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BOUNDARY BEHAVIOR OF A CONFORMAL MAPPING

JOHN MARAFINO

Let D be a simply connected plane domain, not the whole plane. Let R^* denote those accessible boundary points of D such that D twists violently about them; that is, if $\alpha \in R^*$ and $w(\alpha)$ denotes its complex coordinate, then

$$\liminf_{\substack{w\to\alpha\\ w\in D}} \arg(w - w(\alpha)) = -\infty \quad \textbf{and}$$
$$\limsup_{\substack{w\to\alpha\\ w\in D}} \arg(w - w(\alpha)) = +\infty,$$

where $\arg(w - w(\alpha))$ is defined and continuous in D. We show that if a certain geometric condition holds at each point of a set $W^* \subset R^*$, then W^* is a D-conformal null set. Let L_ν denote the ray with terminal point $w(\alpha)$, $\alpha \in R^*$, having inclination ν, $0 \le \nu < 2\pi$. Let m denote Lebesgue measure on L_ν and set

$$u(\nu) = \limsup_{r\to 0} \frac{m((L_\nu \cap D) \cap (w(\alpha), w(\alpha) + re^{i\nu}))}{r}.$$

Let $W^* = \{\alpha \in R^*$: there exists L_{ν_i}, $i = 1, 2, 3$, at $w(\alpha)$ such that $|\nu_i - \nu_j| = (2/3)\pi$, $1 \le i < j \le 3$, and $u(\nu_i) < 1$ for $i = 1, 2, 3\}$.

THEOREM. *W^* is a D-conformal null set.*

Introduction. Let D be a simply connected plane domain, not the whole plane, and let $w = f(z)$ be a one to one conformal map from the open unit disk onto D. It is well-known that for almost every θ, $0 \le \theta < 2\pi$, $f(z)$ has a finite radial limit $f(e^{i\theta})$ at $e^{i\theta}$. By [4, pp. 311–312] we also have for almost every θ that the image of the radius at $e^{i\theta}$ is a rectifiable curve. Thus, for almost every θ, $0 \le \theta < 2\pi$, the image of the radius at $e^{i\theta}$ determines a (ideal) rectifiably accessible boundary point α_θ of D whose complex coordinate $w(\alpha_\theta) = f(e^{i\theta})$ is finite. The set of all such α_θ is denoted by A^*. In fact, using Theorem 1 in [2, p. 37], Theorem 9.3 in [4, p. 268], and Theorem 10.9 in [4, p. 316], it follows that A^* is the set of all rectifiably accessible boundary points of D. On $D^* = D \cup A^*$ we define the arc-length distance l_{D^*} between two points as the infimum of the euclidean lengths of arcs that lie in D and join these two points.

It can be shown that l_{D^*} is a metric for D^*. Any limits involving accessible boundary points are taken in l_{D^*}.

A set $N^* \subset A^*$ is said to be a D-conformal null set provided that the set $\{\theta : \alpha_\theta \in N^*\}$ has measure zero. We note that this definition is independent of f. We shall let T^* be those $\alpha \in A^*$ at which the inner tangent to the boundary of D exists. Thus, if $\alpha \in T^*$ then there exists a unique $\nu(\alpha)$, $0 \leq \nu(\alpha) < 2\pi$, such that for each $\varepsilon > 0$ $(\varepsilon < \pi/2)$ there exists a $\delta > 0$ such that

$$\Delta = \{w(\alpha) + \rho e^{i\nu} : 0 < \rho < \delta,\ |\nu - \nu(\alpha)| < \pi/2 - \varepsilon\} \subset D$$

and $w \to \alpha$ as $w \to w(\alpha)$, $w \in \Delta$. We denote by R^* the set of rectifiably accessible boundary points of D such that

$$\liminf_{\substack{w \to \alpha_\theta \\ w \in D}} \arg(w - w(\alpha_\theta)) = -\infty \quad \text{and} \quad \limsup_{\substack{w \to \alpha_\theta \\ w \in D}} \arg(w - w(\alpha_\theta)) = +\infty,$$

where $\arg(w - w(\alpha_\theta))$ is defined and continuous in D.

