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LEVEL SET MAXIMA AND QUASILINEAR ELLIPTIC PROBLEMS

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The present paper studies existence of solutions to the problem $\rho A'(x) = B'(x)$ where A and B are Fréchet differentiable functionals on a Banach space. For every given value of $A(x) = t$ we prove existence of a solution x and present an expression for the eigenvalue $\rho = \rho(t)$. The result is applied to quasilinear elliptic equations.

1. Introduction. A typical problem of the second order studied below is

$$-\operatorname{div}\mathscr{A}_\xi'(x, \nabla u) = f(x, u), \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is an open bounded domain, $\mathscr{A}(x, \xi) : \Omega \times \mathbf{R}^n \to \mathbf{R}$ and "ellipticity" of $\mathscr{A}$ is defined as:

$$\begin{gathered}\mathscr{A}(x, \cdot) \text{ is convex for any } x \in \Omega, \\ \mathscr{A}_\xi'(x, \xi) \cdot \xi \geq c|\xi|^p, \qquad p > 1,\ c > 0,\ (x, \xi) \in \Omega \times \mathbf{R}^n.\end{gathered} \tag{1.2}$$

When $\mathscr{A}$ is a quadratic form of ξ and $p = 2$, (1.1) becomes a semilinear equation. The study extends also to equations of higher order and to systems.

Our approach follows the approach of Browder [2] and Berger [1] with a subsequent refinement due to [3] and [4]. Let X be a Banach space, $A, B \in C^1(X \to \mathbf{R})$. We consider a critical point equation

$$\rho A'(x) = B'(x). \tag{1.3}$$

This equation might be associated with the maximum problem

$$\sigma(t) = \sup_{A(x)=t} B(x). \tag{1.4}$$

Under general compactness conditions, the maximum in (1.4) is attained and provides (1.3). Remarkably the eigenvalues ρ are derivatives of the function of critical values $\sigma(t)$. More precisely, since we prove that σ might have different right and left hand derivatives $\sigma_\pm'(t)$, both of them are eigenvalues. Under additional conditions we prove that (1.3) is solvable for any ρ between $\inf_t \sigma_-'(t)$ and

$\sup_t \sigma'_+(t)$. Thus, if the graph of $\sigma(t)$ has a slope less than 1 and a slope greater than 1, then $A'(u) = B'(u)$ will be solvable. Applications to quasilinear elliptic problems then follow.

2. Maxima on the level sets. Let X be a reflexive Banach space and let $A, B \in C^1(X \to \mathbf{R})$. Let us list several conditions to be used later. Let

$$S_t = \{u \in X : A(u) = t\}, \tag{2.1}$$

$$\omega_t = \{u \in X : A(u) \le t\} \tag{2.2}$$

and define for those t when $S_t \ne \varnothing$,

$$\sigma(t) = \sup_{u \in S_t} B(u), \tag{2.3}$$

$$\beta(t) = \sup_{u \in \omega_t} B(u) = \sup_{\tau \le t} \sigma(\tau). \tag{2.4}$$

LEMMA 2.1. *Assume the following conditions:*
(A1) *A is coercive, i.e., $\|u_j\| \to \infty \Rightarrow A(u_j) \to \infty$.*
(A2) *A is weakly lower semicontinuous.*
(A3) *$\langle A'(u), u\rangle > 0$ for $u \in X \backslash \{0\}$.*
(B1) *B has no local maxima on X.*
(B2) *B is weakly continuous.*
Then

(2.5) *$\sigma(t)$ is increasing on $[0, \infty)$,*

(2.6) *$\sigma(t)$ is continuous on $[0, \infty)$,*

(2.7) *the maximum in (2.3) is attained for every $t \ge 0$.*

(An immediate consequence of (2.5) is that $\sigma(t) = \beta(t)$.)
Without loss of generality one can assume from now that

$$A(0) = 0. \tag{2.8}$$

Then, due to (A3)

$$\begin{aligned} A(u) &= \int_0^1 \frac{d}{dt} A(tu)\,dt = \int_0^1 \langle A'(tu), u\rangle\,dt \\ &= \int_0^1 t^{-1} \langle A'(tu), tu\rangle\,dt > 0, \quad \text{unless } u = 0. \end{aligned} \tag{2.9}$$

By (A1) the range of A is $[0, \infty)$.

Proof of Lemma 2.1. Consider a maximizing sequence $u_j \in \omega_t$ in (2.4). By (A1) it is bounded in norm. Then there is a weakly convergent renamed subsequence $u_j \xrightarrow{w} u_0$. By (A2) $u_0 \in \omega_t$. By (B2) $B(u_j) \to B(u_0) = \beta(t)$. If $u_0 \notin S_t$, then it is a point of interior maximum for B in ω_t. This contradicts (B1). Consequently, $u_0 \in S_t$. This implies in turn that $\sigma(t) = \beta(t)$ and it is a monotone increasing function. For the given t, the point u_0 is a point of maximum over S_t. Thus, (2.5) and (2.7) are proved.

