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ON THE REPRESENTATION OF THE DETERMINANT OF HARISH-CHANDRA'S C-FUNCTION OF $\mathrm{SL}(n, \mathbb{R})$

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This paper gives the explicit representation of the determinant of the Harish-Chandra C-function of $\mathrm{SL}(n, \mathbb{R})$ $(n \geq 3)$ and some application.

1. Introduction. Let G be a semisimple Lie group with finite center, K a maximal compact subgroup of G. Let θ be the Cartan involution of G fixing K. Let P be a cuspidal parabolic subgroup and $P = MAN$ its Langlands decomposition. For σ in $\widehat{M_d}$ and γ in $\widehat{K}$, we set $\tau = (\gamma, \gamma)$ and denote the space of the τ_M-spherical cusp forms on M by ${}^0\mathfrak{C}_M(M, \tau_M)$. The Harish-Chandra C-function $C_{\overline{P}|P}(1 : \nu)$ has important information in the representation theory.

In the determinant of $C_{\overline{P}|P}(1 : \nu)$, L. Cohn has proved the following results.

THEOREM (*see* [2], *p.* 129). *There exist functions* $\mu_1, \dots, \mu_r \in \mathfrak{a}^*$ *and constants* $p_{i,j}$, $q_{i,j}$ $(i = 1, \dots, r,\ j = 1, \dots, j_i)$ *depending on* τ *such that*

$$\det C_{\overline{P}|P}(1 : \nu) = \mathrm{const} \cdot \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma\left(\frac{\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j}\right)}{\Gamma\left(\frac{\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j}\right)},$$

where $\alpha_1, \dots, \alpha_r$ *are reduced* $\mathfrak{a}$-*roots.*

He gives a conjecture that the constants $p_{i,j}$ and $q_{i,j}$ are rational numbers and depending linearly on the highest weight of the irreducible components of τ.

Let G be $\mathrm{SL}(n, \mathbb{R})$ and P the minimal parameter subgroup of G. In the case that $n = 2$, the Harish-Chandra C-function and determinant of it are well known explicitly. If n is 3 or 4, in [4] Eguchi and the author give the explicit formula of the determinant of Harish-Chandra's C-function of G, which solves Cohn's conjecture affirmatively. The purpose of this paper is to extend the result in [4] to G and apply it to the study of the reducibility of $\pi_{P,\sigma,\nu}$. The application does not give any new result but it gives another proof of Speh-Vogan's reducibility condition ([**12**], [**13**]).

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2. Notation and preliminaries. Let G be a semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{l}$ be a maximal compact subalgebra of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ the corresponding Cartan decomposition and θ the Cartan involution defining the decomposition. We introduce an inner product B_θ on $\mathfrak{g}$ in the standard way such that $B_\theta(X, Y) = -B(X, \theta Y)$, where B is the Killing form on $\mathfrak{g}$. Let $\mathfrak{a}_p$ be a maximal abelian subgroup of $\mathfrak{p}$. We fix an order in the dual space $(\mathfrak{a}_p)^*$ of $\mathfrak{a}_p$, and put $\mathfrak{n}_p = \sum_{\alpha>0} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ denotes the root space of the $\mathfrak{a}_p$-root α, and we let $\mathfrak{v}_p = \theta\mathfrak{n}_p$. Then we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{a}_p + \mathfrak{n}_p$ of $\mathfrak{g}$. Let $\mathfrak{m}_p = Z_{\mathfrak{l}}(\mathfrak{a}_p)$ the centralizer of $\mathfrak{a}_p$ in $\mathfrak{l}$.

We now let $K = N_G(\mathfrak{l})$ be the normalizer of $\mathfrak{l}$ in G, $M_p = Z_K(\mathfrak{a}_p)$ the centralizer of $\mathfrak{a}_p$ in K and $M_p' = N_K(\mathfrak{a}_p)$ the normalizer of $\mathfrak{a}_p$ in K. Let A_p, N_p and V_p be the analytic subgroups of G corresponding to $\mathfrak{a}_p$, $\mathfrak{n}_p$ and $\mathfrak{v}_p$ respectively.

