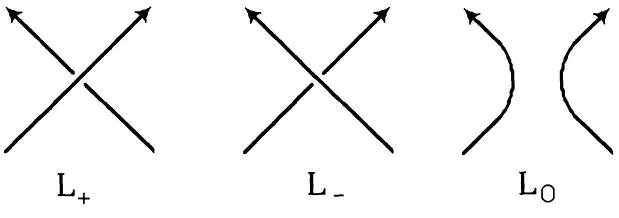


FIGURE 3

Note that this model combines both local (what is going on at each crossing) and global (how the crossings are joined up to give distinct components in $\nu(\mathscr{D})$) information about $\mathscr{D}$.

2. Two new states models. We proceed in a manner very similar to that used above, first defining a polynomial $[\mathscr{D}]$ for each link diagram $\mathscr{D}$ in such a way that $[\mathscr{D}]$ is invariant under Reidemeister moves II and III and then multiplying by an appropriate factor to ensure invariance under Reidemeister move I as well. We first define a $[\cdot]$-state of $\mathscr{D}$ to be a labelling of each connected component of ($\mathscr{D}$-crossing points) with either 1 or 2. A $[\cdot]$-state is legal if at any given crossing, each label occurs an even number of times. Each crossing must then take one of the forms illustrated in Figure 4 (a and b are to be interpreted as distinct labels), and we take the value $\nu(c)$ of the state at that crossing to be as shown.

Now defining $[\mathscr{D}]$ by

$$[\mathscr{D}] = \sum_{\text{legal states } \nu} \prod_{\text{crossings } c} \nu(c)$$

we obtain different possibilities for states models depending on the particular values assigned to A, B, C and D. Of course, any values give a perfectly good function on link diagrams, but in order to obtain link invariants it is necessary that the result be unchanged by the three Reidemeister moves.

THEOREM 1. *Let $A = B = 0$ and $D = C^{-1}$. Then $R = [\cdot]C^2$ is a link invariant, where w is the writhe. Furthermore,*

$$\frac{1}{2}R(C) = (-1)^{c(L)-1}F(iC^{-1}, iC - iC^{-1})$$

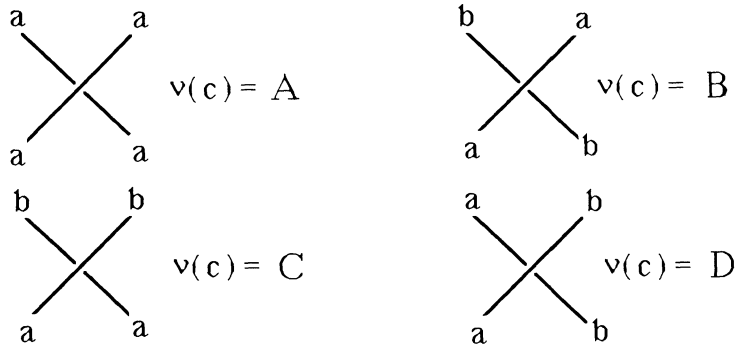


FIGURE 4

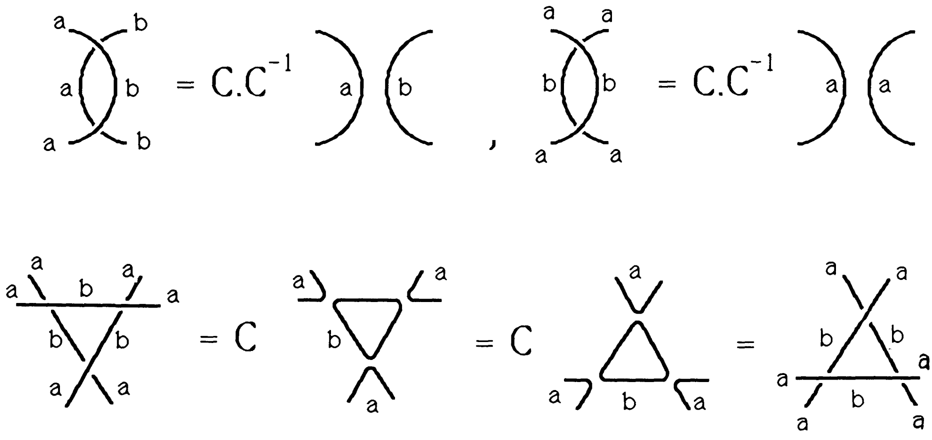


FIGURE 5

where $F(a, z)$ is the Kauffman link invariant and $c(L)$ is the number of components of the link L.