In [3, p. 44] a diameter metric is used and it is shown that $A^* = T^* \cup R^* \cup N^*$, where N^* is a D-conformal null set. Also, an example is given [3, p. 65] of a domain D where $A^* = R^* \cup N^*$. We note that the same characterization of A^* holds using l_{D^*} and that in the cited example each point of A^* is rectifiably accessible.

In this paper we shall restrict our attention to R^* and show that if a certain geometric condition holds at each point of a set $W^* \subset R^*$, then W^* is a D-conformal null set. Let L_ν denote the ray with terminal point $w(\alpha)$, $\alpha \in R^*$, having inclination ν, $0 \leq \nu < 2\pi$. Let m denote Lebesgue measure on L_ν and set

$$u(\nu) = \limsup_{r \to 0} \frac{m((L_\nu \cap D) \cap (w(\alpha), w(\alpha) + re^{i\nu}))}{r}.$$

Let $W^* = \{\alpha \in R^* :$ there exists L_{ν_i}, $i = 1, 2, 3$, at $w(\alpha)$ such that $|\nu_i - \nu_j| = (2/3)\pi$, $1 \leq i < j \leq 3$, and $u(\nu_i) < 1$ for $i = 1, 2, 3\}$.

THEOREM. *W^* is a D-conformal null set.*

The proof of this theorem will be a consequence of the lemma stated below. Given $\alpha \in R^*$ and positive numbers ν, δ', δ'', ε with $\delta' < \delta''$ and $\varepsilon < \pi/2$, we define $A(\alpha, \nu, \delta', \delta'', \pi/2 - \varepsilon) = \{w(\alpha) + \rho e^{i\phi} : |\phi - \nu| < \pi/2 - \varepsilon,\ \delta' < \rho < \delta''\}$.

LEMMA. *Except for a D-conformal null set of R^*, for each $\alpha \in R^*$ there exists a $\nu(\alpha)$, $0 \leq \nu(\alpha) \leq 2\pi$, and a sequence of pairwise disjoint* *disks* $\{O_n\}$ *with radii* r_n *and center* w_n *such that for each* $\varepsilon > 0$, $\varepsilon < \pi/2$, *there exists sequences* $\{\delta'_n\}$, $\{\delta''_n\}$ *of positive real numbers such that*

(1) $A(\alpha, \nu(\alpha), \delta'_n, \delta''_n, \pi/2 - \varepsilon) \subset O_n \subset D$ *for all sufficiently large* n.

(2) $l_{D^*}(w_n, \alpha) \to 0$ *as* $n \to \infty$. *Consequently,* r_n, δ'_n, $\delta''_n \to 0$ *as* $n \to \infty$.

(3)

$$\lim_{n\to\infty} \frac{\delta''_n - \delta'_n}{r_n} = 2\cos(\pi/2 - \varepsilon) \quad \textit{and} \quad \lim_{n\to\infty} \frac{\delta''_n - \delta'_n}{\delta''_n} = 1.$$

Note that as we approach $\alpha \in T^*$ through the regions Δ, the measure of the angle at the vertex approaches π. Because of this property our theorem is trivially true on T^*. Our lemma on R^* asserts that the regions $A(\alpha, \nu(\alpha), \delta'_n, \delta''_n, \pi/2 - \varepsilon)$ are mimicking this property of the Δ's and as a consequence our theorem should hold on R^*.

Proof of Lemma. Our lemma follows from a result due to Gerasch [**1**] which we shall indicate in our discussion below.

Let η_k and δ_k, $k = 1, 2, \ldots$, be sequences of positive numbers decreasing to zero. For any sequence $\{w_n\}$ of points in D we define the *corresponding sequence of disks* $\{O_n\}$ as follows: O_n is the largest disk centered at w_n having radius r_n that is contained in D. We will say that a sequence $\{w_n\} \subset D$ is a boundary sequence provided that the Euclidean distance from w_n to the boundary of D approaches zero as $n \to \infty$. We let $F^*_{k,p}$ be those points $\alpha \in R^*$ such that for all boundary sequences $\{w_n\}$ the corresponding sequence of disks $\{O_n\}$ satisfy $l_{D^*}(O_n, \alpha) > \eta_k r_n$, $r_n < \delta_p$. Set $F^*_k = \bigcup_p F^*_{k,p}$.

