PACIFIC JOURNAL OF MATHEMATICS
Vol. 152, No. 2, 1992

PULLING BACK BUNDLES

GEORGE R. KEMPF

Let D be an ample divisor on a smooth projective algebraic variety X. We will define the notion of a vector bundle $\mathscr{W}$ on X to be strongly stable with respect to D. If X has characteristic zero this definition is the same as the usual definition of stability. In general it implies stability.

Let $f: Y \to X$ be a finite morphism. Then we have the bundle $f^*\mathscr{W}$ on Y which has the ample divisor $f^{-1}D$. If $\mathscr{W}$ is stable with respect to D, we will prove

THEOREM 1 (Characteristic zero). *$f^*\mathscr{W}$ is the direct sum of stable bundles of the same slope with respect to $f^{-1}D$, i.e. $f^*\mathscr{W}$ is poly-stable.*

Consider the special case of a finite morphism $f: \mathbb{P}^n \to \mathbb{P}^n$. For instance f is given by raising the homogeneous coordinates to the kth power. Then we have an essentially unique choice of D and $f^{-1}D$. Our result is a strong version of the above problem. When rank $\mathscr{W} = 2$ this is due to Barth [**5**].

THEOREM 2. *If $\mathscr{W}$ is a strongly stable bundle on $\mathbb{P}^n$, then $f^*\mathscr{W}$ is strongly stable.*

By Theorem 1 in characteristic zero we need only see that $f^*\mathscr{W}$ is indecomposable. One may apply this in particular to the Mumford-Horrocks' bundle on $\mathbb{P}^4$ and thereby produce many other rank two bundles on $\mathbb{P}^4$ with larger Chern classes. See [**6**].

1. Stability and strong stability. Let D be an ample divisor on a smooth projective variety X. Let $\mathscr{W}$ be a torsion-free coherent sheaf on X. The slope $\mu(\mathscr{W}) = \deg\mathscr{W}/\operatorname{rank}\mathscr{W}$ where $\deg\mathscr{W} = [c_1(\mathscr{W}) \cdot D^{\dim X - 1}]$.

Then $\mathscr{W}$ is stable with respect to D if $\mu(\mathscr{F}) < \mu(\mathscr{W})$ for all non-zero coherent subsheaves $\mathscr{F} \subsetneq \mathscr{W}$.

For strong stability we will assume that $\mathscr{W}$ is locally free. When $\mathscr{W}$ is strongly free if for all $0 < i < \operatorname{rank}\mathscr{W}$, $\Gamma(X, \mathscr{L}^{\otimes -1} \otimes \bigwedge^i \mathscr{W}) = 0$ for all invertible sheaves $\mathscr{L}$ on X such that $\deg\mathscr{L} \geq i\mu(\mathscr{W})$.

LEMMA 3. *Strongly stable implies stable.*

Proof. Let $0 \neq \mathscr{F} \subsetneq \mathscr{W}$ be a coherent subsheaf of $\mathscr{W}$ of rank i. Then we have the obvious homomorphism $i\colon \bigwedge^i \mathscr{F} \to \bigwedge^i \mathscr{W}$. Let $\mathscr{L}$ be $(\bigwedge^i \mathscr{F}/\text{torsion})^{\text{double dual}}$. Then i induces an inclusion $\mathscr{L} \subset \bigwedge^i \mathscr{W}$. Thus if $\mathscr{W}$ is strongly stable then $\deg \mathscr{L} < i\mu(\mathscr{W})$ but $\mu(\mathscr{F}) = \deg \mathscr{L}/i$. Thus $\mu(\mathscr{F}) < \mu(\mathscr{W})$ and hence $\mathscr{W}$ is stable. □

Thus one easily checks that stable means that no section of $\mathscr{L}^{\otimes -1} \otimes \bigwedge^i \mathscr{W}$ satisfies the Plücker relations at the generic point X if $\deg \mathscr{L} \geq i\mu(\mathscr{W})$.

Next we will use some analysis.

