

## COMPACT PERTURBATIONS OF ORTHOGONAL POLYNOMIALS

PAUL NEVAI AND WALTER VAN ASSCHE

**We investigate orthogonal polynomials on the real line defined by a recurrence relation for which the recurrence coefficients behave asymptotically like a given system of recurrence coefficients. We give the asymptotic behavior of the orthogonal polynomials (relative to the given comparison system of orthogonal polynomials) and from this we deduce properties of the orthogonality measure.**

**1. Introduction.** Let  $\mu_0$  be a positive unit measure on the real line such that its support contains infinitely many points and all the moments are finite. Assume moreover that the Hamburger moment problem is determined. Denote by  $p_n(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) the normalized orthogonal polynomials for the measure  $\mu_0$ , i.e.

$$\int p_n(x; \mu_0)p_m(x; \mu_0) d\mu_0(x) = \delta_{m,n}, \quad m, n > 0,$$

$$p_n(x; \mu_0) = \gamma_n^0 x^n + \dots, \quad \gamma_n^0 > 0.$$

Let  $a_n^0 > 0$  ( $n = 1, 2, \dots$ ) and  $b_n^0$  ( $n = 0, 1, 2, \dots$ ) be the recurrence coefficients for these orthogonal polynomials, so that

$$(1.1) \quad xp_n(x; \mu_0) = a_{n+1}^0 p_{n+1}(x; \mu_0) + b_n^0 p_n(x; \mu_0) + a_n^0 p_{n-1}(x; \mu_0), \quad n \geq 0,$$

with initial values  $p_{-1}(x; \mu_0) = 0$  and  $p_0(x; \mu_0) = 1$ . We will study a new sequence of orthogonal polynomials  $p_n(x; \mu)$  ( $n = 0, 1, 2, \dots$ ) with recurrence coefficients  $a_n$  ( $n = 1, 2, \dots$ ) and  $b_n$  ( $n = 0, 1, 2, \dots$ ) such that

$$(1.2) \quad \lim_{n \rightarrow \infty} |a_n - a_n^0| = 0, \quad \lim_{n \rightarrow \infty} |b_n - b_n^0| = 0.$$

The orthogonal polynomials  $p_n(x; \mu)$  ( $n = 0, 1, 2, \dots$ ) are said to be a *compact perturbation* of the comparison system  $p_n(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) because the corresponding Jacobi matrix  $J$  with entries  $b_n$  on the diagonal and  $a_n$  on the subdiagonals is a compact perturbation of the Jacobi matrix  $J_0$  with entries  $b_n^0$  and  $a_n^0$ . As a special case one may consider *finite perturbations* (of order  $m \in \mathbb{Z}^+$ )

$p_n(x; \mu_m)$  ( $n = 0, 1, 2, \dots$ ) for which the recurrence coefficients satisfy

$$(1.3) \quad a_{n+1} = a_{n+1}^0, \quad b_n = b_n^0, \quad n \geq m.$$

Notice that for  $m = 0$  we get the comparison system itself. One of the most relevant questions is to find a closed expression for the orthogonality measure  $\mu_m$  for these orthogonal polynomials and the weak limit as  $m \rightarrow \infty$ . This problem has been treated for some specific comparison systems: if  $p_n(x; \mu_0) = U_n(x)$ , the *Chebyshev polynomials of the second kind*, then  $a_{n+1}^0 = 1/2$  and  $b_n^0 = 0$  for  $n \geq 0$ , and a detailed study has been done in [21]. A scattering method was used by Geronimo and Case [11] for finite perturbations of the Chebyshev polynomials of the second kind and there are explicit formulas in [9]. These authors also have results when  $m$  tends to infinity. For a survey of these results we refer to [28]. Geronimo et al. [12] treat the case where the recurrence coefficients of the comparison system are periodic with period  $N > 1$ . Geronimus [14] was one of the first authors to give a detailed account of finite perturbations of Chebyshev polynomials of the second kind and of orthogonal polynomials with asymptotically periodic recurrence coefficients. See also Grosjean [15], [16] for such finite perturbations.

In what follows we will frequently use *associated polynomials* of order  $k \in \mathbb{Z}^+$ . These polynomials  $p_n^{(k)}(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) are defined by the shifted recurrence relation

$$(1.4) \quad xp_n^{(k)}(x; \mu_0) = a_{n+k+1}^0 p_{n+1}^{(k)}(x; \mu_0) + b_{n+k}^0 p_n^{(k)}(x; \mu_0) + a_{n+k}^0 p_{n-1}^{(k)}(x; \mu_0), \quad n \geq 0,$$

with initial values  $p_{-1}^{(k)}(x; \mu_0) = 0$  and  $p_0^{(k)}(x; \mu_0) = 1$ . For  $k = 0$  these associated polynomials reduce to the comparison system and we therefore omit the superscript for  $k = 0$ . One can easily verify that

$$(1.5) \quad p_{n-k}^{(k)}(x; \mu_0) = a_k^0 \int \frac{p_n(x; \mu_0) - p_n(t; \mu_0)}{x - t} p_{k-1}(t; \mu_0) d\mu_0(t).$$

It is important to realize that  $p_{n-k}^{(k)}(x; \mu_0)$  is, for every  $k \in \mathbb{Z}^+$ , a solution of the recurrence relation (1.1) but with initial conditions  $p_{-1}^{(k)}(x; \mu_0) = 0$  and  $p_0^{(k)}(x; \mu_0) = 1$ . We will also need the *functions of the second kind*  $q_n(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) defined as

$$(1.6) \quad q_n(x; \mu_0) = \int \frac{p_n(t; \mu_0)}{x - t} d\mu_0(t), \quad x \in \mathbb{C} \setminus \text{supp}(\mu_0).$$

A straightforward analysis reveals that these functions of the second kind are also a solution of the recurrence relation (1.1), but with initial conditions  $a_0^0 q_{-1}(x; \mu_0) = 1$  and  $q_0(x; \mu_0) = \int \frac{1}{x-t} d\mu_0(t)$ . This sequence is important because this is, for  $x \in \mathbb{C} \setminus \text{supp}(\mu_0)$ , the *minimal solution* of the recurrence relation (1.1). Observe that  $q_0(x; \mu_0)$  is the *Stieltjes transform* of the measure  $\mu_0$ . If one knows  $q_0(x; \mu_0)$  for  $x \in \mathbb{C} \setminus \mathbb{R}$ , then one also knows the measure  $\mu_0$  by Stieltjes' inversion formula (Wintner [30, pp. 93–96], [27, p. 175]):

$$(1.7) \quad \frac{1}{2} \mu_0(\{x\}) + \frac{1}{2} \mu_0(\{y\}) + \mu_0(]x, y[) \\ = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_x^y \{q_0(t - i\varepsilon; \mu_0) - q_0(t + i\varepsilon; \mu_0)\} dt.$$

Associated polynomials and functions of the second kind already appear in Stieltjes' fundamental work [25]. Some interesting properties and formulas may be found in the works of Perron [23] and Geronimus [14]. Functions of the second kind and associated polynomials are usually only studied for classical orthogonal polynomials (Askey and Wimp [2], Barrucand and Dickinson [5], Chihara [7], Grosjean [17], Sherman [24], Szegő [26] and Wimp [29]) but recently more general orthogonal polynomials have also been considered ([4], [6], [10], [22]). If the measure  $\mu_0$  satisfies some extra regularity conditions, then the functions of the second kind may be extended to the support of  $\mu_0$  and then there is a simple relationship with the measure  $\mu_0$  and the function  $q_n(x; \mu_0)$ , which is essentially given by Privalov's theorem [31, §7.4]:

LEMMA 1. *Suppose  $\mu_0$  is an absolutely continuous measure on the interval  $[a, b]$  ( $a$  and  $b$  finite), with weight function  $w_0(x)$ . If  $w_0$  is of Dini type on  $[a, b]$ , i.e.*

$$(1.8) \quad \int_0^{b-a} \frac{\omega(w_0; t)}{t} dt < \infty,$$

where  $\omega(w_0; t)$  is the modulus of continuity of  $w_0$  on  $[a, b]$ , defined by

$$\omega(w_0; t) = \sup\{|w_0(x) - w_0(y)| : x, y \in [a, b], |x - y| < t\},$$

then the Cauchy principal value integral

$$\int_a^b \frac{f(t)w_0(t)}{x - t} dt$$

exists for every  $x \in ]a, b[$  and every polynomial  $f$  and

$$(1.9) \quad q_n(x; \mu_0) = \lim_{\varepsilon \rightarrow 0^+} q_n(x + i\varepsilon; \mu_0) \\ = \int_a^b \frac{p_n(t; \mu_0)w_0(t)}{x - t} dt - i\pi p_n(x; \mu_0)w_0(x), \\ a < x < b.$$

**2. The comparison equation.** It is very natural to compare  $p_n(x; \mu)$  with  $p_n(x; \mu_0)$  because the recurrence coefficients of both systems are asymptotically equal. The general setup for such a comparison is given by Geronimo et al. [12, Eq. III.8] (see also [27, p. 54]). Define

$$(2.1) \quad \hat{p}_n(x; \mu) = \left( \prod_{j=1}^n \frac{a_j}{a_j^0} \right) p_n(x; \mu),$$

then  $\hat{p}_n(x; \mu)$  and  $p_n(x; \mu_0)$  have the same leading coefficient and  $\hat{p}_n(x; \mu)$  satisfies the recurrence relation

$$(2.2) \quad x\hat{p}_k(x; \mu) = a_{k+1}^0 \hat{p}_{k+1}(x; \mu) + b_k \hat{p}_k(x; \mu) + \frac{a_k^2}{a_k^0} \hat{p}_{k-1}(x; \mu).$$

A linear recurrence relation of second order cannot have three linearly independent solutions; therefore there exist  $A_k$  and  $B_k$ —independent of  $n$ —such that

$$p_{n-k-2}^{(k+2)}(x; \mu_0) = A_k p_{n-k-1}^{(k+1)}(x; \mu_0) + B_k p_{n-k}^{(k)}(x; \mu_0).$$

Setting  $n = k$  and  $n = k + 1$  gives  $B_k = -a_{k+2}^0/a_{k+1}^0$  and  $A_k = a_{k+2}^0/(a_{k+1}^0)^2(x - b_k^0)$ , so that

$$(2.3) \quad xp_{n-k-1}^{(k+1)}(x; \mu_0) = a_{k+1}^0 p_{n-k}^{(k)}(x; \mu_0) + b_k^0 p_{n-k-1}^{(k+1)}(x; \mu_0) \\ + \frac{(a_{k+1}^0)^2}{a_{k+2}^0} p_{n-k-2}^{(k+2)}(x; \mu_0).$$

(Equation (2.3) can also be derived from (1.5) by using the recurrence formula for  $p_{k-1}(t; \mu_0)$  in the integrand.) Multiply (2.2) by  $p_{n-k-1}^{(k+1)}(x; \mu_0)$  and (2.3) by  $\hat{p}_k(x; \mu)$ , then subtract the obtained equa-

tions to find

$$\begin{aligned}
 0 &= a_{k+1}^0(p_{n-k}^{(k)}(x; \mu_0)\hat{p}_k(x; \mu) - p_{n-k-1}^{(k+1)}(x; \mu_0)\hat{p}_{k+1}(x; \mu)) \\
 &\quad + (b_k^0 - b_k)p_{n-k-1}^{(k+1)}(x; \mu_0)\hat{p}_k(x; \mu) \\
 &\quad + \frac{(a_{k+1}^0)^2}{a_{k+2}^0}p_{n-k-2}^{(k+2)}(x; \mu_0)\hat{p}_k(x; \mu) \\
 &\quad - \frac{(a_k^0)^2}{a_k^0}p_{n-k-1}^{(k+1)}(x; \mu_0)\hat{p}_{k-1}(x; \mu).
 \end{aligned}$$

Divide every term by  $a_{k+1}^0$  and sum from  $k = 0$  to  $k = n - 1$ . Then one has

$$\begin{aligned}
 (2.4) \quad \hat{p}_n(x; \mu) &= p_n(x; \mu_0) \\
 &\quad + \sum_{k=0}^{n-1} \left\{ \frac{b_k^0 - b_k}{a_{k+1}^0} p_{n-k-1}^{(k+1)}(x; \mu_0) \right. \\
 &\quad \left. + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0 a_{k+2}^0} p_{n-k-2}^{(k+2)}(x; \mu_0) \right\} \hat{p}_k(x; \mu).
 \end{aligned}$$

This is the comparison equation that plays a fundamental role in our investigation of perturbations of  $p_n(x; \mu_0)$ . The equation shows that  $\hat{p}_n(x; \mu)$  is equal to  $p_n(x; \mu_0)$  plus a perturbation. The perturbation contains the differences  $a_{k+1}^0 - a_{k+1}$  and  $b_k^0 - b_k$  and all the associated polynomials  $p_{n-k}^{(k)}(x; \mu_0)$  of order  $k = 1, 2, \dots, n+1$ . This comparison equation enables us to find bounds for the polynomials  $p_n(x; \mu)$  if bounds are known for the comparison system  $p_n(x; \mu_0)$ . Note that it is sufficient to study the orthogonal polynomials  $p_n(x)$  and functions of the second kind  $q_n(x)$  for  $x \in \mathbb{C}^+ = \{x \in \mathbb{C} : \Im x \geq 0\}$ <sup>1</sup> because  $p_n(\bar{x}) = \overline{p_n(x)}$  and  $q_n(\bar{x}) = \overline{q_n(x)}$ .

LEMMA 2. Suppose  $K$  is a compact set in  $\mathbb{C}^+$  such that

$$(2.5) \quad a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) \geq C$$

where  $0 < C < \infty$  is some positive constant. Then

$$\begin{aligned}
 (2.6) \quad &|\hat{p}_n(x; \mu)q_n(x; \mu_0)| \\
 &\leq \frac{1}{C} \exp \left\{ \sum_{k=0}^{n-1} \left( \frac{|b_k^0 - b_k|}{C} + \frac{|(a_{k+1}^0)^2 - (a_{k+1})^2|}{C^2} \right) \right\}
 \end{aligned}$$

for every  $x \in K$ .

<sup>1</sup>  $\Im x$  denotes the imaginary part of the complex number  $x$ .