Consider

$$\sigma(t_0 + 0) = \liminf_{t \to t_0,\, t > t_0} \sigma(t), \tag{2.10}$$

$$\sigma(t_0 - 0) = \limsup_{t \to t_0,\, t < t_0} \sigma(t). \tag{2.11}$$

Let $t_j > t_0$, $t_j \to t_0$ and let $\sigma(t_j)$ be attained an $u_j \in S_{t_j}$. Then by (A1) u_j has a renamed weakly convergent subsequence: $u_j \xrightarrow{w} u_0 \in \omega_{t_0}$. Therefore,

$$\sigma(t_0 + 0) = B(u_0) \le \sup_{u \in \omega_{t_0}} B(u) = \sigma(t_0). \tag{2.12}$$

The converse inequality is true by monotonicity of σ. Thus

$$\sigma(t_0 + 0) = \sigma(t_0). \tag{2.13}$$

Let now u_0 be a point of maximum in (2.1) at $t = t_0$. Then by (A3)

$$t(s) := A(su_0) \tag{2.14}$$

is a monotone increasing function and

$$t(s) \to t_0, \quad t(s) < t_0 \quad \text{as } s \to 1,\ s < 1.$$

Therefore,

$$\sigma(t_0 - 0) \ge \limsup_{\substack{s \to 1 \\ s < 1}} B(su_0) = \sigma(t_0). \tag{2.15}$$

The converse inequality is due to monotonicity of σ. Thus $\sigma(t_0 - 0) = \sigma(t_0)$ and (2.6) is proved. □

LEMMA 2.2. *Assume the conditions of Lemma* 2.1. *Let* u_0 *be a point of maximum in* (2.3), $t > 0$. *Then there is a* $\rho \ge 0$, *such that*

$$\rho A'(u_0) = B'(u_0). \tag{2.16}$$

Proof. Let $v \in X$ be such that

$$\langle A'(u_0), v \rangle < 0. \tag{2.17}$$

By Lemma 2.1 u_0 is a point of maximum of B in ω_t and

$$B(u_0 + \theta v) \leq B(u_0) \tag{2.18}$$

for $\theta > 0$ sufficiently small. Thus,

$$\langle B'(u_0), v\rangle \leq 0 \tag{2.19}$$

for any v satisfying (2.17). From (2.17), (2.19) a routine argument shows that $A'(u_0)$ and $B'(u_0)$ are parallel. By (A3), $A'(u_0) \neq 0$ and (2.16) follows immediately. □

Let us define a set $\Lambda_t \subset S_t \times [0, \infty)$, $t \in (0, \infty)$:

$$\Lambda_t = \{(u, \rho) : B(u) = \sigma(t), \rho A'(u) = B'(u)\}. \tag{2.20}$$

By Lemmas 2.1, 2.2 Λ_t is nonempty. Let

$$\Lambda_t^{(\rho)} = \{\rho \geq 0 : \exists u \in S_t, (u, \rho) \in \Lambda_t\}, \tag{2.21}$$

$$\Lambda_t^{(u)} = \{u \in S_t : \exists \rho \geq 0, (u, \rho) \in \Lambda_t\}. \tag{2.22}$$

By (2.20), one has also:

$$\Lambda_t^{(u)} = \{u \in S_t : B(u) = \sigma(t)\}$$

and

$$\Lambda_t^{(\rho)} = \{\rho = \langle B'(u), u\rangle / \langle A'(u), u\rangle, \ u \in \Lambda_t^{(u)}\}. \tag{2.23}$$

LEMMA 2.3. *Assume the conditions of Lemma* 2.1 *and in addition*
(A4) *If* $u_j \xrightarrow{w} u_0$, *then*

$$A(u_j) \to A(u_0) \Leftrightarrow \langle A'(u_j), u_j\rangle \to \langle A'(u_0), u_0\rangle.$$

(B3) $\langle B'(u), u\rangle$ *is weakly continuous.*
Let $T \subset (0, \infty)$ *and*

$$\Lambda_T = \bigcup_{t \in T} \Lambda_t. \tag{2.24}$$

If T *is compact in* $(0, \infty)$, *then* Λ_T *is weakly compact in* $X \times (0, \infty)$.

Proof. Let $(u_j, \rho_j) \in \Lambda_{t_j}$, $t_j \in T$. Consider a renamed convergent subsequence $t_j \to t_0 \in T$. Then by (A1), (A2) u_j has a weakly convergent (renamed) subsequence $u_j \xrightarrow{w} u_0 \in \omega_{t_0}$. By (B2), $B(u_j) \to B(u_0)$, by Lemma 2.1 $B(u_0) = \sigma(t_0)$ and $u_0 \in S_{t_0}$, i.e.,

$u_0 \in \Lambda_{t_0}^{(u)}$. By Lemma 2.2 u_j satisfy (2.16) with eigenvalues, say, ρ_j, u_0 satisfies (2.16) with some ρ_0, and

$$\rho_j = \langle B'(u_j), u_j\rangle / \langle A'(u_i), u_j\rangle, \qquad (2.25)$$
$$\rho_0 = \langle B'(u_0), u_0\rangle / \langle A'(u_0), u_0\rangle.$$

Note now that $\langle B'(u_j), u_j\rangle \to \langle B'(u_0), u_0\rangle$ by (B3) and $\langle A'(u_j), u_j\rangle \to \langle A'(u_0), u_0\rangle$ by (A4). Thus $\rho_j \to \rho_0$. □