In [**1**, p. 204] Gerasch uses an extremal length argument to show that a set, whose definition is similar to $F^*_{k,p}$, is D-conformal null. The steps of his proof can be used here to show $F^*_{k,p}$ is D-conformal null. In the following three paragraphs we outline this procedure. For more details we refer the reader to Gerasch's paper. We also note that the extremal length argument used by Gerasch was also used by McMillan in [**3**, pp. 58–61] to establish his characterization of A^*.

Let $E = \{e^{i\theta}: \alpha_\theta \in F^*_{k,p}\}$. It is well known that almost every point of a set of positive measure is a point of outer density. If we show that each point $e^{i\theta} \in E$ is not a point of outer density then it follows that $m(E) = 0$. In order to do this we will first establish a relationship between the extremal length of a family of curves in $\{|z| < 1\}$ and the density of E near $e^{i\theta} \in E$. We use the following result which is due to Gerasch [**1**, p. 202]; McMillan proves a similar result on the upper half plane [**3**, p. 56]:

> Let $0 < \delta < \pi/2$ and let $A \subset (0, \delta)$ with outer measure $m^*(A)$. For each θ such that $0 < \theta < \delta$, let γ_θ denote the arc of the circle orthogonal to $|z| = 1$ at $e^{i\theta}$ and at $e^{-i\theta}$ which is contained in the unit disk. Set $\Gamma = \{\gamma_\theta : \theta \in A\}$. Then the extremal length $\lambda(\Gamma)$ of the family of curves satisfies
>
> $$\lambda(\Gamma) \leq \pi/\log k,$$
>
> where $k = \sin\delta/\sin[\delta - m^*(A)]$.

Fix $e^{i\theta_0} \in E$ and let $\{z_n\}$ be a radial sequence approaching $e^{i\theta_0}$. The sequence $\{w_n\}$, $w_n = f(z_n)$, satisfies $l_{D^*}(w_n, \alpha_{\theta_0}) \to 0$ because f has radial limit α_{θ_0}, and so $r_n \to 0$ as $n \to \infty$. Thus $\{w_n\}$ is a boundary sequence whose corresponding sequence of disks $\{O_n\}$ satisfy

$$(1) \qquad \text{For all } \theta \in E, \quad l_{D^*}(O_n, \alpha_\theta) > \eta_k r_n \quad \text{for } r_n < \delta_p.$$

For each n the closure of O_n contains a point on the boundary of D. Let b_n be the radius connecting w_n to this point and define b_n^z by requiring $f(b_n^z) = b_n$. Since f is normal it has no Koebe arcs [**4**, pp. 262–267]. Thus b_n^z has an end point $e^{i\theta_n}$ on $\{|z| = 1\}$, $e^{i\theta_n} \notin E$, and it follows that the image of b_n^z and the image of the radius at $e^{i\theta_n}$ under f determine the same accessible boundary point whose complex coordinate we can now denote by $f(e^{i\theta_n})$. In addition, since $\operatorname{diam} b_n \to 0$ as $n \to \infty$, one has by Koebe's lemma [**2**, p. 31] that $e^{i\theta_n} \to e^{i\theta_0}$ as $n \to \infty$. By choosing a subsequence and relabeling it one can suppose without loss of generality that $\theta_0 < \theta_n < \theta_0 + \pi/2$ for all n. Setting $E_n = \{e^{i(\theta_n - \zeta)} \in E : 0 < \zeta < \theta_n - \theta_0\}$, we define Γ_n to be those circular arcs contained in $\{|z| < 1\}$ which are orthogonal to $|z| = 1$ at $e^{i(\theta_n + \zeta)}$ and $e^{i(\theta_n - \zeta)}$ for some $e^{i(\theta_n - \zeta)} \in E_n$. It follows from above that for all n,

$$(2) \qquad \lambda(\Gamma_n) \leq \pi/\log k_n,$$

where $k_n = \sin(\theta - \theta_0)/\sin[(\theta_n - \theta_0) - m^*(E_n)]$. Using (2) and the fact that $\sin x \leq x$ for $0 \leq x < \pi/2$, we have

$$\sin(\theta_n - \theta_0) \leq e^{\pi/\lambda(\Gamma_n)}[(\theta_n - \theta_0) - m^*(E_n)],$$

and hence

$$(3)\qquad \frac{m^*(E_n)}{\theta_n-\theta_0}\le 1-e^{-\pi/\lambda(\Gamma_n)}[\sin(\theta_n-\theta_0)/(\theta_n-\theta_0)].$$