*Proof.* Denote by  $\mu_0^{(k)}$  the orthogonality measure for the  $k$ th associated polynomials; then  $p_n(x; \mu_0^{(k)}) = p_n^{(k)}(x; \mu_0)$ . The functions of the second kind  $q_{n-k}(x; \mu_0^{(k)})$  are a solution of the recurrence relation (1.1). By minimality and by checking the initial conditions, one finds

$$(2.7) \quad a_k^0 q_{n-k}(x; \mu_0^{(k)}) = \frac{q_n(x; \mu_0)}{q_{k-1}(x; \mu_0)}.$$

By using the Wronskian formula

$$(2.8) \quad a_m [p_m(x; \mu) q_{m-1}(x; \mu) - p_{m-1}(x; \mu) q_m(x; \mu)] = 1$$

for  $m = n - k$  and  $\mu = \mu_0^{(k)}$  one finds

$$\begin{aligned} \left| \frac{a_k^0 q_{k-1}(x; \mu_0)}{p_{n-k}^{(k)}(x; \mu_0) q_n(x; \mu_0)} \right| &= \left| \frac{1}{p_{n-k}(x; \mu_0^{(k)}) q_{n-k}(x; \mu_0^{(k)})} \right| \\ &= a_n^0 \left| \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} - \frac{p_{n-k-1}^{(k)}(x; \mu_0)}{p_{n-k}^{(k)}(x; \mu_0)} \right| \\ &\geq a_n^0 \left| \Im \left\{ \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} - \frac{p_{n-k-1}^{(k)}(x; \mu_0)}{p_{n-k}^{(k)}(x; \mu_0)} \right\} \right|. \end{aligned}$$

The zeros of  $p_{n-k-1}^{(k)}(x; \mu_0)$  and  $p_{n-k}^{(k)}(x; \mu_0)$  interlace. This immediately implies

$$\Im \left( \frac{p_{n-k-1}^{(k)}(x; \mu_0)}{p_{n-k}^{(k)}(x; \mu_0)} \right) \leq 0, \quad x \in \mathbb{C}^+.$$

From (2.7) it follows that the ratio  $q_n(x; \mu_0)/q_{n-1}(x; \mu_0)$  is the Stieltjes transform of a positive measure, and hence

$$\Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) \geq 0, \quad x \in \mathbb{C}^+.$$

Therefore we have for every  $x \in K$

$$(2.9) \quad |p_{n-k}^{(k)}(x; \mu_0) q_{n-k}(x; \mu_0^{(k)})| \leq \left\{ a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) \right\}^{-1} \leq \frac{1}{C}.$$

Now multiply both sides of (2.4) by  $q_n(x; \mu_0)$  and use (2.7) to obtain

$$\begin{aligned}
 (2.10) \quad & \hat{p}_n(x; \mu) q_n(x; \mu_0) \\
 &= p_n(x; \mu_0) q_n(x; \mu_0) \\
 &+ \sum_{k=0}^{n-1} \left\{ (b_k^0 - b_k) p_{n-k-1}^{(k+1)}(x; \mu_0) q_{n-k-1}(x; \mu_0^{(k+1)}) \right. \\
 &\quad \left. + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0} \frac{q_{k+1}(x; \mu_0)}{q_k(x; \mu_0)} \right. \\
 &\quad \left. \times p_{n-k-2}^{(k+2)}(x; \mu_0) q_{n-k-2}(x; \mu_0^{(k+2)}) \right\} \\
 &\quad \times \hat{p}_k(x; \mu) q_k(x; \mu_0).
 \end{aligned}$$

By using the bounds (2.5) and (2.9) and

$$(2.11) \quad \left| \frac{q_{k+1}(x; \mu_0)}{q_k(x; \mu_0)} \right| \leq \frac{a_{k+1}^0}{C}$$

one finds

$$\begin{aligned}
 |\hat{p}_n(x; \mu) q_n(x; \mu_0)| &\leq \frac{1}{C} + \sum_{k=0}^{n-1} \left( \frac{|b_k^0 - b_k|}{C} + \frac{|(a_{k+1}^0)^2 - (a_{k+1})^2|}{C^2} \right) \\
 &\quad \times |\hat{p}_k(x; \mu) q_k(x; \mu_0)|.
 \end{aligned}$$

The upper bound (2.6) then follows by using Gronwall's inequality (see e.g. [28, p. 440]).  $\square$

The condition (2.5) is a very natural one and is true for a large class of orthogonal polynomials. Note that

$$\Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) = \frac{\overline{q_n(x; \mu_0)} q_{n-1}(x; \mu_0) - q_n(x; \mu_0) \overline{q_{n-1}(x; \mu_0)}}{2i |q_n(x; \mu_0)|^2},$$

and by combining the recurrence relations for  $q_n(x; \mu_0)$  and  $\overline{q_n(x; \mu_0)}$  one has

$$\begin{aligned}
 (2.12) \quad & a_{n+1}^0 \{ q_{n+1}(x; \mu_0) \overline{q_n(x; \mu_0)} - \overline{q_{n+1}(x; \mu_0)} q_n(x; \mu_0) \} \\
 &= (x - \bar{x}) |q_n(x; \mu_0)|^2 + a_n^0 \{ q_n(x; \mu_0) \overline{q_{n-1}(x; \mu_0)} \\
 &\quad - \overline{q_n(x; \mu_0)} q_{n-1}(x; \mu_0) \}.
 \end{aligned}$$

If  $x \in \mathbb{C}^+ \setminus \mathbb{R}$  then iterating up gives

$$\begin{aligned}
 (2.13) \quad & a_n^0 \{ q_n(x; \mu_0) \overline{q_{n-1}(x; \mu_0)} - \overline{q_n(x; \mu_0)} q_{n-1}(x; \mu_0) \} \\
 &= (\bar{x} - x) \sum_{k=n}^{\infty} |q_k(x; \mu_0)|^2
 \end{aligned}$$

which shows that  $a_n^0 \Im\{q_{n-1}(x; \mu_0)/q_n(x; \mu_0)\}$  is bounded from below on every compact set  $K$  in  $\mathbb{C}^+ \setminus \mathbb{R}$ . If one iterates (2.12) down then

$$(2.14) \quad a_n^0 \{q_n(x; \mu_0) \overline{q_{n-1}(x; \mu_0)} - \overline{q_n(x; \mu_0)} q_{n-1}(x; \mu_0)\} \\ = (x - \bar{x}) \sum_{k=0}^{n-1} |q_k(x; \mu_0)|^2 + \{q_0(x; \mu_0) - \overline{q_0(x; \mu_0)}\};$$

hence

$$(2.15) \quad a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) = - \frac{(\Im x) \sum_{k=0}^{n-1} |q_k(x; \mu_0)|^2 + \Im q_0(x; \mu_0)}{|q_n(x; \mu_0)|^2}.$$

For real  $x$  the right-hand side of equation (2.15) can only vanish when  $\Im q_0(x; \mu_0)$  vanishes. If  $\mu_0$  is absolutely continuous on an interval  $[a, b]$  and if the weight function  $w_0$  is of Dini type on  $[a, b]$  (see Lemma 1), then  $\Im q_0(x; \mu_0) = -\pi w_0(x)$ . This means that the conditions of Lemma 2 are satisfied when  $\mu_0$  is absolutely continuous on  $[a, b]$  with a weight function  $w_0$  of Dini type on  $[a, b]$  and with  $K$  a compact set of  $\mathbb{C}^+ \setminus \{x \in \mathbb{R} : w_0(x) = 0\}$  not including the endpoints  $a$  and  $b$ . Other conditions on the weight functions  $w_0$  (e.g.  $w_0$  may have a finite number of (integrable) singularities as is the case for generalized Jacobi weights) also lead to (2.5) for an appropriate choice of the set  $K$ .