3. Critical values and eigenvalues.

THEOREM 3.1. *Assume*
(A1) *A is coercive, i.e., $\|u_j\| \to \infty \Rightarrow A(u_j) \to \infty$.*
(A2) *A is weakly lower semicontinuous.*
(A3) *$\langle A'(u), u\rangle > 0$ for $u \in X\backslash\{0\}$.*
(A4) *If $u_j \xrightarrow{w} u_0$, then*

$$A(u_j) \to A(u_0) \Leftrightarrow \langle A'(u_j), u_j\rangle \to \langle A'(u_0), u_0\rangle.$$

(B1) *B has no local maxima on X.*
(B2) *B is weakly continuous.*
(B3) *$\langle B'(u), u\rangle$ is weakly continuous.*
Then for every $t > 0$ there exist left and right derivatives $\sigma'_\pm(t)$, $\sigma'_-(t) \le \sigma'_t(t)$. Moreover,

$$\sigma'_+(t) = \sup\{\rho \in \Lambda_t^{(\rho)}\}, \qquad (3.1)$$

$$\sigma'_-(t) = \inf\{\rho \in \Lambda_t^{(\rho)}\}. \qquad (3.2)$$

Before we prove the theorem, we wish to note that supremum in (3.1) and infimum in (3.2) are attained on some $u_\pm \in \Lambda_t^{(u)}$ due to Lemma 2.3. As a result one has

THEOREM 3.2. *Under conditions of Theorem* 3.2 *for every $t > 0$ there exist $u_\pm \in S_t$, such that*

$$\sigma'_+(t) A'(u_+) = B'(u_+), \qquad (3.3)$$

$$\sigma'_-(t) A'(u_-) = B'(u_-). \qquad (3.4)$$

Proof of Theorem 3.1. 1. Let $u_0 \in \Lambda_{t_0}^{(u)}$. Let $\theta_j \to 1$, $t_j = A(\theta_j u_0)$. Then by continuity of A, $t_j \to A(u_0) = t_0$. Moreover,

$$t_j - t_0 = A(\theta_j u_0) - A(u_0) = \langle A'(u_0), u_0\rangle(\theta_j - 1) + o(\theta_j - 1). \qquad (3.5)$$

Consequently,

$$(3.6)\quad \begin{aligned}\sigma(t_j)-\sigma(t_0) &\geq B(\theta_j u_0)-B(u_0)\\ &= \langle B'(u_0), u_0\rangle(\theta_j-1)+o(\theta_j-1)\\ &= (\langle B'(u_0), u_0\rangle/\langle A'(u_0), u_0\rangle)(t_j-t_0)+o(t_j-t_0).\end{aligned}$$

Fro (3.6) and (2.23) one has immediately,

$$(3.7)\qquad D_+\sigma(t_0)\geq \sup\{\rho\in\Lambda_{t_0}^{(\rho)}\},$$

$$(3.8)\qquad D^-\sigma(t_0)\leq \inf\{\rho\in\Lambda_{t_0}^{(\rho)}\}.$$

2. Let now $u_j\in\Lambda_{t_j}^{(u)}$, $t_i\to t_0$. By Lemma 2.3 a renamed sequence u_j converges weakly to $u_0\in\Lambda_{t_0}^{(u)}$. Let us define $\theta_j>0$ by

$$(3.9)\qquad A(\theta_j u_j)=t_0.$$

By (A3), the function $\theta\to A(\theta u_j)$ is monotone for any $u\neq 0$ and by (A1) the range of it is $[0,\infty)$. Thus, for given $t_0>0$ and $u_j\neq 0$, (3.9) has a unique solution $\theta_j>0$. Since $A\in C^1$, there exist $\eta_j\in[\theta_j, 1]$, such that

$$(3.10)\qquad t_j-t_0=A(u_j)-A(\theta_j u_j)=\langle A'(\eta_j u_j), u_j\rangle(1-\theta_j).$$

From (3.9) it follows that θ_j is a bounded sequence. Let us consider a renamed convergent subsequence: $\theta_j\to\theta_0$. Then $\theta_j u_j\xrightarrow{w}\theta_0 u_0$ and, necessarily, $\theta_0=1$. Therefore $\eta_j\to 1$ and

$$(3.11)\qquad t_j-t_0=\langle A'(u_j), u_j\rangle(1-\theta_j)+o(1-\theta_j).$$

Similarly,

$$(3.12)\qquad B(u_j)-B(\theta_j u_j)=\langle B'(u_j), u_j\rangle(1-\theta_j)+o(1-\theta_j).$$

Therefore

$$(3.13)\quad \begin{aligned}\sigma(t_j)-\sigma(t_0) &\leq \langle B'(u_j), u_j\rangle(1-\theta_j)+o(1-\theta_j)\\ &= (\langle B'(u_j), u_j\rangle/\langle A'(u_j), u_j\rangle)(t_j-t_0)+o(t_j-t_0).\end{aligned}$$

We have to note only that $\langle A'(u_j), u_j\rangle\to\langle A'(u_0), u_0\rangle$ by (A4) and $\langle B'(u_j), u_j\rangle\to\langle B'(u_0), u_0\rangle$ by (B2). Then from (3.13) follows:

$$(3.14)\qquad D^+\sigma(t_0)\leq \sup\{\rho\in\Lambda_{t_0}^{(\rho)}\},$$

$$(3.15)\qquad D_-\sigma(t_0)\geq \inf\{\rho\in\Lambda_{t_0}^{(\rho)}\}.$$

3. Let us combine (3.7) and (3.14). Then

$$\sup\{\rho\in\Lambda_{t_0}^{(\rho)}\}\leq D_+\sigma(t_0)\leq D^+\sigma(t_0)\leq \sup\{\rho\in\Lambda_{t_0}^{(\rho)}\},$$

i.e., $\sigma'_+(t_0)$ exists and satisfies (3.1). Similarly, (3.2) follows from (3.8) and (3.15). □

4. Continua of solutions. Range of solvability. In this section we assume the conditions of Theorem 3.1.

PROPOSITION 4.1. *Assume that for every* $t \in (s_1, s_2) \subset (0, \infty)$ *the set* $\Lambda_t^{(u)}$ *consists of a single element* u_t. *Then the map* $t \to u_t$ *is weakly continuous on* (s_1, s_2).