From (3) we will be able to conclude that $e^{i\theta_0}$ is not a point of outer density providing we establish the existence of a positive number c satisfying $\lambda(\Gamma_n)\ge c$ for all n such that $r_n<\delta_p$. We will use (1) and the conformal invariance of λ to do just this.

Let $\Gamma(b_n)=\{\gamma\colon\gamma$ is a curve in D which joins a point of b_n to a point $f(e^{i\theta})$, $e^{i\theta}\in E\}$ and $\Gamma'_n=\{\gamma'\colon \gamma'\subset\{|z|<1\}$ and $f(\gamma')\in\Gamma(b_n)\}$. Then $\lambda(\Gamma(b_n))=\lambda(\Gamma'_n)$ and since every $\gamma\in\Gamma_n$ contains some curve $\gamma'\in\Gamma'_n$ we have by the comparison principle for extremal length that $\lambda(\Gamma'_n)\le\lambda(\Gamma_n)$. Hence to show $e^{i\theta_0}$ is not a point of outer density it suffices to find a positive number c satisfying $\lambda(\Gamma(b_n))\ge c$ for all n such that $r_n<\delta_p$.

For each n such that $r_n<\delta_p$, let $V_n=\{w\in D\colon \operatorname{dist}(w, b_n)<\eta_k l(b_n)\}$, where $\operatorname{dist}(w, b_n)$ denotes the Euclidean distance form the point w to b_n and $l(\ )$ denotes the length of the curve. By (1) it follows that for any $\gamma\in\Gamma(b_n)$,

$$l(\gamma\cap V_n)>\eta_k r_n.$$

Defining $\rho_n(w)=1$ if $w\in V_n$ and 0 elsewhere we have that ρ_n is a measurable function, $\int_\gamma \rho_n|dw|\ge\eta_k r_n$ for all $\gamma\in\Gamma(b_n)$ and $A(\rho_n)$—the area integral for V_n with respect to ρ_n—satisfies $A(\rho_n)\le 2\eta_k r_n^2+\pi\eta_k^2 r_n^2$. Hence for each n such that $r_n<\delta_p$,

$$\begin{aligned}\lambda(\Gamma(b_n))&=\sup_\rho\frac{L(\Gamma(b_n, \rho))^2}{A(\rho)}\ge\frac{L(\Gamma(b_n, \rho_n))^2}{A(\rho_n)}\\&=\frac{\left[\inf_{\gamma\in\Gamma(b_n)}\int_\gamma\rho_n|dw|\right]^2}{A(\rho_n)}\ge\frac{\eta_k}{2+\pi\eta_k}.\end{aligned}$$

Since η_k is independent of n we choose this last value for c.

It follows that F_k^* and $\bigcup_k F_k^*$ are D-conformal null sets and we can conclude that with the exception of a D-conformal null set, for each $\alpha\in R^*$ and k, $k=1, 2, \dots$, there exists a boundary sequence $\{w_{k,n}\}$ in D whose corresponding sequence of disks $\{O_{k,n}\}$ satisfies $l_{D^*}(O_{k,n}, \alpha)\le\eta_k r_{k,n}$ for all n. Since $r_{k,n}\to 0$ as $n\to\infty$, one has $l_{D^*}(w_{k,n}, \alpha)\to 0$ as $n\to\infty$. Using this fact and a relabeling of the sequence we can assume without loss of generality that for each k, $l_{D^*}(w_{k,n}, \alpha)<1/k$ for all n.