**3. Asymptotic results.** An important problem is to find which properties of the comparison system  $p_n(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) are transferred to the system  $p_n(x; \mu)$  ( $n = 0, 1, 2, \dots$ ) when we are dealing with compact perturbations. One such property is the asymptotic behavior of the ratio of two consecutive orthogonal polynomials:

**THEOREM 1.** *Suppose that  $p_n(x; \mu)$  ( $n = 0, 1, 2, \dots$ ) is a compact perturbation of  $p_n(x; \mu_0)$  ( $n = 0, 1, 2, \dots$ ) so that (1.2) holds, and assume that the recurrence coefficients  $a_{n+1}^0$  ( $n = 0, 1, 2, \dots$ ) are bounded. Let  $S \subset \mathbb{N}$  be an infinite set of positive integers and define  $X(S)$  as the closure of the set of all zeros of  $p_n(x; \mu)$  and  $p_n(x; \mu_0)$  as  $n$  runs through  $S$ , then*

$$(3.1) \quad \lim_{n \rightarrow \infty, n \in S} a_{n+1}^0 \left\{ \frac{p_{n+1}(x; \mu_0)}{p_n(x; \mu_0)} - \frac{\hat{p}_{n+1}(x; \mu)}{\hat{p}_n(x; \mu)} \right\} = 0,$$

uniformly on compact subsets of  $\mathbb{C} \setminus X(S)$ .

*Proof.* Multiply (2.4) by  $p_{n+1}(x; \mu_0)$ , then change  $n$  to  $n + 1$  in (2.4) and multiply by  $p_n(x; \mu_0)$  and subtract the two resulting equa-



tions to find

$$\begin{aligned} & \hat{p}_{n+1}(x; \mu)p_n(x; \mu_0) - \left(\frac{a_{n+1}}{a_{n+1}^0}\right)^2 \hat{p}_n(x; \mu)p_{n+1}(x; \mu_0) \\ &= \sum_{k=0}^n \hat{p}_k(x; \mu) \left\{ \frac{b_k^0 - b_k}{a_{k+1}^0} [p_{n-k}^{(k+1)}(x; \mu_0)p_n(x; \mu_0) \right. \\ & \quad \left. - p_{n-k-1}^{(k+1)}(x; \mu_0)p_{n+1}(x; \mu_0)] \right. \\ & \quad \left. + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0 a_{k+2}^0} \right. \\ & \quad \left. \cdot [p_{n-k-1}^{(k+2)}(x; \mu_0)p_n(x; \mu_0) \right. \\ & \quad \left. - p_{n-k-2}^{(k+2)}(x; \mu_0)p_{n+1}(x; \mu_0)] \right\}. \end{aligned}$$

Recall that  $p_{n-k-1}^{(k+1)}(x; \mu_0)$  (for fixed  $k$ ) and  $p_n(x; \mu_0)$  are two solutions of the recurrence relation (1.1). Their Wronskian is given by

$$(3.2) \quad a_{n+1}^0 \{p_{n-k-1}^{(k+1)}(x; \mu_0)p_n(x; \mu_0) - p_{n-k-1}^{(k+1)}(x; \mu_0)p_{n+1}(x; \mu_0)\} \\ = a_{k+1}^0 p_k(x; \mu_0).$$

If one uses this Wronskian formula in the previous equation, then

$$(3.3) \quad a_{n+1}^0 \left\{ \hat{p}_{n+1}(x; \mu)p_n(x; \mu_0) - \left(\frac{a_{n+1}}{a_{n+1}^0}\right)^2 \hat{p}_n(x; \mu)p_{n+1}(x; \mu_0) \right\} \\ = \sum_{k=0}^n \left\{ (b_k^0 - b_k)p_k(x; \mu_0) \right. \\ \left. + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0} p_{k+1}(x; \mu_0) \right\} \hat{p}_k(x; \mu).$$

Divide by  $\hat{p}_n(x; \mu)p_n(x; \mu_0)$ . Then

$$(3.4) \quad a_{n+1}^0 \left\{ \frac{\hat{p}_{n+1}(x; \mu)}{\hat{p}_n(x; \mu)} - \left(\frac{a_{n+1}}{a_{n+1}^0}\right)^2 \frac{p_{n+1}(x; \mu_0)}{p_n(x; \mu_0)} \right\} \\ = \frac{1}{\hat{p}_n(x; \mu)p_n(x; \mu_0)} \left( \sum_{k=0}^n c_k p_k(x; \mu_0) \hat{p}_k(x; \mu) \right. \\ \left. + \sum_{k=0}^n d_k p_{k+1}(x; \mu_0) \hat{p}_k(x; \mu) \right)$$

where

$$c_k = b_k^0 - b_k, \quad d_k = \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0}.$$

Notice that  $c_k \rightarrow 0$  and  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ . We show that

$$\lim_{n \rightarrow \infty, n \in S} \frac{1}{\hat{p}_n(x; \mu) p_n(x; \mu_0)} \sum_{k=0}^n c_k p_k(x; \mu_0) \hat{p}_k(x; \mu) = 0$$

uniformly on compact subsets of  $\mathbb{C} \setminus X(S)$ ; a similar reasoning holds for the sum involving  $d_k$ . By Schwarz' inequality we have

$$\left| \sum_{k=0}^n c_k p_k(x; \mu_0) \hat{p}_k(x; \mu) \right|^2 \leq \sum_{k=0}^n |c_k| |p_k(x; \mu_0)|^2 \sum_{k=0}^n |c_k| |\hat{p}_k(x; \mu)|^2,$$

hence it is sufficient to prove that

$$(3.5) \quad \lim_{n \rightarrow \infty, n \in S} \frac{1}{|p_n(x; \mu_0)|^2} \sum_{k=0}^{n-1} |c_k| |p_k(x; \mu_0)|^2 = 0,$$

and the same with  $\hat{p}_k(x; \mu)$  instead of  $p_k(x; \mu_0)$ . By standard techniques (3.5) follows if we can show that there is a constant  $C$  such that

$$(3.6) \quad \frac{1}{|p_n(x; \mu_0)|^2} \sum_{k=0}^{n-1} |p_k(x; \mu_0)|^2 \leq C,$$

uniformly on compact sets of  $\mathbb{C} \setminus X(S)$ . Let  $K$  be a compact set in  $\mathbb{C} \setminus X(\mathbb{N})$ , then the distance from  $K$  to  $X(\mathbb{N})$  is strictly positive and we denote this distance by  $\delta$ . Recall the partial fractions decomposition

$$\frac{p_{n-1}(x; \mu_0)}{p_n(x; \mu_0)} = a_n^0 \sum_{j=1}^n \frac{\lambda_{j,n} p_{n-1}^2(x_{j,n}; \mu_0)}{x - x_{j,n}},$$

where  $\lambda_{j,n}$  ( $1 \leq j \leq n$ ) are the Christoffel numbers and  $x_{j,n}$  ( $1 \leq j \leq n$ ) are the zeros of  $p_n(x; \mu_0)$  (see e.g. [26, Theorem 3.3.5], [27, p. 9]). All the zeros of  $p_n(x; \mu_0)$  ( $n = 1, 2, \dots$ ) are in  $X(\mathbb{N})$ ; hence  $|x - x_{j,n}| \geq \delta$  and therefore

$$(3.7) \quad \left| \frac{p_{n-1}(x; \mu_0)}{p_n(x; \mu_0)} \right| \leq \frac{a_n^0}{\delta} \leq \frac{M}{\delta},$$

where  $M$  is a constant such that  $a_n^0 \leq M$  for all  $n$ . From this one finds

$$\left| \frac{p_k(x; \mu_0)}{p_n(x; \mu_0)} \right| \leq (M/\delta)^{n-k}$$

and thus

$$\frac{1}{|p_n(x; \mu_0)|^2} \sum_{k=0}^{n-1} |p_k(x; \mu_0)|^2 \leq \sum_{k=0}^{n-1} (M/\delta)^{n-k}.$$