Proof. It is once again fairly easy to show that $[\mathscr{D}]$ is invariant under the second and third Reidemeister moves (see Figure 5. There are two possible labellings to consider for Reidemeister move II, both shown, and four possibilities for Reidemeister move III, of which one is shown. The reader may easily check the others).

It is convenient to allow all possible labellings of arcs of ($\mathscr{D}$-crossing points) but to assign values of 0 to crossings of types not appearing in Figure 4 so that illegal states do not appear in the sum. Now, as before, Reidemeister move I just multiplies $[\mathscr{D}]$ by a power of C, so setting $R(\mathscr{D}) = [\mathscr{D}]C^{w}$ it follows that the polynomial R in C is a link invariant. It remains only to identify it with the claimed evaluation of F. Suppose given four unoriented link diagrams which are identical except in the vicinity of one particular crossing where they are as shown in Figure 6.

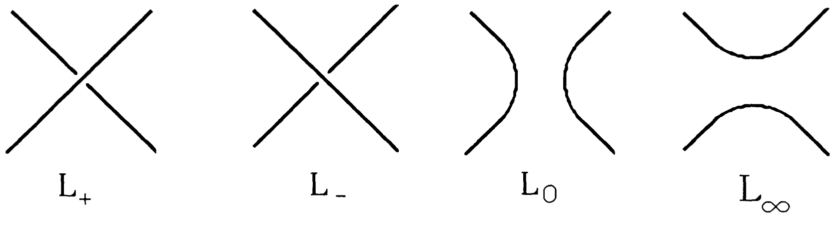
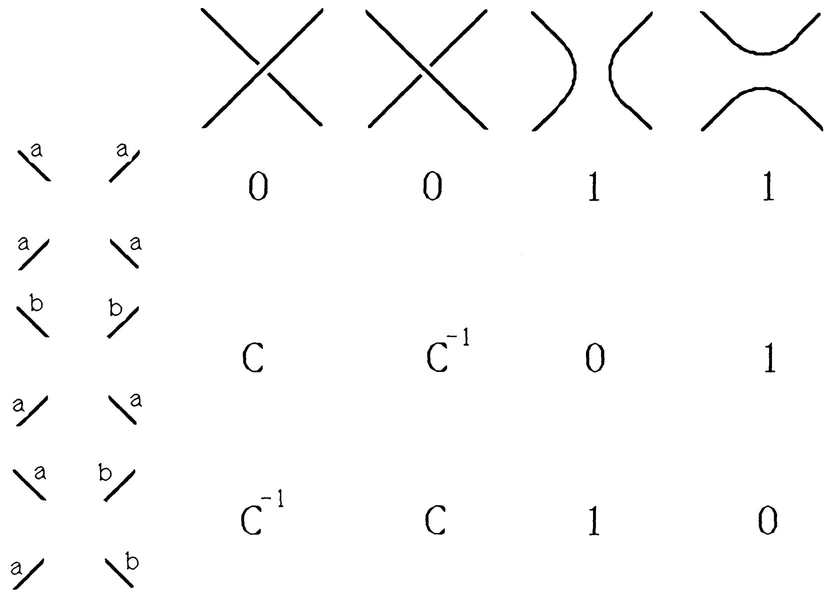


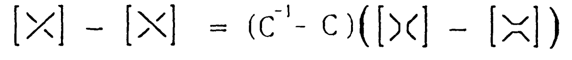
FIGURE 6

TABLE 1

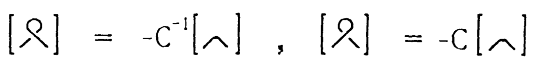


0	0	1	1
C	C^{-1}	0	1
C^{-1}	C	1	0

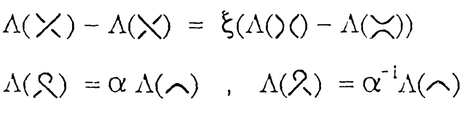
Now evaluating the bracket polynomial $[\cdot]$ on these four link diagrams we find its values are in the ratios given in Table 1 and it easily follows that



and we have already seen that



This makes $R = [\cdot]C^{w}$ a case of the so-called 'Dubrovnik polynomial' $D(\alpha, \xi)$ defined by



$$D(\alpha,\xi) = \Lambda(\alpha,\xi)\cdot\alpha^{-w}$$

In fact, $\frac{1}{2}R(C) = D(C^{-1}, C^{-1} - C)$. Now W. B. R. Lickorish pointed out some time ago that $D(\alpha, \xi)$ is really $F(a, z)$ in disguise via

$$D(\alpha, \xi) = (-1)^{c(L)-1}F(i\alpha, -i\xi)$$

where $c(L)$ is the number of components of the link L. Hence,

$$R(C) = w(-1)^{c(L)-1}F(iC^{-1}, iC - iC^{-1}).$$

This concludes the proof of Theorem 1.