Proof. Let $t_0 \to t_0 \in (s_1, s_2)$. Then $\overline{\{t_j\}}$ is compact in $(0, \infty)$ and by Lemma 2.3 u_{t_j} has a (renamed) weakly convergent subsequence $u_{t_j} \xrightarrow{w} u_0 \in \Lambda_{t_0}^{(u)}$. Since $\Lambda_{t_0}^{(u)} = \{u_{t_0}\}$, the original sequence u_{t_j} must be weakly convergent to the same element u_{t_0}. □

COROLLARY 4.2. *Under conditions of Proposition* 4.1 *the problem*

$$\rho A'(u) = B'(u) \tag{4.1}$$

possesses a weakly continuous family of eigenfunctions $t \to u_t$ *corresponding to eigenvalues* $\rho_t = \langle B'(u_t), u_t\rangle / \langle A'(u_t), u_t\rangle$ *continuous in* $t \in (s_1, s_2)$. *The function* (2.3) *has a continuous derivative on* (s_1, s_2) *and* $\rho_t = \sigma'(t)$.

The proof follows from Theorems 3.1 and 3.2.

PROPOSITION 4.3. *Let* $h: X^2 \to X$ *be a map, such that*

(Ah) $A(h(u, v)) > \max\{A(u), A(v)\}$ *for* $u \neq v$.

(Bh) $B(h(u, v)) \geq \min\{B(u), B(v)\}$ *for* $u, v \in X$.

Then the assertions of Proposition 4.1 *and Corollary* 4.2 *hold for any* $t > 0$.

Proof. Let $u, v \in \Lambda_t^{(u)} u \neq v$. Then by (Ah), $h(u, v) \in \omega_{t'}$, $t' < t$, but by (Bh) $B(h(u, v)) \geq B(u) = \sigma(t)$. This contradicts Lemma 2.1. Thus the conditions of Lemma 4.1 are satisfied at any $t > 0$. □

A simple example when (Ah), (Bh) are satisfied, can be provided by

COROLLARY 4.4. *Let* A *be strictly convex and* B *be concave on* X. *Then the assertions of Proposition* 4.1 *and Corollary* 4.2 *hold for* $t \in (0, \infty)$.

Proof. Take $h(u, v) = \lambda u + (1 - \lambda)v$, $\lambda \in (0, 1)$. □

If $\sigma \in C^1_{\text{loc}}(0, \infty)$, then (4.1) is solvable (with $u \neq 0$) for any $\rho \in I(A, B)$, where

$$I(A, B) = \left(\inf_t \sigma'_-(t), \sup_t \sigma'_+(t)\right). \tag{4.2}$$

However, σ does not generally have a continuous derivative (cf. [**4**]). Thus we wish to answer the question, for what subsets of $I(a, b)$ does (4.1) still have a non-zero solution.

PROPOSITION 4.5. *Let the function* $\rho t - \sigma(t)$ *have a local minimum on* $(0, \infty)$. *Then* (4.1) *has a non-zero solution.*

Proof. Let $\rho t - \sigma(t)$ have a local minimum at $t_0 > 0$. Then $\sigma'_-(t_0) \geq \rho \geq \sigma'_+(t_0)$. By Theorem 3.1, $\sigma'_-(t_0) \leq \sigma'_+(t_0)$. Thus, σ is differentiable at t_0 and $\sigma'(t_0) = \rho$. Then by Theorem 3.2, (4.1) has a solution with $A(u) = t_0$. □

In order to get a more extensive result we use a mountain pass theorem from [**5**].

THEOREM 4.6. *Let* $G \in C^1(X \to \mathbf{R})$ *and let* $u_0 \in X\backslash\{0\}$. *Let* $N \subset X$ *be an open bounded set, such that* $0 \in N$ *but* $u_0 \notin \overline{N}$. *Assume that*

$$G(u) \geq 0, \qquad u \in \partial N, \tag{4.3}$$

$$G(0) \leq 0, \qquad G(u_0) \leq 0. \tag{4.4}$$

Then there is a sequence $u_k \in X$ *and* $\gamma \geq 0$ *such that*

$$G(u_k) \to \gamma, \tag{4.5}$$

$$\|G'(u_k)\|_{X^*} \|u_k\|_X \to 0. \tag{4.6}$$

Let now

$$A_\theta = \theta\langle A'(u), u\rangle - A(u), \tag{4.7}$$

$$B_\theta = \theta\langle B'(u), u\rangle - B(u), \tag{4.8}$$

$$\Phi = \{\theta \in \mathbf{R} \colon |A_\theta| \text{ is coercive}\}, \tag{4.9}$$

and let

(4.10) $Q(\theta)$ be set of limit points for $B_\theta(u)/A_\theta(u)$ when $\|u\| \to \infty$, $\theta \in \Phi$.

