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THE STANDARD DUAL OF AN OPERATOR SPACE

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The notion of a dual of an operator space which is again an operator space has been introduced independently by Vern Paulsen and the author, and by Effros and Ruan. Its significance in the theory of tensor products of operator spaces has already been partially explored by the aforementioned. Here we establish some other fundamental properties of this dual construction, and examine how it interacts with other natural categorical constructs for operator spaces. We define and study a notion of projectivity for operator spaces, and give a noncommutative version of Grothendieck's characterization of $l^1(I)$ spaces for a discrete set I.

1. Introduction. If V is a vector space then we write $M_n(V)$ for the vector space of $n \times n$ matrices with entries in V. Let E and F be normed spaces and suppose that for each positive integer n there are norms $\|\cdot\|_n$ and $|||\cdot|||_n$ defined on the spaces $M_n(E)$ and $M_n(F)$ respectively. A linear map $T: E \to F$ has a natural n-fold amplification $T_n: M(E) \to M_n(F)$ given by $T_n([e_{ij}]) = [T(e_{ij})]$. We say that T is *completely bounded* if

$$\|T\|_{cb} = \sup\{\|T_n\|: \text{positive integers } n\} < \infty,$$

completely contractive if each T_n is a contraction, and a *complete isometry* if each T_n is an isometry. We write $CB(E, F)$ for the space of completely bounded maps from E to F, with norm $\|\cdot\|_{cb}$. An *operator space* $(E, \|\cdot\|_n)$ is a pair consisting of a vector space E, and a sequence of norms $\|\cdot\|_n$ defined on the spaces $M_n(E)$, such that there exists a complete isometry from E into the space $B(H)$ of bounded operators on a Hilbert space H. Recall that $B(H)$ has a natural sequence of matrix norms obtained by identifying $M_n(B(H))$ with $B(H^{(n)})$ in the obvious way. Thus an operator space is merely a subspace E of some $B(H)$, with the commitment to keep track of the associated norms of matrices with entries in E.

The theory of quantized functional analysis [E]—the study of operator spaces and their completely bounded maps—is intended to be a generalization of ordinary functional analysis—the study of normed spaces and bounded linear maps. For instance, the extension theorem for completely bounded maps **[A]** generalizes the Hahn-Banach theorem. The reason why it is a strict generalization is because there is a full embedding $\mathfrak{J}$ of the category **NS** of normed spaces and bounded linear maps into a full subcategory of the category **OS** of operator spaces and completely bounded maps. Restated in less flowery language: corresponding to every normed space E there is a canonical operator space $\mathfrak{J}(E)$; and every bounded map $T: E \to F$ of normed spaces induces a canonical completely bounded map $\mathfrak{J}(T)$ from $\mathfrak{J}(E)$ to $\mathfrak{J}(F)$ with $\|\mathfrak{J}(T)\|_{cb} = \|T\|$. In fact there are two such natural embeddings, MIN and MAX, which we describe below (see also **[BP, ER1]**) since they are of interest in the sequel.

A normed space E may be considered as a subspace of the commutative C^*-algebra of bounded functions on the unit ball of its dual space; this subspace is an operator space with the natural sequence of matrix norms inherited from the C^*-algebra. We call this operator space $\mathrm{MIN}(E)$. The following property, which also defines the operator space $\mathrm{MAX}(E)$, holds: For any linear isometry T from E into some $B(H)$ we have

$$\mathrm{BALL}(M_n(\mathrm{MIN}(E))) \supset \mathrm{BALL}(M_n(T(E))) \supset \mathrm{BALL}(M_n(\mathrm{MAX}(E))).$$

The MIN embedding behaves well with respect to injective constructs, the MAX with respect to projective constructs (see §§2 and 3, and **[BP]**). With respect to the MIN embedding C^*-algebras correspond to $C(X)$ spaces, algebras of operators to function algebras **[B]**, the spatial tensor norm of operator spaces to the injective tensor norm of normed spaces **[BP]**, injective operator spaces to injective normed spaces **[R2]**, etc. With respect to the MAX embedding the projective operator space tensor norm corresponds to the projective tensor norm of normed spaces **[BP]**, projective operator spaces to projective Banach spaces (§3), etc.

One obstacle to the success of this "quantized" program has been the absence of an appropriate duality theory for operator spaces. Previous candidates for the dual of an operator space **[CE]** force one to leave the category **OS**. Recently however the notion of a dual of an operator space which is again an operator space has been introduced independently in **[BP]** and **[ER3]**. The significance of this new duality has already been partially explored via its relationship with the theory of tensor products of operator spaces **[BP, ER3, ER4]**. In particular it has made possible a natural generalization of the Grothendieck tensor norm program to the non-commutative scenario.