If  $\delta > M$  then the geometric series on the right-hand side converges and hence (3.6) is true for  $\delta$  large enough. Therefore we have shown that the theorem is true on compact sets  $K$  which are far enough away from  $X(\mathbb{N})$ . The left-hand side of (3.1) is uniformly bounded on compact sets of  $\mathbb{C} \setminus X(S)$  (this follows easily from (3.7)), and since the full sequence converges to zero on a set with an accumulation point, we can use the Stieltjes-Vitali theorem to complete our proof.  $\square$

In view of the upper bound (2.6) it is natural to suppose that

$$(3.8) \quad \sum_{n=0}^{\infty} (|b_n^0 - b_n| + |(a_{n+1}^0)^2 - (a_{n+1})^2|) < \infty.$$

Let us introduce the function

$$(3.9) \quad \phi(x) = 1 + \sum_{k=0}^{\infty} \left\{ (b_k^0 - b_k) q_k(x; \mu_0) + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0} q_{k+1}(x; \mu_0) \right\} \hat{p}_k(x; \mu).$$

If  $K$  is a compact set of  $\mathbb{C}^+$  such that (2.5) holds, then from Lemma 2, (3.8) and (2.11) we deduce that  $\phi$  is a continuous function in  $K$  which is analytic in the interior of  $K$ . In particular one has

$$\lim_{y \rightarrow 0^+} \phi(x + iy) = \phi(x),$$

for every  $x \in K \cap \mathbb{R}$ .

**THEOREM 2.** *Suppose that (3.8) holds and that  $K$  is a compact set in  $\mathbb{C}^+$  such that (2.5) holds for every  $x \in K$ ; then*

$$(3.10) \quad \lim_{n \rightarrow \infty} a_{n+1}^0 \left\{ \hat{p}_{n+1}(x; \mu) q_n(x; \mu_0) - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu) q_{n+1}(x; \mu_0) \right\} = \phi(x)$$

*holds uniformly on  $K$ .*

*Proof.* First multiply (2.4) by  $q_{n+1}(x; \mu_0)$ , then change  $n$  to  $n+1$  in (2.4) and multiply by  $q_n(x; \mu_0)$ . Subtract the two resulting equations and use the Wronskian

$$(3.11) \quad a_{n+1}^0 \{p_{n-k}^{(k+1)}(x; \mu_0)q_n(x; \mu_0) - p_{n-k-1}^{(k+1)}(x; \mu_0)q_{n+1}(x; \mu_0)\} \\ = a_{k+1}^0 q_k(x; \mu_0)$$

to find

$$(3.12) \quad a_{n+1}^0 \left\{ \hat{p}_{n+1}(x; \mu)q_n(x; \mu_0) - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu)q_{n+1}(x; \mu_0) \right\} \\ = 1 + \sum_{k=0}^n \left\{ (b_k^0 - b_k)q_k(x; \mu_0) \right. \\ \left. + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0} q_{k+1}(x; \mu_0) \right\} \hat{p}_k(x; \mu).$$

The result then follows immediately.  $\square$

**COROLLARY 1.** *Suppose that (3.8) holds. Let  $S$  be an infinite set of positive integers and define  $X(S)$  as the closure of the zeros of  $p_n(x; \mu)$  and  $p_n(x; \mu_0)$  as  $n$  runs through  $S$ . Then*

$$(3.13) \quad \lim_{n \rightarrow \infty, n \in S} \frac{\hat{p}_n(x; \mu)}{p_n(x; \mu_0)} = \phi(x)$$

uniformly for  $x$  on compact subsets of  $\mathbb{C}^+ \setminus X(S)$ .

*Proof.* The left-hand side of (3.10) can be written as

$$a_{n+1}^0 \left\{ \hat{p}_{n+1}(x; \mu)q_n(x; \mu_0) - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu)q_{n+1}(x; \mu_0) \right\} \\ = a_{n+1}^0 \frac{\hat{p}_n(x; \mu)}{p_n(x; \mu_0)} \left\{ \frac{\hat{p}_{n+1}(x; \mu)}{\hat{p}_n(x; \mu)} p_n(x; \mu_0)q_n(x; \mu_0) \right. \\ \left. - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 p_n(x; \mu_0)q_{n+1}(x; \mu_0) \right\}.$$

The result follows from (3.11) (with  $k = -1$ ), Lemma 2 and Theorem 1.  $\square$

**COROLLARY 2.** *Suppose that (3.8) holds and that  $\mu_0$  is absolutely continuous on  $[a, b]$  with a weight function  $w_0$  which is of Dini type on  $[a, b]$ . Let  $K$  be a closed interval in  $[a, b]$  such that (2.5) is satisfied; then*

$$(3.14) \quad \pi \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu) \sqrt{w_0(x)} - \frac{|q_n(x; \mu_0)\phi(x)|}{\sqrt{w_0(x)}} \sin(\arg \phi(x) - \arg q_n(x; \mu_0)) = O \left( \sum_{k=n+1}^{\infty} (|b_k^0 - b_k| + |(a_{k+1}^0)^2 - (a_{k+1})^2|) \right)$$

holds uniformly in  $K$ .

*Proof.* Multiply both sides of (3.12) by  $\overline{q_n(x; \mu_0)}$  and then take the imaginary part of both sides of the equation to find

$$- a_{n+1}^0 \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu) \Im\{q_{n+1}(x; \mu_0)\overline{q_n(x; \mu_0)}\} = \Im\{\overline{q_n(x; \mu_0)}\phi(x)\} - R_n(x)$$

where

$$R_n = \Im \left\{ \overline{q_n(x; \mu_0)} \sum_{k=n+1}^{\infty} \left( (b_k^0 - b_k)q_k(x; \mu_0) + \frac{(a_{k+1}^0)^2 - (a_{k+1})^2}{a_{k+1}^0} q_{k+1}(x; \mu_0) \right) \hat{p}_k(x; \mu) \right\}.$$

The remainder  $R_n(x)$  can easily be estimated by using Lemma 2. Recall that

$$a_{n+1}^0 \Im\{q_{n+1}(x; \mu_0)\overline{q_n(x; \mu_0)}\} = \Im q_0(x) = -\pi w_0(x),$$

and by (2.15)

$$a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) = \frac{\pi w_0(x)}{|q_n(x; \mu_0)|^2} \geq C;$$

hence  $|q_n(x; \mu_0)|^2 \leq \pi w_0(x)/C$ , and the result follows. □

**4. The orthogonality measure.** In this section we will show that the function  $\phi(x)$  defined in (3.9) contains a lot of information concerning the orthogonality measure  $\mu$ .