If we now consider the rectangular array of points $[w_{k,n}]$ and form the sequence $\{w_{n,n}\}$ we arrive at the following result: Except for a D-conformal null set of R^*, for each $\alpha \in R^*$ there exists a sequence $\{w_n\}$ $(w_{n,n} = w_n)$ along with its corresponding sequence of disks $\{O_n\}$ which satisfy the following:

(4) O_n is contained in D for each n,

(5) $l_{D^*}(w_n, \alpha) \to 0$ as $n \to \infty$,

(6) $l_{D^*}(O_n, \alpha) \leq \eta_n r_n$ for all n.

Since $\alpha \in R^*$ we have that the Euclidean distance from O_n to $w(\alpha)$, $\operatorname{dist}(O_n, w(\alpha))$, is positive for all n. Using (5) there exists a point of the sequence $\{w_n\}$, which we denote by w_{n_1} such that $l_{D^*}(w_{n_1}, \alpha) < \operatorname{dist}(O_1, w(\alpha))/2$. It follows that $\operatorname{dist}(w_{n_1}, w(\alpha)) < \operatorname{dist}(O_1, w(\alpha))/2$ and $r_{n_1} < \operatorname{dist}(O_1, w(\alpha))/2$. Hence the disks O_{n_1} and O_1 are disjoint. Using the fact that $\operatorname{dist}(O_{n_1}, w(\alpha))$ is positive we repeat the above argument to get a point w_{n_2}, $n_1 < n_2$ from the sequence $\{w_n\}$ whose corresponding disk O_{n_2} is disjoint from O_{n_1} and O_1. Inductively we are able to define a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ whose corresponding sequence of disks $\{O_{n_k}\}$ satisfy (4), (5), (6) in addition to being pairwise disjoint. By relabeling this sequence one can assume without loss of generality that the O_n are pairwise disjoint. Since the sequence

$$\{\nu_n = \arg(w_n - w(\alpha)) \operatorname{mod}(2\pi),\ n = 1, 2, \dots\}$$

is bounded, it has a subsequence that converges to some ν_0 where either $0 < \nu_0 < 2\pi$ or $\nu_0 =$ mod (2π). Thus, we can further assume without loss of generality that the sequence $\{w_n\}$ is such that

$$\text{(7)} \qquad \arg(w_n - w(\alpha)) \mod 2\pi \to \nu_0 \quad \text{as } n \to \infty.$$

We shall set $\nu(\alpha) = \nu_0$.

For any positive numbers δ and ε, $\varepsilon < \pi/2$, we know from (5) and (7) that for n sufficiently large, O_n is contained in $\{|w - w(\alpha)| < \delta\}$ and O_n intersects the sector $\Delta = \{w(\alpha) + \rho e^{i\phi}: \rho < \delta$ and $|\phi - \nu_0| < \pi/2 - \varepsilon\}$. We show that one can choose n sufficiently large so that the boundary of O_n intersects the boundary of Δ at four distinct points. Let L_n denote the ray through w_n having terminal point $w(\alpha)$. Let T'_n, T''_n denote the two rays tangent to O_n having terminal point $w(\alpha)$ and let ε_n denote the measure of the angle formed by L_n and T'_n and L_n and T''_n. See Figure 1. If d_n denotes the Euclidean distance from O_n to $w(\alpha)$, we have that

$$\sin \varepsilon_n = \frac{r_n}{r_n + d_n} > \frac{r_n}{r_n + l_{D^*}(O_n, \alpha)} = \frac{1}{1 + l_{D^*}(O_n, \alpha)/r_n}.$$