The purpose of this note is to establish some fundamental properties of this dual construction, and to examine how it interacts with some other natural categorical constructs for spaces of operators. We show for example that the new dual gives an explicit natural relation between MIN and MAX, and that the new matrix norms on the second dual coincide with the older version. The extension theorem for completely bounded maps may be reformulated as the fact that if T is a complete isometry then T^* is a complete quotient map; this is now explicitly a generalization of the Hahn-Banach theorem.

In §3 we use the duality to characterize projective operator spaces. Grothendieck showed in **[G]** that a Banach space F with the property that each bounded map from F into a quotient space X/Y lifts to a bounded map from F to X, and such that F^* is injective, is isometric with $l^1(I)$ for some discrete set I. The converse obviously holds. We show here that the only operator spaces with an analogous completely bounded lifting property for quotients, and whose duals are W^*-algebras, are the L^1 direct sums of finite dimensional trace class algebras. Since an injective Banach space with predual is a W^*-algebra this is indeed a generalization of Grothendieck's result. We say that a W^*-algebra M is *weak*-injective* if each weak* continuous completely bounded map from a weak* closed subspace Y of a dual space X into M extents to a weak* continuous completely bounded map from X to M. The last result may be rephrased as follows: a W^*-algebra is weak*-injective if and only if it is W^*-algebraically isomorphic to a direct sum of finite dimensional matrix algebras. In particular this shows that the hyperfinite II_1 factor is not σ-weakly linearly isomorphic to a direct sum of matrix algebras, which complements a result in **[CS]** asserting that they are completely boundedly isomorphic.

2. Definitions and basic properties. Let X be a vector space, and let $\|\cdot\|_n$ be a sequence of norms defined on the space $M_n(X)$ of $n \times n$ matrices with entries in X. If X^* is the dual normed space of X then we define on the space $M_n(X^*)$ the following norm:

$$\|[f_{ij}]\|_n = \sup\{\|[f_{ij}(x_{kl})]\| : \text{positive integers } m, \\ [x_{kl}] \in \mathrm{BALL}(M_m(X))\}.$$

This is equivalent to equating $M_n(X^*)$ with $CB(X, M_n)$ via the usual identification of a matrix of linear maps with a matrix valued linear map. It is not hard to see that $\|\cdot\|_1$ coincides with the usual norm. There is a matrix norm structure on $CB(X, Y)$, defined in exactly the same way, if Y is any operator space [**ER2**, **BP**].

For X as above let M be the W^*-algebra

$$\bigoplus_{\infty}\{M_n\colon \text{positive integers } n\text{, and each } x \in \text{BALL}(M_n(X))\},$$

and define a linear complete isometry $T\colon X^* \to M$ by

$$T(f) = \bigoplus\{f_n(x)\},$$

where f_n denotes the usual n-fold amplification of f. It is easy to check that if X is a uniformly closed matrix normed space then T is actually a $\sigma(X^*, X) - \sigma(M, M_*)$ homeomorphism onto a weak*-closed subspace of M. We have shown:

PROPOSITION 2.1. *With the sequence of matrix norms defined above, X^* is completely isometrically isomorphic to an operator space. Moreover X^* has a weak*-homeomorphic completely isometric representation on a Hilbert space.*

REMARK. This observation provides a direct proof of [**ER4**, Proposition 5.1]. We also remark that there is a result corresponding to Proposition 2.1 for the "classical matrix normed dual" of an L^1 matrix normed space [**ER2**, Theorem 3.3] which is considerably more difficult to prove.

We call the operator space $(X^*, \|\cdot\|_n)$ defined above the *standard dual of* X. We shall always write X^* for this operator space, unless stated otherwise. Similarly one can show that $CB(X, Y)$ is an operator space for any operator space Y.

If X is an operator space then it follows from a result of Roger Smith [**S1**] that we may as well take $m = n$ in the definition of X^* above. The finite dimensional trace class algebra M_n^* is denoted by T_n. It is shown in [**BP**] that $T_n = R_n \otimes_h C_n$ completely isometrically, where R_n and C_n are respectively the operator spaces which are first row and column of M_n, and where $\otimes_h$ denotes the Haagerup tensor norm. Also, as observed in [**ER4**], as an operator space R_n^* is C_n, and C_n^* is R_n.