PROPOSITION 4. *If* $\operatorname{char}(X) = 0$ *then strongly stable* $\Leftrightarrow$ *stable.*

Proof. Assume that $\mathscr{W}$ is stable. Let $\mathscr{W}$ be a Kähler metric with $c_1(D)$ as cohomology class. Then by the theorem of Donaldson-Uhlenberg-Yau $\mathscr{W}$ admits a Kähler-Einstein metric. As mentioned in [4] $\bigwedge^i \mathscr{W}$ has a Kähler-Einstein metric of slope $i\mu(\mathscr{W})$. Thus by Kobayashi's theorem $\bigwedge^i \mathscr{W}$ is the direct sum of stable bundles $\mathscr{F}_*$ of slope $i\mu(\mathscr{W})$. In particular each $\mathscr{F}_*$ does not contain an invertible sheaf $\mathscr{L}$ of $\deg \geq i\mu(\mathscr{W})$. Hence $\bigwedge^i \mathscr{W}$ has the same property. □

2. The proof of Theorem 1. We will prove Theorem 1 by induction of dimension $X = n$. Let $h = \dim \mathscr{W}$.

If $n = 1$ then $\mathscr{W}$ has a Hermitian-Einstein metric for some Hermitian metric ω_X on X. Thus $f^*\mathscr{W}$ has a Hermitian-Einstein metric for the degenerate metric $f^*\omega_Z$ on Y which vanishes at the ramification points of f. Let $\mathscr{F} \subset f^*\mathscr{W}$ be a coherent sheaf of rank f, which we may assume is a subbundle as Y is a smooth curve. Thus $\mathscr{L} = \bigwedge^h \mathscr{F} \subset \bigwedge^h f^*\mathscr{W}$ is a subbundle. Hence the curvature of $\mathscr{L}$ is pointwise smaller than that of $\bigwedge^h f^*\mathscr{W}$.

We immediately conclude that $f^*\mathscr{W}$ is semi-stable. If the $\deg \mathscr{L} = \text{slope} \bigwedge^h f^*\mathscr{W}$, then $\mathscr{L}$ has a Hermitian-Einstein metric with respect to $f^*\omega_X$. Then we have a section of $\mathscr{L}^{\otimes -1} \otimes \bigwedge^h f^*\mathscr{W}$ corresponding to the inclusion but this sheaf has zero curvature. As usual we see that $\mathscr{F}$ is a direct summand of $f^*\mathscr{W}$.

For the inductive step let X' be a general hyperplane section of X of large degree. Then $\mathscr{W}|_{X'}$ is stable by the restriction theorem of Mehta-Ramanathan [**1**]. By Bertini $f^{-1}(X') = Y'$ is smooth. Trivially $f'\colon Y' \to X'$ is finite. Then $f'^*(\mathscr{W}|_{X'}) = f'^*\mathscr{W}|_{Y'}$ is poly-stable. Say

$f'^*\mathscr{W}|_{Y'} = \bigoplus \mathscr{V}_i^{\oplus n_i}$ where the $\mathscr{V}_i$ are non-isomorphic bundles with the same slope. It follows that

$$\operatorname{End}(f'^*\mathscr{W}|_{Y'}) = \bigoplus \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{\oplus n_i})$$

and each direct summand is given by a idempotent.

By Serre's vanishing theorem $\operatorname{End}(f^*\mathscr{W}) \to \operatorname{End}(f^*\mathscr{W}|_{Y'})$ is an isomorphism because Y' has large degree. Thus we have a decomposition $f^*\mathscr{W} = \bigoplus \mathscr{W}_i^{\oplus n_i}$ which extends to the one above and this decomposition is independent of the choice of Y'. Thus each $\mathscr{W}_i$ is stable and they have the same slope by the trivial direction of the reasoning of the restriction theorem. Thus Theorem 1 is here.

3. Endomorphisms of $\mathbb{P}^n$. Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-constant morphism. Then $f^*\mathscr{O}_{\mathbb{P}^n}(1) = \mathscr{O}_{\mathbb{P}^n}(k)$ where k is positive. Now $f(x_0, \dots, x_n) = (F_0(x), \dots, F_n(X))$ where $F_0, \dots, F_n$ are homogeneous polynomials of degree k with no common zero.

Let $i\colon k[Y_0, \dots, Y_n] \to k[X_0, \dots, X_n]$ be the homomorphism sending Y_i to F_i. Then by the argument in invariant theory [3] we may conclude that i is injective and $k[X_0, \dots, X_n]$ is a free $k[Y_0, \dots, Y_n]$-module with a basis $r_1, \dots, r_d$ of homogeneous elements. This implies

LEMMA 5. (a) *f is a flat finite morphism.*

(b) *for all l, $f_*(\mathscr{O}_{\mathbb{P}^n}(l)) = \bigoplus_{m\in S(l)} \mathscr{O}_{\mathbb{P}^n}(m)$ where the finite set $S(l)$ satisfies*

(c) *$S(0)$ has only one non-negative element which is zero and $S(l)$ has non-negative elements if $l < 0$.*

Proof. The point (c) follows from (b) by looking to the isomorphism of global sections

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(l)) = \bigoplus_{m\in S(l)} \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)).$$