LEMMA 3. *Suppose (3.8) holds and that  $K$  is a compact set in  $\text{supp}(\mu_0)$  such that (2.5) is true. Then  $\phi(x) \neq 0$  for all  $x \in K$ .*

*Proof.* Define

$$\psi_n(x) = a_{n+1}^0 \left\{ \hat{p}_{n+1}(x; \mu) q_n(x; \mu_0) - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu) q_{n+1}(x; \mu_0) \right\}.$$

Then by (3.12) we have

$$(4.1) \quad \psi_n(x) - \psi_{n-1}(x) = \left\{ (b_n^0 - b_n) q_n(x; \mu_0) + \frac{(a_{n+1}^0)^2 - (a_{n+1})^2}{a_{n+1}^0} q_{n+1}(x; \mu_0) \right\} \hat{p}_n(x; \mu).$$

From (2.5) it follows that  $q_n(x) \neq 0$  for  $x \in K$ . One easily finds the bounds

$$(4.2) \quad \begin{aligned} \left| \frac{\psi_n(x)}{q_n(x; \mu_0)} \right| &\geq \left| \Im \frac{\psi_n(x)}{q_n(x; \mu_0)} \right| \\ &= a_{n+1}^0 \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 |\hat{p}_n(x; \mu)| \left| \Im \frac{q_{n+1}(x; \mu_0)}{q_n(x; \mu_0)} \right| \\ &= a_n^0 \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 |\hat{p}_n(x; \mu)| \left| \Im \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right| \\ &\geq C \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 |\hat{p}_n(x; \mu)| \end{aligned}$$

where we have used the equality

$$a_{n+1}^0 \Im \left( \frac{q_{n+1}(x; \mu_0)}{q_n(x; \mu_0)} \right) = -a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right).$$

By using the bound (4.2) in (4.1) and by the bound

$$\left| \frac{q_{n+1}(x; \mu_0)}{q_n(x; \mu_0)} \right| \leq \frac{a_{n+1}^0}{C}$$

one obtains

$$\begin{aligned}
 (4.3) \quad |\psi_{n-1}(x)| &\leq |\psi_n(x)| \left\{ 1 + \left( |b_n^0 - b_n| \right. \right. \\
 &\quad \left. \left. + \frac{|(a_{n+1}^0)^2 - (a_{n+1})^2|}{C} \right) \frac{1}{Ca_n^0} \left( \frac{a_{n+1}^0}{a_{n+1}} \right)^2 \right\} \\
 &\leq |\psi_n(x)| \exp \left\{ \left( |b_n^0 - b_n| \right. \right. \\
 &\quad \left. \left. + \frac{|(a_{n+1}^0)^2 - (a_{n+1})^2|}{C} \right) \frac{1}{Ca_n^0} \left( \frac{a_{n+1}^0}{a_{n+1}} \right)^2 \right\} \\
 &\leq |\psi_m(x)| \exp \sum_{k=n}^m \left( |b_k^0 - b_k| \right. \\
 &\quad \left. + \frac{|(a_{k+1}^0)^2 - (a_{k+1})^2|}{C} \right) \frac{1}{Ca_k^0} \left( \frac{a_{k+1}^0}{a_{k+1}} \right)^2.
 \end{aligned}$$

which holds for every  $m \geq n$ . Let  $m \rightarrow \infty$ ; then the sum on the right-hand side converges and  $\psi_m(x) \rightarrow \phi(x)$  by Theorem 2. Hence if  $\phi(x) = 0$  then  $\psi_n(x) = 0$  for infinitely many  $n$ . But  $\psi_n(x) = 0$  if and only if both  $p_n(x; \mu)$  and  $p_{n+1}(x; \mu)$  vanish simultaneously, which is impossible because of the interlacing property of the zeros of orthogonal polynomials.  $\square$

**THEOREM 3.** *Suppose that  $K$  is a compact set in  $\text{supp}(\mu_0)$  such that (2.5) holds and assume moreover that (3.8) is true. Then  $\mu$  is absolutely continuous with respect to  $\mu_0$  on  $K$  and there exists a version of the Radon-Nikodym derivative  $d\mu(x)/d\mu_0(x)$  which is continuous on  $K$ . One has the explicit formula*

$$(4.4) \quad |\phi(x)|^2 = \frac{d\mu_0(x)}{d\mu(x)}, \quad x \in K.$$

*Proof.* Consider the Jacobi matrices

$$\begin{aligned}
 J &= \left( \int xp_i(x; \mu)p_j(x; \mu) d\mu(x) \right)_{i,j=0,1,2,\dots}, \\
 J_0 &= \left( \int xp_i(x; \mu_0)p_j(x; \mu_0) d\mu_0(x) \right)_{i,j=0,1,2,\dots},
 \end{aligned}$$

then  $J - J_0 : l_2 \rightarrow l_2$  is a compact operator. If  $\pi(x)$  is a polynomial, then

$$\pi(J) = \left( \int \pi(x) p_i(x; \mu) p_j(x; \mu) d\mu(x) \right)_{i,j=0,1,2,\dots}$$

and similarly for  $\pi(J_0)$ . From

$$J^m - J_0^m = \sum_{k=0}^{m-1} J^k (J - J_0) J_0^{m-k-1}$$

it follows that  $\pi(J) - \pi(J_0)$  is a compact operator for every polynomial  $\pi(x)$  (see Kato [18, p. 158]). Now both  $\pi(J)$  and  $\pi(J_0)$  are banded matrices; hence compactness implies

$$\lim_{n \rightarrow \infty} \left| \int \pi(x) p_n(x; \mu) p_{n+k}(x; \mu) d\mu(x) - \int \pi(x) p_n(x; \mu_0) p_{n+k}(x; \mu_0) d\mu_0 \right| = 0,$$

for every integer  $k \geq 0$ . Introduce the bounded linear functionals

$$\Lambda_n f = \int_K f(x) p_n(x; \mu) p_{n+k}(x; \mu) d\mu(x),$$

$$\Lambda_n^0 f = \int_K f(x) p_n(x; \mu_0) p_{n+k}(x; \mu_0) d\mu_0(x),$$

where  $f \in C_K$ , the Banach space of continuous functions on  $K$  equipped with the maximum norm, then it follows from the Banach-Steinhaus theorem that  $\|\Lambda_n - \Lambda_n^0\|_{C'_K} \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\|\cdot\|_{C'_K}$  is the operator norm on  $C'_K$  associated with the maximum norm  $\|\cdot\|_\infty$  on  $C_K$ . From (2.5) we have

$$C \leq a_n^0 \Im \left( \frac{q_{n-1}(x; \mu_0)}{q_n(x; \mu_0)} \right) = \frac{-\Im q_0(x; \mu_0)}{|q_n(x; \mu_0)|^2} = \frac{\pi w_0(x)}{|q_n(x; \mu_0)|^2},$$

hence

$$\begin{aligned} & |\Lambda_n(f q_n(x; \mu_0) \overline{q_{n+k}(x; \mu_0)}) - \Lambda_n^0(f q_n(x; \mu_0) \overline{q_{n+k}(x; \mu_0)})| \\ & \leq \|\Lambda_n - \Lambda_n^0\|_{C'_K} \|f w_0\|_\infty / C \\ & \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