The following is proven in [**BP**, **ER3**]:

PROPOSITION 2.2. *If X is an operator space then X is completely isometrically contained in the standard second dual X^{**}.*

For the reader who does not have immediate access to these references we note that the result also follows from Theorem 2.5. We now give a construction due to Stephen Montgomery-Smith which explicitly displays Proposition 2.2 as a type of Bourbaki-Alaoglu theorem. Let X be an operator space, and let $M_\infty(X^*)$ be the space of countably infinite matrices with entries in X^*, whose finite truncations are uniformly bounded; the norm on $M_\infty(X^*)$ is given by least upper bound. This space has a predual **[ER4]**. Put $M_\infty = M_\infty(\mathbf{C})$, and consider the C^*-algebra $C(\mathrm{BALL}(M_\infty(X^*)); M_\infty)$ of functions from $\mathrm{BALL}(M_\infty(X^*))$ to M_∞ which are continuous when each of the spaces are endowed with their weak* topology. Define a map j from X to $C(\mathrm{BALL}(M_\infty(X^*)); M_\infty)$ by setting $j(x)([f_{ij}])$ equal to $[f_{ij}(x)]$. Proposition 2.2 is equivalent to the statement that j is a complete isometry.

Notice that the previous two propositions give a characterization (independent of **[R1]**) of operator spaces as the vector spaces X with a sequence of norms defined on the space $M_n(X)$ such that X is completely isometrically contained in X^{**}. However there seems to be no easy way of showing directly that an L^∞-matrix normed space is completely isometrically contained in its standard second dual.

PROPOSITION 2.3. *If X and Y are operator spaces and if $S\colon X \to Y$ is a completely bounded linear map then $S^*\colon Y^* \to X^*$ is completely bounded as a map between standard duals, and $\|S^*\|_{cb} = \|S\|_{cb}$. Moreover if S is a complete isometry then S^* is a complete quotient map, and if S is a complete quotient map then S^* is a complete isometry. In particular S is a complete isometry if and only if S^{**} is a complete isometry.*

Proof. The only nontrivial point here is that if S is a complete isometry then S^* is a complete quotient map; but this is equivalent to the extension theorem for completely bounded maps.

REMARK. The map $T \to T^{**}$ from $CB(X, Y)$ to $CB(X^{**}, Y^{**})$ is a complete isometry.

COROLLARY 2.4. *If X and Y are operator spaces with $Y \subset X$ then $Y^* = X^*/Y^\perp$, and $(X/Y)^* = Y^\perp$ completely isometrically*; *here as usual $*$ denotes the standard dual.*

Let X be an operator space. Traditionally the norm on $M_n(X^*)$ has been given by the duality pairing $\langle [f_{ij}], [x_{ij}]\rangle = \sum_{ij}\langle f_{ij}, x_{ij}\rangle$ with $M_n(X)$. Sometimes $\langle [f_{ij}], [x_{ij}]\rangle = \sum_{ij}\langle f_{ij}, x_{ji}\rangle$ or $\langle [f_{ij}], [x_{ij}]\rangle =$

$n^{-1}\sum_{ij}\langle f_{ij}, x_{ji}\rangle$ are used instead. We shall refer to these as the "classically" dual matrix norms. However whichever of these classical pairings one considers one obtains the same operator space as the second dual; we shall denote this classical second dual as X''.

THEOREM 2.5. *If X is an operator space then the standard second dual matrix norm structure coincides with the classical second dual structure. That is, $X^{**} = X''$ completely isometrically.*

Proof. Let us write $|||\cdot|||_n$ for the classical matrix norms on $M_n(X'')$. Choose $F \in M_n(X^{**})$; by the bipolar theorem there exists a net $x^\nu \in M_n(X)$, with $\|x^\nu\|_n = |||F|||_n$, converging in the weak* topology to F. Thus $\sum_{ij} f_{ij}(x^\nu_{ij}) \to \sum_{ij} F_{ij}(f_{ij})$ for all $[f_{ij}] \in M_n(X^*)$. In particular $f(x^\nu_{ij}) \to F_{ij}(f)$ for $f \in X^*$, and so $[f_{kl}(x^\nu_{ij})] \to [F_{ij}(f_{kl})]$ for $[f_{kl}] \in M_n(X^*)$. This shows that $\|\cdot\|_n \leq |||\cdot|||_n$ on $M_n(X^{**})$.

The reverse inequality follows using an argument similar to that used in [**BP**, **ER3**] to prove the complete isometry $X \to X^{**}$. Of course X'' has a weak*-homeomorphic completely isometric representation on a Hilbert space: if $X \subset B(H)$ completely isometrically, then the second dual of the inclusion is a weak*-homeomorphic completely isometric map of X'' into the W^*-algebra $B(H)''$. So we can suppose that X'' acts on a Hilbert space H, and that every functional on X'' of the form $F \to \langle F\zeta, \eta\rangle$, for $\zeta, \eta \in H$, is continuous in the $\sigma(X'', X')$-topology [**T**].