Any conjugate of $\mathfrak{m}_p \oplus \mathfrak{a}_p \oplus \mathfrak{n}_p$ is called a minimal parabolic subalgebra, and any Lie subalgebra $\mathfrak{s}$ that contains a minimal parabolic subalgebra is called parabolic. Then $\mathfrak{s}$ has a Langlands decomposition (relative to θ) $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Here $\mathfrak{m} \oplus \mathfrak{a} = Z_{\mathfrak{g}}(\mathfrak{a})$, and we can impose an ordering on the $\mathfrak{a}$-roots so that $\mathfrak{n}$ is built from the positive $\mathfrak{a}$-roots. Let $\mathfrak{v} = \theta\mathfrak{n}$. If $\mathfrak{a}_M$ is a maximal abelian subspace of $\mathfrak{m} \cap \mathfrak{p}$, then $\mathfrak{a} \oplus \mathfrak{a}_M$ is a maximal abelian subspace of $\mathfrak{p}$ and can be taken as $\mathfrak{a}_p$ in our theory. When we introduce an ordering on the $\mathfrak{a}_p$-roots so that $\mathfrak{a}$ comes before $\mathfrak{a}_M$, then the positive $\mathfrak{a}$-roots are the nonzero restriction to $\mathfrak{a}$ of the positive $\mathfrak{a}_p$-roots. The sum of the root spaces for the positive $\mathfrak{a}_p$-roots that vanish on $\mathfrak{a}$ is denoted by $\mathfrak{n}_M$.

Let M_0, A, A_M, N, V, N_M be analytic subgroups corresponding to $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{a}_M$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{n}_M$ respectively and put $M = M_0 M_p$. The group $P = MAN$ is a parabolic subgroup. The subgroups in our discussion have the following properties (see e.g. [**8**]).

(1.1) (1) $MA = Z_G(\mathfrak{a})$, $MAN = N_G(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$, MAN is closed, and $(m, a, n) \in M \times A \times N \to man \in MAN$ is a diffeomorphism onto,

(2) $\theta|_{\mathfrak{m}}$ is a Cartan involution of $\mathfrak{m}$, and $K_M = K \cap M$ is the corresponding maximal compact subgroup of M,

(3) $M = K_M A_M N_M$ is an Iwasawa decomposition of M,

(4) $A_p = A_M A$ and $N_p = N_M N$ diffeomorphically,

(5) $G = KMAN$ with the KM, A and N components unique,

(6) $K \cap MA = K \cap M$,

(7) $V \cap MAN = \{1\}$,

(8) the M_p group for M equals the M_p group for G.

Two parabolic subgroups with the same MA are associated. The choices for N are in obvious one-to-one correspondence with the Weyl chambers. Let $M' = N_K(\mathfrak{a})M$ and $W(\mathfrak{a}) = M'/M$. If w is in M', then w acts on characters of A and representations of M by

$$w \cdot \nu(a) = \nu(w^{-1}aw), \qquad w \cdot \sigma(m) = \sigma(w^{-1}mw).$$

Then $W(\mathfrak{a})$ acts on characters of A and classes of representations of M. An $\mathfrak{a}$-root is said to be reduced if $r\alpha$ is not a root for $0 < r < 1$ $(r \in \mathbb{R})$. Let β be a reduced $\mathfrak{a}$-root in the dual $\mathfrak{a}^*$, H_β the corresponding member of $\mathfrak{a}$ under the identification set up by B_θ, and $(H_\beta)^\perp$ the orthogonal complement of $\mathbb{R} \cdot H_\beta$ in $\mathfrak{a}$. We set $\mathfrak{n}^{(\beta)} = \sum_{c>0} \mathfrak{g}_{c\beta}$, $\mathfrak{v}^{(\beta)} = \theta\mathfrak{n}^{(\beta)} = \sum_{c<0} \mathfrak{g}_{c\beta}$ and let $\mathfrak{g}^{(\beta)}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}^{(\beta)}$ and $\mathfrak{v}^{(\beta)}$. Let $N^{(\beta)}$, $V^{(\beta)}$ and $G^{(\beta)}$ be the analytic subgroups corresponding to $\mathfrak{n}^{(\beta)}$, $\mathfrak{v}^{(\beta)}$ and $\mathfrak{g}^{(\beta)}$ respectively.

Let $\widehat{K}$ and $\widehat{M}$ be the set of all equivalence classes of the irreducible unitary representations of K and M respectively. For each $\sigma \in \widehat{M}$ we fix a representation $(\tilde{\sigma}, H^{\tilde{\sigma}})$ in σ and, abusing notation, we use also σ for $\tilde{\sigma}$. For each γ in $\widehat{K}$ we fix an element (π_γ, H^γ) in γ.