This implies that the difference of

$$\begin{aligned} & (a_{n+1}^0)^2 \int_K f(x) \left| \hat{p}_{n+1}(x; \mu) q_n(x; \mu_0) \right. \\ & \quad \left. - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu) q_{n+1}(x; \mu_0) \right|^2 d\mu(x) \end{aligned}$$



and

$$(a_{n+1}^0)^2 \int_K f(x) |p_{n+1}(x; \mu_0)q_n(x; \mu_0) - p_n(x; \mu_0)q_{n+1}(x; \mu_0)|^2 d\mu_0(x)$$

converges to zero as  $n \rightarrow \infty$  whenever  $\|fw_0\|_\infty$  is finite. From Theorem 2 we have

$$(4.5) \quad \lim_{n \rightarrow \infty} (a_{n+1}^0)^2 \int_K f(x) \left| \hat{p}_{n+1}(x; \mu)q_n(x; \mu) - \left( \frac{a_{n+1}}{a_{n+1}^0} \right)^2 \hat{p}_n(x; \mu)q_{n+1}(x; \mu) \right|^2 d\mu(x) = \int_K f(x) |\phi(x)|^2 d\mu(x),$$

and by the Wronskian formula (3.11) (with  $k = -1$ ) one has

$$(4.6) \quad (a_{n+1}^0)^2 \int_K f(x) |p_{n+1}(x; \mu_0)q_n(x; \mu_0) - p_n(x; \mu_0)q_{n+1}(x; \mu_0)|^2 d\mu_0(x) = \int_K f(x) d\mu_0(x).$$

By comparing the right-hand side of (4.5) and (4.6) we find that  $\mu_0$  is absolutely continuous with respect to  $\mu$  on  $K$  and since  $\phi(x) \neq 0$  on  $K$  (by Lemma 3) it also follows that  $\mu$  is absolutely continuous with respect to  $\mu_0$  on  $K$ . This proves the theorem.  $\square$

### 5. Examples.

5.1. *Finite perturbations.* Suppose that the recurrence coefficients of  $p_n(x; \mu_m)$  agree with the recurrence coefficients of  $p_n(x; \mu_0)$  from a certain index  $m$  onward, i.e.

$$(5.1) \quad a_{n+1} = a_{n+1}^0, \quad b_n = b_n^0 \quad n \geq m.$$

It is clear that (3.8) is true whenever one deals with these *finite perturbations*. From (3.9) and (3.12) it follows that the function  $\phi(x) = \phi_m(x)$  is given by

$$(5.2) \quad \phi_m(x) = a_n^0 \left\{ \hat{p}_n(x; \mu_m)q_{n-1}(x; \mu_0) - \left( \frac{a_n}{a_n^0} \right)^2 \hat{p}_{n-1}(x; \mu_m)q_n(x; \mu_0) \right\}, \quad n \geq m.$$

Choose  $n = m + 1$ ; then

$$(5.3) \quad |\phi_m(x)|^2 = (a_{m+1}^0)^2 \{ \hat{p}_{m+1}^2(x; \mu_m) |q_m(x; \mu_0)|^2 \\ + \hat{p}_m^2(x; \mu_m) |q_{m+1}(x; \mu_0)|^2 - 2\hat{p}_m(x; \mu_m) \\ \cdot \hat{p}_{m+1}(x; \mu_m) \Re[q_m(x; \mu_0) \overline{q_{m+1}(x; \mu_0)}] \}.$$

The Radon-Nikodym derivative  $d\mu_0/d\mu$  can therefore be computed explicitly by using Lemma 1. This requires the computation of Cauchy principal values and thus one needs a numerical technique that deals with singular integrals of this type (see e.g. [8]).

5.2. *Perturbations of Jacobi polynomials.* The recurrence coefficients for the orthonormal Jacobi polynomials  $p_n^{(\alpha, \beta)}(x)$  are given by

$$(a_n^0)^2 = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)} \\ = \frac{1}{4} + \frac{1-2(\alpha^2+\beta^2)}{16n^2} + O(n^{-3}), \\ b_n^0 = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} = \frac{\beta^2 - \alpha^2}{4n^2} + O(n^{-3}),$$

where  $\alpha, \beta > -1$  and if  $\alpha = -\beta$  one has  $b_0 = \beta$ . If (3.8) is satisfied, then this means that the recurrence coefficients of  $p_n(x; \mu)$  satisfy

$$\sum_{k=0}^{\infty} (|1 - 4a_{k+1}^2| + 2|b_k|) < \infty$$

but then it is more convenient to use the constant recurrence coefficients  $a_n^0 = 1/2$ ,  $b_n = 0$  as comparison system, and such perturbations have been studied quite well ([11], [21], [28]). It would be of interest to find results under the stronger condition

$$(5.4) \quad \sum_{n=0}^{\infty} n^\gamma \left( |b_n^0 - b_n| + \frac{|(a_{n+1}^0)^2 - (a_{n+1})^2|}{a_{n+1}^0} \right) < \infty$$

with  $\gamma > 0$ , but in order to do this we need bounds of the type

$$(5.5) \quad |p_{n-k}(x; \mu_0^{(k)}) q_{n-k}(x; \mu_0^{(k)})| \leq C(n+1)^\gamma, \quad x \in K,$$

where  $\mu_0^{(k)}$  is the orthogonality measure for the associated Jacobi polynomials of order  $k$ .

**THEOREM 4.** *Suppose that  $|q_0(x; \mu_0^{(n+1)})| \leq C$  is valid on an interval  $K$  of the real line and that (5.5) holds. Then*

$$(5.6) \quad |\hat{p}_n(x; \mu)q_n(x; \mu_0)| \leq C(n+1)^\gamma \exp \left\{ \sum_{k=0}^{n-1} (k+1)^\gamma \cdot (C|b_k^0 - b_k| + C^2|(a_{k+1}^0)^2 - (a_{k+1})^2|) \right\}.$$

Moreover the function  $\phi$  defined in (3.9) is continuous on  $K$  and Theorem 2 (with its corollaries) remains valid.

*Proof.* The bound follows by inserting the upper bound (5.5) into (2.10) and by using

$$\frac{q_{k+1}(x; \mu_0)}{q_k(x; \mu_0)} = a_{k+1}^0 q_0(x; \mu_0^{(k+1)}).$$

One can then repeat the proof of Theorem 2 with these bounds.  $\square$

5.3. *Perturbations of Pollaczek polynomials.* The recurrence coefficients for orthonormal *Pollaczek polynomials* are given by

$$\begin{aligned} (a_n^0)^2 &= \frac{n(n+2\lambda-1)}{4(n+\lambda+a-1)(n+\lambda+a)} \\ &= \frac{1}{4} - \frac{a}{2n} + O(n^{-2}), \\ b_n^0 &= \frac{-b}{n+\lambda+a} = -\frac{b}{n} + O(n^{-2}), \end{aligned}$$

where  $\lambda > 0$  and  $a + \lambda > 0$  or  $-1/2 < \lambda < 0$  and  $-1 < \lambda + a < 0$ . It would be of interest to find bounds of the type (5.5) with  $K = [-1, 1]$ , where  $\mu_0^{(k)}$  is the orthogonality measure for associated *Pollaczek polynomials* of order  $k$ , because  $[-1, 1]$  is the essential spectrum for *Pollaczek polynomials*.