Let $[F_{kl}] \in M_n(X'')$, let $\varepsilon > 0$ be given, and then choose $\zeta_1, \dots, \zeta_n$ and $\eta_1, \dots, \eta_n \in H$ with $|\sum_{kl}\langle F_{kl}\zeta_l, \eta_k\rangle| \geq |||[F_{kl}]|||_n - \varepsilon$ and $\sum\|\zeta_i\|^2 = \sum\|\eta_i\|^2 = 1$. Define a map T on X'' by $T(\cdot) = P_K \cdot_{|K}$, where $K = \operatorname{span}\{\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n\}$. Then T may be regarded as a completely contractive map from X'' into M_k, say. We may identify $T(\cdot)$ with $[\langle\cdot\xi_j, \xi_i\rangle]$ where $\{\xi_i\}$ is an orthonormal basis for K. Also

$$\begin{aligned}\|T_n([F_{kl}])\| &= \|[\langle F_{kl}\xi_j, \xi_i\rangle]\| \\ &\geq \left|\sum_{kl}\langle T(F_{kl})\zeta_l, \eta_k\rangle\right| \geq |||[F_{kl}]|||_n - \varepsilon.\end{aligned}$$

Define $f_{ij}(x) = \langle x^\frown\xi_j, \xi_i\rangle$ for $x \in X$. Then $F_{kl}(f_{ij}) = \langle F_{kl}\xi_j, \xi_i\rangle$ by the weak* continuity of the functionals $\langle\cdot\xi_j, \xi_i\rangle$. Thus

$$\|[F_{kl}]\|_n \geq \|[F_{kl}(f_{ij})]\| \geq |||[F_{kl}]|||_n - \varepsilon,$$

which completes the proof since ε was arbitrary.

Now if A is a C^*-algebra we know that the normed space A^{**} is again a C^*-algebra [T] and consequently has a canonical predefined operator space structure.

Corollary 2.6. *If A is a C^*-algebra then the standard second dual operator space structure of A^{**} coincides with its operator space structure as a C^*-algebra.*

Corollary 2.6 and Theorem 2.5 are equivalent. It is curious that if A is an operator algebra it seems one cannot prove by a direct calculation that the natural multiplication on the standard second dual A^{**} is completely contractive. In fact one can show that this latter fact is also equivalent to Theorem 2.5.

It might be suspected at this point that there are other notions of a dual which are equally (or perhaps more) useful. However the standard dual is determined uniquely by its functorial properties; it is the unique functor d from **OS** to **OS** such that

(i) $X^d = X^*$ isometrically for operator spaces X,

(ii) for all operator spaces X and Y the canonical imbedding of $X \otimes Y$ in the space of linear maps from X^d to Y gives an *isometry* from the spatial tensor product of X and Y into $CB(X^d, Y)$,

(iii) if $X \subset Y$ completely isometrically then $X^{dd} \subset Y^{dd}$ completely isometrically, and

(iv) if A is a C^*-algebra then the operator space structure on A^{dd} matches its operator space structure as a C^*-algebra.

To see this observe that (iii) and (iv) imply that $X^{dd} = X^{**}$ completely isometrically. Now put $Y = M_n$ and replace X with X^d in (ii). Alternatively the standard dual is characterized by the isometric isomorphism from the spatial tensor product of X^d and Y onto $CB(X, Y)$ for finite dimensional Y.

We recall that a commutative operator space is an operator space which is completely isometrically isomorphic to a subspace of a commutative C^*-algebra; or equivalently, an operator space of the form $\mathrm{MIN}(X)$ for some normed space X.

Corollary 2.7. *The standard second dual of a commutative operator space is again a commutative operator space.*

Corollary 2.8. *Let X be a normed space. Then* $\mathrm{MIN}(X)^* = \mathrm{MAX}(X^*)$ *and* $\mathrm{MAX}(X)^* = \mathrm{MIN}(X^*)$ *completely isometrically.*

Proof. Observe that

$$M_n(\mathrm{MAX}(X)^*) = CB(\mathrm{MAX}(X), M_n) = B(X, M_n) \\ = X^* \otimes_\lambda M_n = M_n(\mathrm{MIN}(X^*))$$

via the canonical isometric isomorphisms. This proves the second assertion. Therefore, $\mathrm{MAX}(X^*)^* = \mathrm{MIN}(X^{**}) = \mathrm{MIN}(X)^{**}$ completely isometrically, after using the previous corollary. Thus $\mathrm{MAX}(X^*)$ and $\mathrm{MIN}(X)^*$ are two isometric operator spaces with the same standard dual, and are consequently completely isometrically isomorphic by Proposition 2.1.

A von Neumann algebra M has a unique Banach space predual **[D]**. Endowing this with the obvious "standard predual" matrix norm structure gives us an operator space which we call the *standard predual operator space* of M, and write as M_*. One may ask if the standard dual of M_* is M; a moment's thought will convince the reader that this is not a tautology.