To prove (a) first note that f is affine as $f^{-1}(Y_i \neq 0) = (F_i \neq 0)$ is affine. Thus (a) follows from (b). For (b) we compute

$$\begin{aligned}\Gamma(F_i) \neq 0, \mathscr{O}_{\mathbb{P}^n}(l) &= [k[X_0, \dots, X_n]_{F_0})]_{\text{degree } l} \\ &= \bigoplus [r_i, k[Y_0, \dots, Y_n]_{(X_i)}]_{\text{degree } l} \\ &= \bigoplus [r_i, k[Y_0, \dots, Y_n]_{(X_i)}]_{\text{some degree depending on } r_i}.\end{aligned}$$

As this isomorphism is global (b) follows. □

4. The proof of Theorem 2. Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a non-constant morphism. Let $\mathscr{W}$ be a strongly stable vector bundle on $\mathbb{P}^n$ of slope μ with respect to D where D is a hyperplane section.

Now we want to prove that $f^*\mathscr{W}$ is strongly stable of slope $k \cdot \mu$ with respect to D where $kD \sim f^*\mathscr{W}$. Let $\mathscr{L}$ be an invertible sheaf on $\mathbb{P}^n$ such that $\Gamma(\mathbb{P}^n, \mathscr{L}^{\otimes -1} \otimes \bigwedge^j f^*\mathscr{W}) \neq 0$ for $0 < i < \operatorname{rank}\mathscr{W}$. Then we need to show that $\deg\mathscr{L} < ik\mu$. Let $\mathscr{L} = \mathscr{O}_{\mathbb{P}^n}(l)$.

Write $l = kr - s$ where $0 \leq s < k$. Then

$$\begin{aligned}\Gamma(\mathbb{P}^n, \mathscr{L}^{\otimes -1} \otimes \textstyle\bigwedge^i f^*\mathscr{W}) &= \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-s)) \otimes f^*(\mathscr{O}_{\mathbb{P}^r}(-r) \otimes \textstyle\bigwedge^i \mathscr{W}) \\ &= \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^r}(-s)) \oplus (\mathscr{O}_{\mathbb{P}^n}(-r) \otimes \textstyle\bigwedge^i \mathscr{W}) \\ &= \bigoplus_{m \in S(-s)} \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-rm) \otimes \textstyle\bigwedge^i \mathscr{W}).\end{aligned}$$

As $\mathscr{W}$ is strongly stable we get $+rm < i\mu$ for some $m \in S(-s)$ where $m < 0$ unless $s = 0$ then $m \leq 0$. Thus $\deg\mathscr{L} = l = kr + s = k(r + s/k) \leq k(rm) \leq k(i\mu) = i(k\mu)$ which is what we wanted.

5. Splitting of bundles. Let $\mathscr{W}$ be a bundle on $\mathbb{P}^n$. Then $\mathscr{W}$ is split if and only if $\mathscr{W} = \bigoplus \mathscr{O}_{\mathbb{P}^n}(l_i)$ for some l_i. Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a finite morphism.

LEMMA 6. *$f^*\mathscr{W}$ is split iff $\mathscr{W}$ is.*

Proof. The "if" part is trivial.

To prove the other way note that $H^i(\mathbb{P}^n, \mathscr{W}(i))$ is a direct summand of $H^i(\mathbb{P}^n, f^*\mathscr{W}(ki))$ by §2. Thus Horrocks' criterion [2] for $f^*\mathscr{W}$ implies the same condition for $\mathscr{W}$. Hence $\mathscr{W}$ is split if $f^*\mathscr{W}$ is. □

REFERENCES

[1] V. B. Mehta and A. Ramanathan, *Semistable sheaves on projective varieties and their restrictions to curves*, Math. Ann., **258** (1982), 213–224.

[2] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. (3), **14** (1958), 689–713.

[3] G. Kempf, *Computing Invariants*, in *Invariant Theory* (S. S. Koh), Lecture Notes in Math., vol. 1278, Springer-Verlag, New York, 1980, pp. 8–94.

[4] M. Lübke, *Stability of Einstein-Hermitian vector bundle*, Manuscripta Math., **42** (1983), 245–257.

[5] W. Barth, *Some properties of stable rank-2 vector bundles on* $\mathbb{P}^n$, Math. Ann., **226** (1977), 125–150.

[6] W. Decker and F. O. Schreyer, *Pullbacks of the Horrocks-Mumford bundle*, J. Reine Angew Math., **382** (1978), 215–220.

Received November 20, 1989. Partially supported by an NSF grant.

THE JOHN'S HOPKINS UNIVERSITY
BALTIMORE, MD 21218