5.4. *Generalized Jacobi weights, sieved weights.* A generalized Jacobi weight [3], [21] is of the form

$$w_0(x) = \varphi(x)(1+x)^{\gamma_1} \prod_{k=2}^{N-1} |t_k - x|^{\gamma_k} (1-x)^{\gamma_N}, \quad -1 < x < 1,$$

where  $\gamma_k > -1$ ,  $-1 < x_2 < x_3 < \dots < x_{N-1} < 1$  and  $\varphi$  is a positive function on  $[-1, 1]$  of Dini type. The recurrence coefficients of generalized Jacobi polynomials are not known in general, but for some special cases (such as sieved ultraspherical [1] or sieved Jacobi polynomials) explicit formulas are known which show some periodic oscillations. For sieved orthogonal polynomials obtained through a polynomial transformation of an ultraspherical weight, i.e. when considering the weight

$$\begin{aligned} w_0(x) &= |U_{k-1}(x)|(1 - T_k^2(x))^\lambda \\ &= (1 - x^2)^\lambda |U_{k-1}(x)|^{2\lambda+1}, \quad -1 < x < 1, \end{aligned}$$

one has [13]

$$\begin{aligned} b_n &= 0, \quad a_{nk+j} = 1/2 \quad (j = 2, \dots, k-1), \\ a_{kn}^2 &= \frac{n(n+2\lambda)}{2[4(n+\lambda)^2-1]} \frac{c_{n-1}(1)}{c_n(1)}, \quad a_{kn+1}^2 = \frac{1}{2} - a_{kn}^2. \end{aligned}$$

where  $c_n(x)$  are the orthonormal ultraspherical polynomials with weight  $(1 - x^2)^\lambda$ . This means that when recurrence coefficients oscillate around their limiting values, then the weight may have singularities inside the interval on which orthogonality holds.

#### REFERENCES

- [1] W. Al-Salam, W. Allaway and R. Askey, *Sieved ultraspherical polynomials*, Trans. Amer. Math. Soc., **284** (1984), 39–55.
- [2] R. Askey and J. Wimp, *Associated Laguerre and Hermite polynomials*, Proc. Royal Soc. Edinburgh, **96A** (1985), 15–37.
- [3] V. Badkov, *Convergence in mean and almost everywhere of Fourier series in orthogonal polynomials*, Mat. Sb., **95** (137) (1974), 229–262.
- [4] W. Barrett, *An asymptotic formula relating to orthogonal polynomials*, J. London Math. Soc., (2) **6** (1973), 701–704.
- [5] P. Barrucand and D. Dickinson, *On the associated Legendre polynomials*, in *Orthogonal Expansions and their Continuous Analogs* (D. T. Haimo, ed.), Southern Illinois Univ. Press, Edwardsville, 1968, pp. 43–50.
- [6] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys., **15** (1974), 2166–2174.
- [7] T. S. Chihara, *Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [8] G. Criscuolo and G. Mastroianni, *On the convergence of an interpolatory product rule for evaluating Cauchy principal value integrals*, Math. Comp., **48** (1987), 725–735.

- [9] J. Dombrowski and P. Nevai, *Orthogonal polynomials, measures and recurrence relations*, SIAM J. Math. Anal., **17** (1986), 752–759.
- [10] J. S. Geronimo, *On the spectra of infinite dimensional Jacobi matrices*, J. Approx. Theory, **53** (1988), 251–265.
- [11] J. S. Geronimo and K. M. Case, *Scattering theory and polynomials orthogonal on the real line*, Trans. Amer. Math. Soc., **258** (1980), 467–494.
- [12] J. S. Geronimo and W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. Approx. Theory, **46** (1986), 251–283.
- [13] —, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc., **308** (1988), 559–581.
- [14] Ya. L. Geronimus, *On some finite difference equations and corresponding systems of orthogonal polynomials*, Zap. Mat. Otd. Fiz.-Mat. Fak. i Kharkov. Mat. Obšč., (4) **25** (1957), 87–100 (in Russian).
- [15] C. C. Grosjean, *The measure induced by orthogonal polynomials satisfying a recursion formula with either constant or periodic coefficients, part I: constant coefficients*, Acad. Analecta, Koninkl. Acad. Wetensch. Lett. Sch. Kunsten België, **48** (3) (1986), 39–60.
- [16] —, *The measure induced by orthogonal polynomials satisfying a recursion formula with either constant or periodic coefficients, part II: pure or mixed periodic coefficients*, Acad. Analecta, Koninkl. Acad. Wetensch. Lett. Sch. Kunsten België, **48** (5) (1986), 55–94.
- [17] —, *The weight functions, generating functions and miscellaneous properties of the sequences of orthogonal polynomials of the second kind associated with the Jacobi and the Gegenbauer polynomials*, J. Comput. Appl. Math., **16** (1986), 259–307.
- [18] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1966.
- [19] A. Máté and P. Nevai, *Sublinear perturbations of the differential equation  $y^{(n)} = 0$  and of the analogous difference equation*, J. Differential Equations, **53** (1984), 234–257.
- [20] A. Máté, P. Nevai and V. Totik, *Extensions of Szegő's theory of orthogonal polynomials, II*, Constr. Approx., **3** (1987), 51–72.
- [21] P. G. Nevai, *Orthogonal Polynomials*, Memoirs Amer. Math. Soc., vol. 231, Amer. Math. Soc., Providence, R.I., 1979.
- [22] —, *A new class of orthogonal polynomials*, Proc. Amer. Math. Soc., **91** (1984), 409–415.
- [23] O. Perron, *Die Lehre von den Kettenbrüchen*, Chelsea Publ. Company, New York, 1950.
- [24] J. Sherman, *On the numerators of the convergents of the Stieltjes continued function*, Trans. Amer. Math. Soc., **35** (1933), 64–87.
- [25] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse, **8** (1894), J1–122; **9** (1895), A1–47.
- [26] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 4th edition 1975.
- [27] W. Van Assche, *Asymptotics for Orthogonal Polynomials*, Lecture Notes in Mathematics, vol. 1265, Springer-Verlag, Berlin, 1987.
- [28] —, *Asymptotics for orthogonal polynomials and three-term recurrences*, in “Orthogonal Polynomials: Theory and Practice” (P. Nevai, ed.), NATO ASI Series C **294**, Kluwer Academic Publishers, Dordrecht, 1990, pp. 435–462.
- [29] J. Wimp, *Explicit formulas for the associated Jacobi polynomials and some applications*, Canad. J. Math., **39** (1987), 983–1000.

- [30] A. Wintner, *Spektraltheorie der unendlichen Matrizen*, Hirzel, Leipzig, 1929.  
[31] A. Zygmund, *Trigonometrical Series*, Chelsea Publishing Company, New York, 1952.

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THE OHIO STATE UNIVERSITY  
COLUMBUS, OH 43210-1174

AND

KATHOLIEKE UNIVERSITEIT LEUVEN  
CELESTIJNENLAAN 200B  
B-3001 HEVERLEE (LEUVEN) BELGIUM