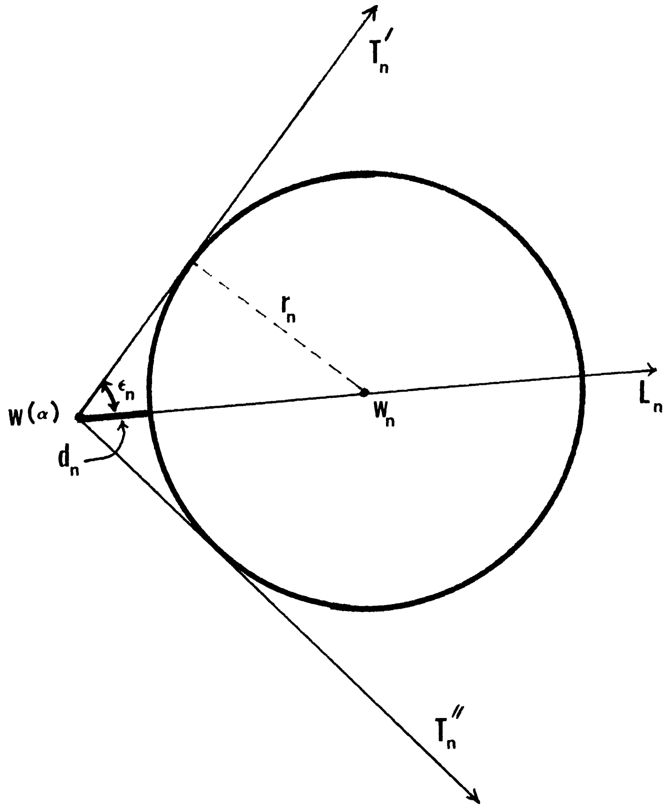

FIGURE 1

Using (6) we see that as $n \to \infty$, $\sin \varepsilon_n \to 1$ and so $\varepsilon_n \to \pi/2$. If we choose n sufficiently large so that

$$\varepsilon_n > \pi/2 - \varepsilon/2 \quad \text{and} \quad |\arg L_n - \nu_0| < \varepsilon/4$$

then our claim follows. Note that one pair of the four points lies on the segment $w(\alpha) + \rho e^{i(\nu_0 + (\pi/2 - \varepsilon))}$: $0 \le \rho \le \delta$ and the other pair lies on the segment $w(\alpha) + \rho e^{i(\nu_0 - (\pi/2 - \varepsilon))}$: $0 \le \rho \le \delta$. We choose the pair whose Euclidean distance from one another is the smaller and denote their distance from $w(\alpha)$ by δ_n' and δ_n'' where $\delta_n' < \delta_n''$. Doing this for each n, n sufficiently large, defines two sequences $\{\delta_n'\}$, $\{\delta_n''\}$ such that

$$A(\alpha, \nu_0, \delta_n', \delta_n'', \pi/2 - \varepsilon) \subset O_n \subset D \quad \text{for all } n.$$

Letting $\tau_n = [\arg(w_n - w(\alpha)) \bmod 2\pi - \nu_0]$ and referring to Figure