The particular evaluation of F that we have encountered has already been investigated by Lickorish and Millett [**6**]. It is equal to

$$\tfrac{1}{2}(-1)^{c(L)-1}\sum_{X\subset L} a^{-4lk(X,L-X)}$$

the sum being over all sublinks of the link L. This is the probability generating function for the linking number of a randomly-chosen sublink with its complement!

We now look at another set of values for A, B, C and D:

THEOREM 2. *Let* $A = \frac{1}{2}(x + x^{-1})$, $B = -\frac{1}{2}(x + x^{-1})$, $C = \frac{1}{2}(x - x^{-1})$ *and* $D = -\frac{1}{2}(x - x^{-1})$. *Then* $s = [\cdot]x^{w}$ *is a link invariant, where* w *is the writhe. Furthermore,* S *as a polynomial in the variable* x *is identical to the polynomial* R *in the variable* C *of Theorem* 1.

Proof. It is again easy to check that Reidemeister move II has no effect upon the bracket polynomial $[\mathcal{D}]$ of a link diagram $\mathcal{D}$, see Figure 7.

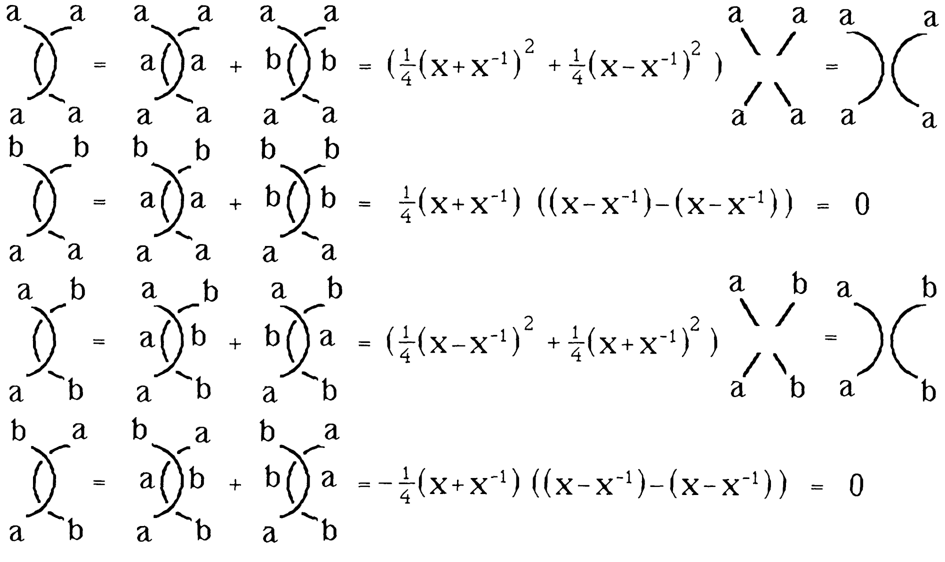
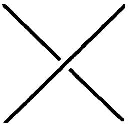
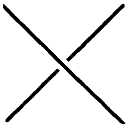
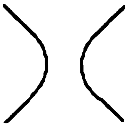
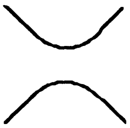
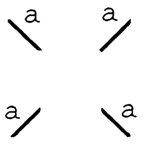
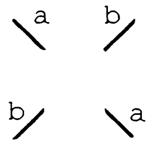
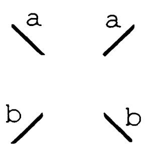
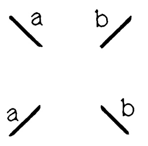


FIGURE 7

Reidemeister move III is equally easy but extremely tedious to check and adds no particular insight, so I leave it as an exercise for the computationally enthusiastic reader. As before, Reidemeister move I just multiplies $[\mathscr{D}]$ by a power of x, so setting $S(\mathscr{D}) = [\mathscr{D}]x^w$ it follows that the polynomial S in x is a link invariant. The rest of the proof is identical to that of Theorem 1 so I will do no more than provide in Table 2 the ratios of values of $\langle\mathscr{D}\rangle$ on the four link diagrams of Figure 6.

In order to explain where these states models come from, I will briefly return to the definition of the new bracket polynomial $[\mathscr{D}]$ in §2.