Set now

$$Q_* = \bigcap_{\theta \in \Phi} Q(\theta). \tag{4.11}$$

THEOREM 4.7. *Assume, in addition to conditions of Theorem* 3.1, *that*:

(ABw) A', B' *are continuous from* X *to* X^* *with regard to respective weak topologies.*

Then for every $\rho \in I(A, B)\backslash Q_*$ *the equation* (4.1) *has a nonzero solution.*

Proof. 1. Let $\rho \in I(A, B)\backslash Q_*$, i.e., ρ is a slope of a secant to the graph of $\sigma(t)$, the functional $|A_\theta|$ is coercive for some θ and ρ is not a limit point of B_θ/A_θ at infinity. If $\rho t - \sigma(t)$ has a local minimum at $t \neq 0$, then ρ is an eigenvalue (Proposition 4.5). Thus we would consider the case when $\rho t - \sigma(t)$ has no local minimum. If $\rho = \sigma'_+(t_0)$ or $\rho = \sigma'_-(t_0)$ for some $t_0 > \theta$, then it is an eigenvalue by Theorem 3.2. The remaining case is: for some $t_0 > 0$, $\sigma'(t_0) < \rho\sigma'_+(t_0)$. This implies that t_0 is a point of local maximum of $\rho t - \sigma(t)$. Since we assume that $\rho t - \sigma(t)$ has no local minimum, t_0 is a point of strict global maximum. In particular, $\delta := \rho t_0 - \sigma(t_0) > 0 - \sigma(0) = 0$, and there exists $t_1 > t_0$, such that $\delta_1 := \rho t_1 - \sigma(t_1) < \delta$.

2. Let

$$G(u) = \rho A(u) - B(u) - \delta_1. \tag{4.12}$$

Then all the conditions of Theorem 4.6 are fulfilled with $N = \{u \in X, A(u) \leq t_0\}$ and $u_0 \in S_{t_1}$. Let $u_k \in X$, $\gamma \geq 0$ satisfy

$$\rho A(u_k) - B(u_k) \to \delta_1 + \gamma \geq \delta_1, \tag{4.13}$$

$$\|\rho A'(u_k) - B'(u_k)\|_{X^*} \|u_k\|_X \to 0. \tag{4.14}$$

Then

$$\rho\langle A'(u_k), u_k\rangle - \langle B'(u_k), u_k\rangle \to 0 \tag{4.15}$$

and, consequently,

$$\rho A_\theta(u_k) - B_\theta(u_k) \to -\gamma - \delta. \tag{4.16}$$

If $\|u_k\| \to \infty$, then $|A_\theta(u_k)| \to \infty$, $\rho = \lim B_\theta(u_k)/A_\theta(u_k)$, which contradicts the assumptions. Thus the sequence u_k is bounded in norm. Let now u_k be a renamed weakly convergent sequence, and $u_0 = w\text{-}\lim u_k$. Then by (ABw) from (4.14) follows:

$$\rho A'(u_0) = B'(u_0). \tag{4.17}$$

Moreover, by (B3), (4.14), (4.17)

$$\begin{aligned} \lim \rho\langle A'(u_k), u_k\rangle &= \lim\langle B'(u_k), u_k\rangle = \langle B'(u_0), u_0\rangle \\ &= \rho\langle A'(u_0), u_0\rangle. \end{aligned} \tag{4.18}$$

Then by (A4)

$$\lim A(u_k) = A(u_0). \tag{4.19}$$

Thus by (4.13), (B2),

$$0 < \delta_1 + \gamma = \lim \rho A(u_k) - B(u_k) = \rho A(u_0) - B(u_0)$$

which proves that $u_0 \neq 0$. □

COROLLARY 4.8. *Let* $\rho_* = \inf_{\theta\in\Phi} \limsup_{\|u\|\to\infty} B_\theta(u)/A_\theta(u)$. *Then* (4.1) *has a non-zero solution for*

$$\rho \in I(A, B) \cap (\rho_*, \infty).$$

5. Applications to quasilinear elliptic problems. Let $\Omega \subset \mathbf{R}^n$ be an open bounded set $X = W_0^{l,p}(\Omega)^k$, $p > 1$, $l \in \mathbf{N}$, $k \in \mathbf{N}$. Let $\nu(l)$ be the number of multi-indices of length not exceeding l. Assume that $\mathscr{A}(x, \{y_\alpha\}_{|\alpha|\le l})$ and $\mathscr{B}(x, \{y_\alpha\}_{|\alpha|\le l-1})$ are C^1 real valued functions of $\{y_\alpha\}$ whose derivatives are Carathéodory functions of $(x, \{y_\alpha\})$. Without loss of generality we assume that

$$\mathscr{A}(x, 0) = 0, \qquad \mathscr{B}(x, 0) = 0. \tag{5.1}$$

We require for the function $\mathscr{A}$ the following ellipticity condition:

$$\mathscr{A}(x, \cdot) \text{ is convex for almost every } x \in \Omega \tag{5.2}$$

and the following coercivity condition:

$$\frac{d}{dt}\mathscr{A}(x, \{ty_\alpha\})|_{t=1} \ge c \sum_{|\alpha|=l} |y_\alpha|^p, \qquad c > 0, x \in \Omega. \tag{5.3}$$