THEOREM 2.9. *A W^*-algebra is completely isometrically isomorphic to the standard dual of its standard predual.*

Proof. By the definition of the standard predual M_* we see that for $[x_{ij}] \in M_n(M)$ we have

$$\|[x_{ij}]\|_n \geq \sup\{\|[f_{kl}(x_{ij})]\| : [f_{kl}] \in \mathrm{BALL}(M_m(M_*))\};$$

the reverse inequality follows as the proof of Theorem 2.5.

It follows from the above that a W^*-algebra M has a unique *standard* predual operator space M_*.

If A is a unital operator algebra then the maps $\pi^*: A \to CB(A^*)$, and $\pi_*: A \to CB(A_*)$ if A has a predual, induced by the left (right) regular representation are easily seen to be completely isometric homomorphisms. More generally if A is an operator space, and an algebra with identity of norm one, such that $\|[a_{ij}b_{kl}]\|_{mn} \leq \|[a_{ij}]\|_m\|[b_{kl}]\|_n$ for all $[a_{ij}] \in M_m(A)$ and $[b_{kl}] \in M_n(A)$, then the homomorphism $\pi^*: A \to CB(A^*)$ is a complete isometry. If in addition A has a predual, and if the homomorphism π^* above compresses to a map $\pi_*: A \to CB(A_*)$, then π_* will also be a unital completely isometric homomorphism, and moreover if A is commutative then $\pi_*(A)$ will be maximally commutative in $CB(A_*)$. Such "matrix normed algebras" seem to occur naturally in many settings. For instance if G is a locally compact group, and if $B(G)$ is defined to be the standard dual of the group C^*-algebra $C^*(G)$ of G, then $B(G)$ has a natural commutative algebra structure **[Ey]**, and one can show using the representation theorem for completely bounded maps **[P1]** that it has the property above. Thus there is a natural completely isometric unital representation of $B(G)$ as a maximally commutative subalgebra of $CB(C^*(G))$. We thank M. E. Walter for raising this question (see also Theorem 1 in **[W]**).

If A is a unital operator algebra one may ask if the natural bilinear maps $A \times A^* \to A^*$ and $A^* \times A \to A^*$ induced by the right and left regular representation are completely contractive. In fact if e_{ij} are the matrix units in $A = M_2$, and if we consider $[e_{ji}] \in M_2(M_2)$ and $[e_{ji}] \in M_2(T_2)$, one can see this is not the case. Modifying this example in an obvious way shows that this is not the case even for $A = l_2^\infty$.

3. Direct sums and projectivity. If $\{X_\alpha : \alpha \in \Lambda\}$ is a collection of operator spaces then $\bigoplus_\infty \{X_\alpha : \alpha \in \lambda\}$ will denote the operator space which is the usual direct sum with matrix norms given by

$$\left\| \left[\bigoplus \{x_\alpha(i, j) : \alpha \in \Lambda\} \right] \right\|_n = \sup\{\|[x_\alpha(i, j)]\|_n : \alpha \in \Lambda\}.$$

In other words we identify $M_n(\bigoplus_\infty X_\alpha)$ with $\bigoplus_\infty M_n(X_\alpha)$. It is clear that $\bigoplus_\infty$ and MIN commute, that is, $\mathrm{MIN}(\bigoplus_\infty X_\alpha) = \bigoplus_\infty \mathrm{MIN}(X_\alpha)$ as operator spaces, for normed spaces X_α.

There is a natural duality $\langle \cdot, \cdot \rangle$ between $\bigoplus X_\alpha$ and $\bigoplus X_\alpha^*$. We define an operator space structure on the L^1-direct sum $\bigoplus_1 \{X_\alpha : \alpha \in \Lambda\}$ by assigning matrix norms

$$\|[u_{ij}]\|_n = \sup\left\{ \|[\langle f_{kl}, u_{ij} \rangle]\| : [f_{kl}] \in \mathrm{BALL}\left(M_m \left(\bigoplus_\infty X_\alpha^* \right) \right) \right\}.$$

It is clear that with these matrix norms $\bigoplus_1 X_\alpha$ is an operator space, and that the canonical maps $\iota_\alpha : X_\alpha \to \bigoplus_1 X_\alpha$ and $q_\alpha : \bigoplus_1 X_\alpha \to X_\alpha$ are complete isometries and complete quotient maps respectively. A simple argument gives the coincidence of $\mathrm{MAX}(\bigoplus_1 X_\alpha)$ and $\bigoplus_1 \mathrm{MAX}(X_\alpha)$ as operator spaces, for normed spaces X_α.