TABLE 2

				
	$\frac{1}{2}(X+X^{-1})$	$\frac{1}{2}(X+X^{-1})$	1	1
	$-\frac{1}{2}(X+X^{-1})$	$-\frac{1}{2}(X+X^{-1})$	0	0
	$\frac{1}{2}(X-X^{-1})$	$\frac{1}{2}(X-X^{-1})$	0	1
	$-\frac{1}{2}(X-X^{-1})$	$-\frac{1}{2}(X-X^{-1})$	1	0

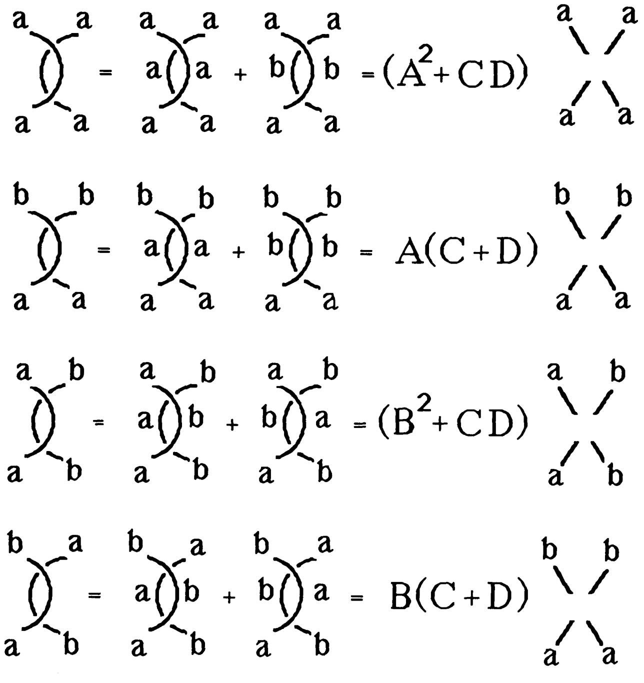


FIGURE 8

It is clear from Figure 8 that to obtain invariance under Reidemeister move II we are led to the relations:

$$\begin{aligned} A^2 + CD &= 1, \\ A(C+D) &= 0, \\ B^2 + CD &= 1, \\ B(C+D) &= 0. \end{aligned}$$

Now there are two cases: either we may assume $A = 0$ and $B = 0$, in which case $CD = 1$, or alternatively we have $C = -D$ and then $B^2 = A^2 = -C^2$. These two solutions lead to the states models of Theorems 1 and 2.

I conclude this section by noting the provocative resemblance between the substitutions in $F(a, z)$ which provide the Jones polynomial, which is reducible to

$$F(a^3, -(a + a^{-1}))$$

and Lickorish/Millett's probability generating function, which is reducible to

$$F(a, -(a + a^{-1})).$$

3. Special evaluations of the Jones polynomial. There are also special similar states models for particular evaluations of the Jones polynomial $V(t)$. Let n be a positive integer greater than 1, and proceed as in §3 to define a bracket polynomial $[\cdot]$ on link diagrams. However, instead of labelling each component of ($\mathscr{D}$-crossing points) in a diagram $\mathscr{D}$ with 1 or 2, we allow ourselves n distinct labels. Again, a legal state is one in which, at each crossing point, each label occurs an even number of times. The conditions put on A, B, C and D by requiring invariance under Reidemeister moves II and III are:

$$\begin{aligned} B &= 0, \\ A &= C + C^{-1}, \\ D &= -C, \\ A^2 &= 2 - n \end{aligned}$$

(there is another solution but it is trivial). Precisely the same arguments as used in §2 to identify R show that the link invariant provided by this model is equivalent to the Jones polynomial $V(C^4)$. Noting that

$$2 - n = A^2 = C^2 + C^{-2} + 2$$

we have

THEOREM 3. *Using n distinct labels on the components of ($\mathscr{D}$-crossing points) and setting $A = C + C^{-1}$, $B = 0$, $D = -C$ and $A^2 = 2 - n$, the bracket polynomial $[\mathscr{D}]$ is invariant under Reidemeister moves* II *and* III *and after renormalising with respect to the writhe provides a link invariant. Furthermore, this invariant is reducible to the Jones polynomial $V(t)$ when $t^{1/2} + t^{-1/2} = -n$.*

It is interesting to note that, for small n, these values of t seem to include some which are already known to be interesting (see, e.g. [5], [7]). Recent work by V. F. R. Jones [2] and V. G. Turaev [9] may show why these values are significant.

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