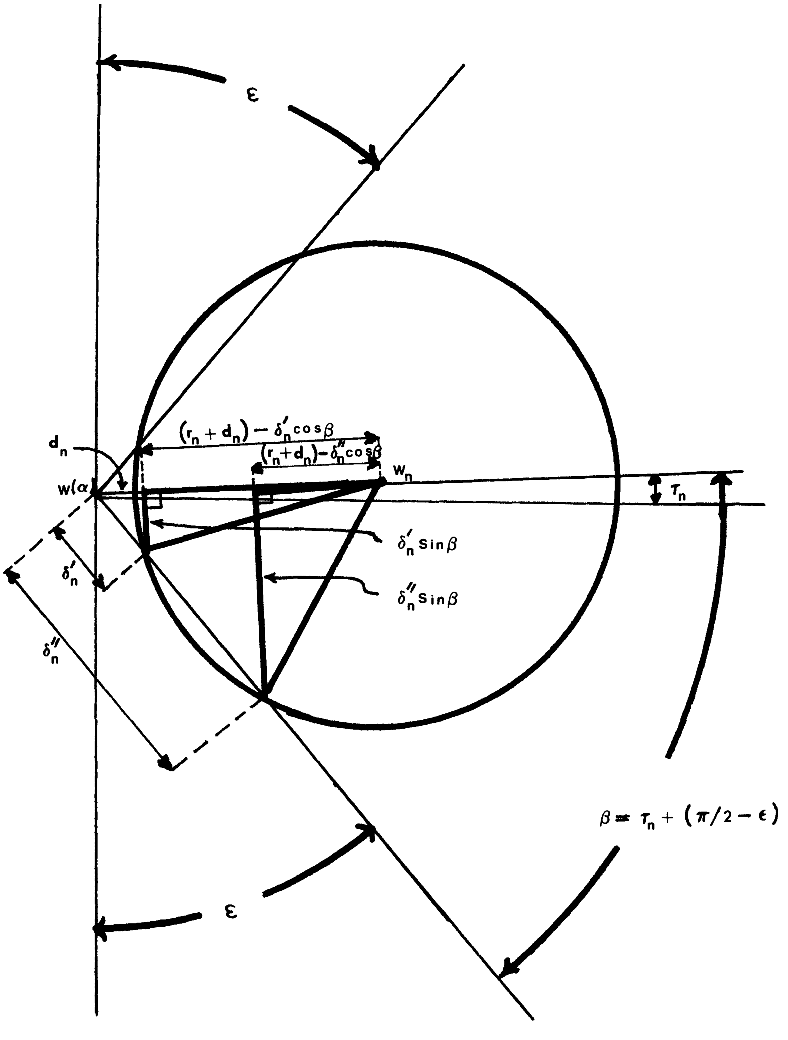

FIGURE 2

2 we have that

$$[\delta'_n \sin(\tau_n + (\pi/2 - \varepsilon))]^2 + [(r_n + d_n) - \delta'_n \cos(\tau_n + (\pi/2 - \varepsilon))]^2 = r_n^2,$$
$$[\delta''_n \sin(\tau_n + (\pi/2 - \varepsilon))]^2 + [(r_n + d_n) - \delta''_n \cos(\tau_n + (\pi/2 - \varepsilon))]^2 = r_n^2.$$

Upon solving for δ'_n and δ''_n we have that

$$\frac{\delta''_n - \delta'_n}{r_n} = 2\frac{r_n + d_n}{r_n}\sqrt{\cos(\tau_n + (\pi/2 - \varepsilon))^2 + r_n^2/(r_n + d_n)^2 - 1}$$

and

$$\frac{\delta_n'' - \delta_n'}{\delta_n''} = \frac{2\sqrt{\cos(\tau_n + (\pi/2 - \varepsilon))^2 + r_n^2/(r_n + d_n)^2 - 1}}{\cos(\tau_n + (\pi/2 - \varepsilon)) + \sqrt{\cos(\tau_n + (\pi/2 - \varepsilon))^2 + r_n^2/(r_n + d_n)^2 - 1}}.$$

Hence

$$\lim_{n\to\infty} \frac{\delta_n'' - \delta_n'}{r_n} = 2\cos(\pi/2 - \varepsilon), \qquad \lim_{n\to\infty} \frac{\delta_n'' - \delta_n'}{\delta_n''} = 1,$$

and the lemma is proven.

Proof of Theorem. Suppose to the contrary that W^* is not D-conformal null. By the lemma, with the exception of a D-conformal null set of W^* we know that for each $\alpha \in W^*$ there exists $\nu(\alpha)$, $0 \leq \nu(\alpha) \leq 2\pi$, such that for $\varepsilon = \pi/12$ there exist sequences $\{\delta_n'\}$, $\{\delta_n''\}$ such that

$$A(\alpha, \nu(\alpha), \delta_n', \delta_n'', 5\pi/12) \subset D \quad \text{for all } n$$

and

$$\lim_{n\to\infty} \frac{\delta_n'' - \delta_n'}{\delta_n''} = 1.$$

Also, from the definition of W^*, for each $\alpha \in W^*$ there exists a k, $k = 1, 2, 3$, such that L_{ν_k} passes through the regions $A(\alpha, \nu(\alpha), \delta_n', \delta_n'', 5\pi/12)$ and $u(\nu_k) < 1$. However,

$$\frac{m((L_{\nu_k} \cap D) \cap (w(\alpha), w(\alpha) + \delta_n'' e^{i\nu_k}))}{\delta_n''} > \frac{\delta_n'' - \delta_n'}{\delta_n''}$$

and this implies that $u(\nu_k) = 1$. This is a contradiction. Thus W^* must be a D-conformal null set.

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JAMES MADISON UNIVERSITY
HARRISONBURG, VA 22807