It is also not hard to show that $(\bigoplus_1 X_\alpha)^* = \bigoplus_\infty X_\alpha^*$ and that $(\bigoplus_\infty X_\alpha)^* \supseteq \bigoplus_1 X_\alpha^*$ completely isometrically. The second result of the previous line needs the fact that all the sums occurring may be approximated by finite sums. Similarly one can see that $\bigoplus_1 X_\alpha^*$ is the standard dual of some operator space.

PROPOSITION 3.1. *Every uniformly closed operator space is completely isometrically isomorphic to a quotient of an L^1-direct sum of finite dimensional trace class algebras.*

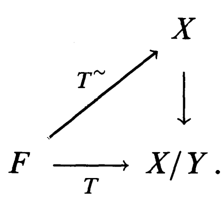
Proof. Let X be a uniformly closed operator space. By Proposition 2.1 there is a completely isometric $\sigma(X^*, X) - \sigma(M, M_*)$ homeomorphism from X^* onto a weak* closed subspace of a direct sum of matrix algebras. This induces the required complete quotient map, after using Corollary 2.4 and the first fact from the last paragraph.

We can explicitly write down the quotient map guaranteed by the proposition above as follows: let Λ be the collection of matrices in $\mathrm{BALL}(M_n(X))$ for all positive integers n. We associate to each $x \in \Lambda$ a finite dimensional trace class algebra T_{n_x} consisting of matrices of the same size as x. There is a natural map $S_x: T_{n_x} \to X: [a_{kl}] \to \sum_{kl} a_{kl} x_{kl}$; define a map S from $\bigoplus_1 \{T_{n_x} : x \in \Lambda\}$ to X by $S(\bigoplus a_x) = \sum_x S_x(a_x)$. Since X is complete S is well defined, and it is easily checked that S is a complete quotient map.

COROLLARY 3.2. *Let X be a uniformly closed operator space. Then X is a quotient of the standard predual $B(H)_*$, for some Hilbert space H.*

REMARK. There are some similar results in **[ER2]**, but with regard to a different notion of duality.

DEFINITION 3.3. A uniformly closed operator space F is said to be *projective* if given an operator space X and a uniformly closed subspace Y of X, and given $\varepsilon > 0$, then each completely contractive linear map $T: F \to X/Y$ lifts to a completely bounded linear map $T^\sim: F \to X$, with $\|T^\sim\|_{cb} \leq 1 + \varepsilon$, so that the following diagram commutes:



An operator space X is said to be a *dual operator space* if there exists an operator space X_* whose standard dual is X.

We say that a dual operator space E is *weak*-injective* if it has the following property: if X is a dual operator space with a weak* closed subspace Y, and if $T: Y \to E$ is a weak* continuous completely bounded map, then for any $\varepsilon > 0$ there is a weak* continuous completely bounded extension $T^{\sim}: X \to E$ of T, with $\|T^{\sim}\|_{cb} < \|T\|_{cb} + \varepsilon$.

It is not hard to see that an operator space F is projective if and only if F^* is weak*-injective.

There are no doubt various other notions of projectivity which one might consider, in terms of approximate liftings and approximate commutative diagrams. The problem of lifting maps on operator systems with some sort of positivity has been extensively studied (see **[S2]** for a survey).

Grothendieck considered the analogous notion for Banach spaces **[G]**, and characterized the projective Banach spaces precisely as the spaces isometrically isomorphic to $l^1(I)$ for a discrete set I. This shows that an operator space of the form $\mathrm{MAX}(F)$, for a Banach space F, is projective precisely when F is isometrically isomorphic to $l^1(I)$ for a discrete set I.

THEOREM 3.4. *The algebra* $\mathbf{C}$ *of complex numbers is the only* C^*-*algebra which is projective as an operator space.*

Proof. It follows from results in **[H]** that if A is a C^*-algebra which is projective as an operator space then A is either $\mathbf{C}$, l_2^∞ or M_2. If M_2 was projective then it is easy to see that l_2^∞ is projective. However, Proposition 3.4 of **[H]** implies that $l_2^\infty = \mathrm{MAX}(l_2^\infty)$ (see **[P2]** for a simpler proof of this). Thus if l_2^∞ is projective then it is projective as a Banach space, which by Grothendieck's result implies in turn that l_2^∞ and l_2^1 are isometric. However it is well known that although l_2^∞ and l_2^1 are isometric if the underlying field is the real numbers, they are not isometric if the field is the complex numbers. This is the desired contradiction.