We also require the following upper bounds for $\mathscr{A}'$ and $\mathscr{B}'$:

$$|\mathscr{A}'_{y_\alpha}(x, \{y_\alpha\})| \le c\left(1 + \sum_{|\alpha|\le l} |y_\alpha|^{p-1}\right), \qquad c > 0, \tag{5.4}$$

$$|\mathscr{B}'_{y_\alpha}(x, \{y_\alpha\})| \le C \sum_{\substack{|\beta|\le l-1 \\ |\beta|\ge l-n/p}} |y_\beta|^{q_{\alpha\beta}} + V_\alpha(\{y_\beta\}_{|\beta|<l-n/p}) + W_\alpha(x), \tag{5.5}$$

where

$$\begin{aligned} &C > 0 \\ &W_\alpha \in L^{r_\alpha},\ r_\alpha = 1 \quad \text{if } l - |\alpha| > n/p, \\ &r_\alpha > pn/(pn - n - p(l - |\alpha|)) \quad \text{if } l - |\alpha| \le n/p, \\ &V_\alpha \text{ is a continuous function,} \end{aligned} \tag{5.6}$$

$$q_{\alpha\beta} > \frac{pn}{n - p(l - |\beta|)} \quad \text{if } l - |\alpha| > np,$$

$$q_{\alpha\beta} < \frac{pn - n + p(l - |\alpha|)}{n - p(l - |\beta|)}, \quad \text{if } l - |\alpha| \leq n/p.$$

PROPOSITION 5.1. *Assume* (5.2)–(5.6). *Then the functionals*

$$A(u) = \int_\Omega \mathscr{A}(x, \{\partial^\alpha u\})\, dx, \tag{5.7}$$

$$B(u) = \int_\Omega \mathscr{B}(x, \{\partial^\alpha u\})\, dx \tag{5.8}$$

satisfy (A1–A4), (B2), (B3), (ABw) *on* $X = W_0^{l,p}(\Omega)^k$.

Proof. The verification of continuity and differentiability properties is standard and based on compactness in the Sobolev embedding theorem. We wish to make remarks on only a few details.

1. Relation (5.3) implies (A3) and also (A1), since it immediately gives

$$\frac{d}{dt}\mathscr{A}(x, \{ty_\alpha\}) \geq ct^{p-1} \sum_{|\alpha|=l} |y_\alpha|^p, \qquad t > 0. \tag{5.9}$$

2. Relation (5.2) implies that the set (2.2) is convex. Thus it is weakly closed and A is lower semicontinuous.

3. Due to (5.9), (5.4) weak convergence of a sequence u_j together with convergence of either $A(u_j)$ or of $\langle A'(u_j), u_j\rangle$ is equivalent to convergence in norm. □

To verify the condition (B1) in most of the applications it suffices to prove that $B'(u) \neq 0$ unless $u = 0$ and that $u = 0$ is not a point of maximum. Two particular cases are given below.

LEMMA 5.2. *Let*

$$\sum_\alpha \mathscr{B}'_{y_\alpha}(x, \{y_\alpha\}) y_\alpha > 0 \quad \textit{for } \{y_\alpha\} \neq 0. \tag{5.10}$$

Then (B1) *holds.*

Proof. From (5.10) it follows that $\langle B'(u), u\rangle > 0$ unless $u = 0$. The point $u = 0$ is not a point of maximum, but rather of minimum, since for every $u \in X\backslash\{0\}$, B increases along the line $t \to tu$, $t > 0$. □

LEMMA 5.3. *Let* $k = 1$, $b(x, \{y_\alpha\}) = F(y_0)$ *and assume that with some* $\varepsilon > 0$,

$$F'(y_0) > 0 \quad \textit{for } y_0 \in (0, \varepsilon) \quad \textit{and} \\ F'(y_0) \neq 0 \quad \textit{for } y_0 \in (-\varepsilon, 0). \tag{5.11}$$

Then (B1) *holds.*

Proof. Let u be the point of maximum of b and let $\overline{u}$ be a decreasing spherical rearrangement for u. Then $\overline{u}$ is a $W_0^{l,p}$-function dependent on the radial variable only. Therefore $\overline{u}$ is continuous away from the origin, the range of $\overline{u}$ is a closed interval I containing zero. Moreover, the range of u is dense in I, for if (s, t) is not in the range of u, (s, t) is not in the range of $\overline{u}$. If u is a point of maximum for B, then $F' = 0$ on the range of u, and since F' is continuous $F' = 0$ on I. By (5.11), therefore, $I \cap (-\varepsilon, \varepsilon) \subset \{0\}$. However, $u = 0$ is not a maximum: one can perturb 0 by a function $v \geq 0$ of an arbitrarily small norm so that $B(v) > B(0) = 0$. □

The following statement is now an immediate corollary of Theorems 3.1, 3.2.

THEOREM 5.4. *Assume* (5.1–5.6) *and* (B1). *Then for every* $t > 0$ *there exists a semistrong solution* $u_t^\pm \in W_0^{l,p}(\Omega)^k$ *satisfying the respective equations:*

$$\sigma'_\pm(t) \sum_{|\alpha|\leq l} (-1)^{|\alpha|} \partial^\alpha \mathscr{A}'_{y_\alpha}(x, \{\partial^\gamma u(x)\}) \\ = \sum_{|\beta|\leq l-1} (-1)^{|\beta|} \partial^\beta \mathscr{B}'_{y_\beta}(x, \{\partial^\gamma u(x)\}). \tag{5.12}$$

Moreover,

$$A(u_t^\pm) = t \tag{5.13}$$

and the function σ *is given by*

$$\sigma(t) = \sup_{A(u)=t} B(u). \tag{5.14}$$

We now will look for realization of conditions (Ah), (Bh) to get continuous curves of eigenfunctions.