THEOREM 3.5. *If* F *is projective then the standard dual* F^* *is an injective operator space.*

Proof. Suppose $Y \subset X$, and that $T: Y \to F^*$ is a complete contraction. Taking duals gives a map $T^*: F^{**} \to Y^*$, and by restriction a map $F \to Y^*$. Using Corollary 2.4 and the projectivity of F we may lift to a map $S_n: F \to X^*$ with $\|S_n\|_{cb} \leq 1 + 1/n$. Taking duals again gives a map $S_n^*: X^{**} \to F^*$, and composing this with the canonical map $X \to X^{**}$ gives a map $T_n^{\sim}: X \to F^*$. Some diagram chasing is necessary to show that $T_n^\sim$ is in fact an extension of T. Let $T^\sim$ be a BW^*-topology limit point **[P1]** of $\{T_n^\sim\}$, it is easy to see that $T^\sim$ is a completely contractive extension of T.

The proof above was found together with V. Paulsen.

PROPOSITION 3.6. *If* $\{F_\alpha : \alpha \in \Omega\}$ *is a collection of projective operator spaces then* $\bigoplus_1 \{F_\alpha : \alpha \in \Lambda\}$ *is projective.*

Proof. Let $T: \bigoplus_1 F_\alpha \to X/Y$ be completely contractive, and let $\varepsilon > 0$ be given. Then the composition $T \circ \iota_\alpha : F_\alpha \to X/Y$ is a complete contraction, and lifts to $T_\alpha^\sim : F_\alpha \to X$ with $\|T_\alpha^\sim\|_{cb} \leq 1 + \varepsilon$. Define $T^\sim : \bigoplus_1 F_\alpha \to X$ by $T^\sim(\bigoplus f_\alpha) = \sum T_\alpha^\sim(f_\alpha)$. Clearly $T^\sim$ is an extension, we must show that $\|T^\sim\|_{cb} \leq 1 + \varepsilon$. Now

$$\begin{aligned}
&\|[T^\sim(\oplus f_\alpha(i, j))]\|_n \\
&\quad = \sup \left\{ \left\| \left[\sum_\alpha \varphi_{kl}(T_\alpha^\sim(f_\alpha(i, j))) \right] \right\| : [\varphi_{kl}] \in \mathrm{BALL}(M_n(X^*)) \right\} \\
&\quad \leq \sup \left\{ \left\| \left[\bigoplus \varphi_{kl} \circ T_\alpha^\sim \right] \right\|_m : [\varphi_{kl}] \in \mathrm{BALL}(M_m(X^*)) \right\} \\
&\qquad\qquad\qquad\qquad \times \left\| \left[\bigoplus f_\alpha(i, j) \right] \right\|_n \\
&\quad \leq \sup\{\|T_\alpha^\sim\|_{cb} : \alpha \in \Lambda\} \left\| \left[\bigoplus f_\alpha(i, j) \right] \right\|_n \\
&\quad \leq (1 + \varepsilon) \left\| \left[\bigoplus f_\alpha(i, j) \right] \right\|_n ,
\end{aligned}$$

which completes the proof.

PROPOSITION 3.7. *The space* T_n *is projective.*

Proof. By results in **[BP]** $CB(T_n, X) = M_n(X)$ completely isometrically. Thus lifting a map in $CB(T_n, X/Y)$ to a map in $CB(T_n, X)$ of nearly the same norm follows from the definition of X/Y as an operator space.

The proof of Proposition 3.7 shows that "corners" of T_n are also projective. More specifically if p and q are orthogonal projections in M_n then $p\, T_n\, q$ is projective. Conversely, one can show that a projective operator space which is a subspace of some T_n is completely isometrically isomorphic to an L^1 direct sum of such "corners". This is because an injective operator space on a finite dimensional Hilbert space is a "corner" of a finite dimensional C^*-algebra **[R2**, Theorem

4.5]. After this paper was submitted Roger Smith has proved **[S3]**, in response to a question of the author, that every finite dimensional injective operator space is of the form pAq, where A is a *finite dimensional* C^*-algebra, and p and q are orthogonal projections in A. Thus we obtain:

THEOREM 3.8 **[S3]**. *The finite dimensional projective operator spaces are (up to complete isometric isomorphism) precisely the operator spaces of the form* $(p_1 T_{n_1} q_1) \oplus_1 \cdots \oplus_1 (p_m t_{n_m} q_m)$, *where* p_i *and* q_i *are orthogonal projections in* M_{n_i}.

DEFINITION 3.9. An operator space X is said to be *almost a direct summand* of an operator space Y if for all $\varepsilon > 0$ there is a subspace F of Y which is the range of a projection P on Y satisfying $\|P\|_{cb} < 1+\varepsilon$; and if there exists an isomorphism $T\colon X \to F$ with $\|T\|_{cb}$ and $\|T^{-1}\|_{cb} < 1+\varepsilon$.