THEOREM 5.5. *Assume* (5.1–5.6) *and* (B1). *Let* $\mathscr{B}(x, \cdot)$ *be concave for a.e.* $x \in \Omega$. *Then there is a continuous family* $t \to (\rho_t, u_t) \in (0, \infty) \times W_0^{l,p}(\Omega)^k$, *such that*

$$\rho_t \sum_{|\alpha|\le l} (-1)^{|\alpha|} \partial^\alpha \mathscr{A}'_{y_\alpha}(x, \{\partial^\gamma u_t(x)\}) = \sum_{|\alpha|\le l-1} (-1)^{|\alpha|} \partial^\alpha \mathscr{B}'_{y_\alpha}(x, \{\partial^\gamma u_t(x)\}). \tag{5.15}$$

Proof. Apply Corollary 4.4. Then u_t satisfies (5.15) and $t \to u_t$ is weakly continuous. However, by (5.4), (5.9) convergence of $t_j = A(u_{t_j})$ together with weak convergence of u_{t_j} is equivalent to convergence in norm and the family (ρ_t, u_t) is continuous. □

THEOREM 5.6. *Let* $\mathscr{A}(x, \{\partial^\alpha u\}) = |\nabla u|^p$, $\mathscr{B}(x, \{y_\alpha\}) = F(y_0)$, $k = 1$. *Assume also that* $F \in C^1$,

$$F'(s) > 0 \quad \text{for } s > 0 \quad \text{and} \quad F'(s) = 0 \quad \text{for } s \le 0, \tag{5.16}$$

and that the map

$$s \to F(s^{1/p}) \text{ is concave for } s > 0. \tag{5.17}$$

Then there is a continuous family $t \to (\rho_t, u_t)$ *satisfying*

$$-\rho_t \operatorname{div}(|\nabla u_t|^{p-2} \nabla u_t) = F'(u_t), \qquad u_t \ge 0. \tag{5.18}$$

Proof. Let

$$h_\theta(u_1, u_2) = (\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p}, \qquad 0 \in (0, 1). \tag{5.19}$$

Then, applying Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned} &|\nabla(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p}|^p \\ &= \left| \frac{1}{p} \frac{\theta p |u_1|^{p-1} \nabla |u_2| + (1-\theta) p |u_2|^{p-1} \nabla |u_2|}{(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p'}} \right|^p \\ &= \left| \frac{\theta^{1/p+1/p'} |u_1|^{p/p'} \nabla |u_1| + (1-\theta)^{1/p+1/p'} |u_2|^{p/p'} \nabla |u_2|}{(\theta|u_1|^p + (1-\theta)|u_2|^p)^{1/p'}} \right|^p \\ &\le \theta |\nabla |u_1||^p + (1-\theta)|\nabla |u_2||^p \\ &= \theta |\nabla u_1|^p + (1-\theta)|\nabla u_2|^p. \end{aligned} \tag{5.20}$$

The relation (5.20) makes sense a.e. when $u_1, u_2 \in C_0^1(\Omega)$ and the equality holds only if $u_1 = u_2$ or one of them vanishes. Then the following will be true on $W_0^{1,p}(\Omega)$:

$$A(h_\theta(u_1, u_2)) \le \theta A(u_1) + (1-\theta) A(u_2) \tag{5.21}$$

with the same remark on equality. By (5.16), (5.17),

$$B(h_\theta(u_1, u_2)) \le \theta B(u_1) + (1-\theta)B(u_2). \tag{5.22}$$

Thus (Ah), (Bh) are satisfied and the assertion follows from Theorem 5.11. One has only to note that $t_j \to t_0$ and $u_{t_i} \xrightarrow{w} u_{t_0}$ implies $u_{t_j} \to t_0$. □

Now we wish to find some realizations of Theorem 4.7.

LEMMA 5.7. *Assume* (5.1–5.6) *and* (B1). *Let*

$$\mathscr{B}'_{y_\alpha}(x, \{y_\alpha\}) = o\left(\sum_{|\alpha|\le l-1} |y_\alpha|^{p-1}\right) \quad \textit{uniformly in } \Omega. \tag{5.23}$$

Then

$$Q_* \subset \{0\}. \tag{5.24}$$

Proof. By (5.3), (5.4), $A_\theta \ge \|u\|^p$ with θ sufficiently large. By (5.23), $B_\theta(u) = o(\|u\|^p)$. Thus $Q_* \subset \{0\}$. □

THEOREM 5.8. *Assume* (5.1–5.6), (B1) *and* (5.23). *Then for every*

$$\rho \in \left(0, \sup_t \sigma'_+(t)\right) \tag{5.25}$$

there is a non-zero solution of

$$\rho \sum_{|\alpha|\le l} (-1)^{|\alpha|}\partial^\alpha \mathscr{A}(x, \{\partial^\gamma u(x)\}) = \sum_{|\beta|\le l-1} (-1)^{|\beta|}\partial^\beta \mathscr{B}(x, \{\partial^\gamma u(x)\}). \tag{5.26}$$

Moreover,

$$\sup_t \sigma'_+(t) \ge \sup_{u\ne 0} B(u)/A(u). \tag{5.27}$$

Proof. By Theorem 4.7, (5.26) is solvable for $\rho \in I(a, b)$. The lower bound in $I(a, b)$ is less than $\sigma(t)/t$ which goes to zero when t tends to ∞. The upper bound of $I(a, b)$ is greater or equal to any given slope of a secant line to the graph of σ, e.g. $\sigma(t)/t$, which implies (5.27). □

THEOREM 5.9. *Assume that* $k = 1$,

$$\mathscr{B}(x, \{y_\alpha\}) = F(y_0), \qquad \mathscr{A}(x, \{\partial_x^\alpha u\}) = |\nabla u|^p.$$

Let $F \in C^1$ *and if* $n \geq p$, *let* $F'(s) = o(|s|^{q-1})$, $q < pn/(n-p)$. *Assume that for some* $\varepsilon > 0$

(5.28) $F'(s)/s^{p-1+\varepsilon}$ *is an increasing function in a neighbourhood of* $+\infty$,

$$F'(s) = o(|s|^{p-1}) \quad \text{as } s \to -\infty .$$

Then for every

$$\rho \in \left(\inf_t \sigma'_-(t), \infty\right) \tag{5.29}$$

there is a solution $u \neq 0$ *for*

$$-\rho p \operatorname{div}(|\nabla u|^{p-2}\nabla u) = F'(u). \tag{5.30}$$

Proof. Note that $\mathscr{A}$, $\mathscr{B}$ satisfy (5.1–5.6) and B1.

From (5.28) it follows that $\mathscr{B}_\theta(u) \leq o(\|u\|^p)$ for $\theta > 1/(p+\varepsilon)$. Thus $Q_* \cap (0, \infty) = \varnothing$, and one can apply Theorem 4.7. By (5.8) $\sup \sigma'_+(t) \geq \sup_t \sigma(t)/t = \infty$. □

As a general realization of 4.7 we state:

THEOREM 5.10. *Assume* (5.1–5.6) *and* (B1). *Then* (5.26) *has a non-zero solution* u *for*

$$\rho \in (\inf \sigma'_-(t), \sup \sigma'_+(t)) \backslash Q_* . \tag{5.31}$$

6. Examples. The following examples illustrate the solvability results of this paper.

EXAMPLE 6.1. Let $\Omega \subset \mathbf{R}^n$ be an open bounded set. Consider

$$\begin{cases} -\rho \operatorname{div}(|\nabla|^{p-2}\nabla u) = u^\alpha + u^\beta , & u \geq 0, p > 1, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{6.1}$$

Case 1. $0 < \alpha < \beta < p - 1$. Then by Theorem 5.6 there is a continuous family (ρ_t, u_t) of eigenfunctions. By Theorem 5.4, the range of eigenvalues ρ is $I(A, B)$ which is here $(0, \infty)$.

Case 2. $0 < \alpha < p - 1 < \beta$. If $n > p$ assume also $\beta < np/(n-p) - 1$. Then (6.1) has a solution for every $\rho \in (\rho_0, \infty)$, $\rho_0 > 0$ and

$$\rho_0 \leq \inf_t \sup_{\|u\|_{1,p}=1} t^{-1} \int_\Omega \left(\frac{p}{\alpha+1} u^{\alpha+1} t^{(\alpha+1)/p} + \frac{p}{\beta+1} u^{\beta+1} t^{(\beta+1)/p} \right) dx . \tag{6.2}$$

Solvability for $\rho \in I(A, B)$ is provided by Theorem 5.9.

Case 3. $p-1 < \alpha < \beta$ and if $n > p$, $\beta < np/(n-p) - 1$. Similarly, (6.1) is solvable with $\rho \in (0, \infty)$ by Theorem 5.9.

Case 4. $0 < \alpha < \beta = p - 1$. The argument is like in Case 1, only $I(A, B) = (0, \rho_0)$,

$$\rho_0 = \sup_{\|u\|_{l,p}=1} \frac{1}{p} \int |u|^p . \tag{6.3}$$

Case 5. $0 < \alpha = p - 1 < \beta$, if $n > p$, $\beta < np/(n-p) - 1$. The argument follows one of Case 3, but $I(a, b) = (\rho_0, \infty)$, where ρ_0 is like in (6.3).

EXAMPLE 6.2. Let $\Omega \subset \mathbf{R}^n$, $n < 6$, be an open bounded domain. Consider a system:

$$\begin{cases} -3\rho \operatorname{div}(|\nabla u| + \frac{1}{2}|\nabla v|^{3/2}|\nabla u|^{-1/2})\nabla u = 5u^4, \\ -3\rho \operatorname{div}(|\nabla v| + \frac{1}{2}|\nabla u|^{3/2}|\nabla v|^{-1/2})\nabla v = 4v^3, \qquad u, v|_{\partial\Omega} = 0. \end{cases}$$

This system corresponds to

$$A(u, v) = \int_\Omega (|\nabla u|^3 + |\nabla v|^3 + |\nabla u|^{3/2}|\nabla v|^{3/2})\, dx,$$
$$B(u, v) = \int_\Omega (u^5 + v^4)\, dx.$$

By Theorem 5.9, it is solvable for $\rho \in I(A, B)$ and $I(A, B) = (0, \infty)$.

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