THEOREM 3.10. *An operator space* X *is projective if and only if* X *is almost a direct summand of an* L^1*-direct sum of finite dimensional trace class algebras.*

Proof. The sufficiency is obvious. The necessity follows easily from Propositions 3.1, 3.6, and 3.7.

REMARK. There is a result dual to 3.10: an operator space X is injective if and only if X is a direct summand of an L^∞-direct sum of finite dimensional matrix algebras.

In light of Corollary 3.2 one might hope for a characterization as a direct summand of some $B(H)_*$; however $B(H)_*$ is not projective if H is infinite dimensional. This is somewhat surprising in light of Proposition 3.7. We recall (see remark after Definition 3.3) that the projectivity of $B(H)_*$ is equivalent to an extension theorem for weak* continuous completely bounded maps into $B(H)$. There is an example in **[ER2]** which shows that no such extension theorem exists. Alternatively, the non-weak*-injectivity of $B(H)$ follows immediately from the following consequence of Theorem 3.10:

COROLLARY 3.11. *Each weakly convergent sequence in a projective operator space converges uniformly.*

Proof. It is sufficient to consider a direct sum of finite dimensional trace class algebras; and the usual proof of the assertion for $l^1(I)$ works.

THEOREM 3.12. *A W^*-algebra M is weak*-injective if and only if M is isomorphic to an L^∞-direct sum of finite dimensional matrix algebras.*

Proof. The sufficiency is clear. To prove the other direction we observe that M decomposes into a direct sum of a finite W^*-algebra and a properly infinite W^*-algebra, each of which must be weak*-injective. We first show that the properly infinite part is empty. Recall that a properly infinite W^*-algebra R is isomorphic to a W^*-algebra tensor product $R\overline{\otimes}B(H)$, where H is a separable Hilbert space [V, Appendix C]. However there are obviously normal projections from $R\overline{\otimes}B(H)$ onto $B(H)$; and thus $R\overline{\otimes}B(H)$ is not weak*-injective since $B(H)_*$ is not projective for infinite dimensional H (see comment before Corollary 3.11).

We have shown that M is finite, and so there exists a normal conditional expectation of N onto the center C [T]. Therefore C is weak*-injective, so the comments before Proposition 3.4 imply that C is linearly isometrically isomorphic, and hence algebraically isomorphic, to $l^\infty(I)$ for some discrete set I. Thus M is a direct sum of finite weak*-injective factors. Since infinite dimensional type I factors are not weak*-injective, if we can show that there are no weak*-injective type II_1 W^*-algebras then we shall have completed the proof.

Suppose that there exists a weak*-injective W^*-algebra N of type II_1, we will obtain a contradiction. Let S be a maximal abelian *-subalgebra of N; since we have the (completely contractive) conditional expectation available [T, V.2.3.6] it suffices to show that S is not isomorphic to $l^\infty(I)$ for some discrete set I. However if it were, and if e is a minimal projection in $l^\infty(I)$, then since N can contain no minimal projection we see that e strictly dominates a nonzero projection f. It is now easy to see that f is not in S but does commute with S, contradicting the maximality of S.

COROLLARY 3.13. *A direct sum of finite dimensional matrix algebras is not σ-weakly linear isomorphic to either the hyperfinite* II_1 *factor, or to $B(H)$ for any infinite dimensional Hilbert space H.*

This last corollary complements a result in [CS] asserting that they are completely boundedly isomorphic. We remark that it is easy to show directly that the hyperfinite II_1 factor is not weak*-injective, by explicitly constructing a weakly convergent sequence in the predual which does not converge uniformly.

COROLLARY 3.14. *An operator space* X *which is the standard predual of a* W^**-algebra is projective if and only if* X *is completely isometric to an* L^1 *direct sum of finite dimensional trace class algebras.*

The proof of Theorem 3.12 shows that a W^*-algebra M is an L^∞ direct sum of finite dimensional matrix algebras if and only if it has the property that every weak*-continuous completely contractive map from a weak* closed subspace Y of a dual operator space X into M extends to some weak* continuous completely bounded map from X to M (we are not asking for control of the cb norm). Or stated in another way, if F is a (uniformly closed) universal projective object in **OS** (that is, each completely bounded map $F \to X/Y$ has some completely bounded lifting $F \to X$ (we are not asking for control of the norm)), whose standard dual is a W^*-algebra, then F is projective in our sense, and consequently completely isometrically isomorphic to an L^1 direct sum of finite dimensional trace class algebras. This is the generalization of Grothendieck's result promised in the introduction.

The results above ought to be viewed as a step towards an ideal characterization of projective operator spaces, perhaps as "corners" of projective preduals of W^*-algebras (see Theorem 3.8 and remarks immediately before). It would be interesting to see the development of a theory of the "projective skeleton" of an operator space, corresponding to the injective envelope **[R2]**.

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