We recall the generalized principal series representations. Let $P = MAN$ be a parabolic subgroup and $\rho_P = \frac{1}{2} \cdot \sum_{\alpha>0}(\dim \mathfrak{g}_\alpha)\alpha$. Let σ be in $\widehat{M}$ and ν in $\mathfrak{a}_{\mathbb{C}}^*$ (the complexification of $\mathfrak{a}^*$). Let $C_{P,\sigma,\nu}(G)$ be the space of all continuous functions f from G to H^σ such that

$$f(xman) = e^{-(\nu+\rho_P)(\log a)}\sigma(m)^{-1}f(x) \qquad (x \in G).$$

Let $h^{P,\sigma,\nu}$ be the completion of $C_{P,\sigma,\nu}(G)$ by the norm

$$\|f\|^2 = \int_K \|f(k)\|^2 \, dk \qquad (f \in C_{P,\sigma,\nu}(G)).$$

The representation $\pi_{P,\sigma,\nu}$ is given by

$$\pi_{P,\sigma,\nu}(g)f(x) = f(g^{-1}x) \qquad (g \in G)'.$$

The compact picture is the restriction of the induced picture to K. Here the dense subspace $C_\sigma(K)$ is

$$\{f\colon K \to H^\sigma | f \text{ is continuous and } f(km) = \sigma(m)^{-1}f(k)\}$$

and is independent of ν. According to the decomposition $G = KMAN$ of (1.1) each $g \in G$ is written as

$$g = \kappa(g)\mu(g)(\exp H(g))n(g),$$
$$(\kappa(g) \in K,\ \mu(g) \in M,\ H(g) \in \mathfrak{a},\ n(g) \in N).$$

Then representation is given by

$$\pi_{P,\sigma,\nu}(g)f(k) = e^{-(\nu+\rho_P)(H(g^{-1}k))}f(\kappa(g^{-1}k)).$$

If γ is in $\widehat{K}$, the projection operator E_γ defined by

$$E_\gamma = d(\gamma)\overline{\chi}_\gamma * f \quad (f \in C_\sigma(K)),$$

where $d(\gamma)$ and χ_γ denote the dimension and the character of γ respectively. For γ in $\widehat{K}$, we put

$$H^{P,\sigma,\nu} = \{f \in H^{P,\sigma,\nu} | E_\gamma f = f\}.$$

3. Some lemmas for the intertwining operators. Let $P = MAN'$ and $P' = MAN'$ be associated parabolic subgroups and let σ be in $\widehat{M}$ and ν in $\mathfrak{a}^*_{\mathbb{C}}$. For f in $C_{P,\sigma,\nu}(G)$ we set

$$A(P' : P : \sigma : \nu)f(x) = \int_{V \cap N'} f(xv)\,dv,$$

where $V = \theta N$ and dv is the normalized Haar measure on $V \cap N'$ by

$$\int_{V\cap N'} e^{-2\rho_P(H(v))}\,dv = 1.$$

The operator $A(P' : P : \sigma : \nu)$ is called the intertwining operator. In this section we shall describe the properties of the intertwining operators, which are well known results (see e.g. **[8]**).

The inner product B_θ on $\mathfrak{g}$ induces an inner product on the dual $\mathfrak{a}^*$ of $\mathfrak{a}$, which we denote by $\langle \cdot, \cdot \rangle$.

Let ρ_M be half the sum of the positive $\mathfrak{a}_M$-roots. Since the parabolic subgroup $P = MAN$ contains the minimal parabolic subgroup $P_p = M_p A_p N_p$ such that $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$.

For each $\mathfrak{a}$-root β, set $C_\beta = \max\{\rho_M(H_\alpha)\}$, where the maximum is taken over all $\mathfrak{a}_p$-roots α satisfying $\alpha|_\mathfrak{a} = \beta$.

LEMMA 3.1. *Let $P = MAN$ and $P' = MAN$ be associated parabolic subgroups and suppose that $\langle \mathrm{Re}\,\nu, \beta\rangle > C_\beta$ for every $\mathfrak{a}$-root β such that $\mathfrak{g}_\beta \subset \mathfrak{n} \cap \mathfrak{v}'$. Then the integral $A(P' : P : \sigma : \nu)f(x)$ $(x \in G, f \in C_{P,\sigma,\nu}(G))$ is a convergent. Moreover, if f is a K-finite function in the compact picture of $\pi_{P,\sigma,\nu}$ then the integral has an analytic continuation to a global meromorphic function in ν.*

LEMMA 3.2. *If σ is in $\widehat{M}$ and ν in $\mathfrak{a}^*_\mathbb{C}$, then we have*

$$A(P' : P : \sigma : \nu)\pi_{P,\sigma,\nu}(g) = \pi_{P',\sigma,\nu}(g)A(P' : P : \sigma : \nu)$$

for all g in G.

For w in M', let $R(w)f(x) = f(xw)$. Then it follows from Lemma 3.2 *that*

$$A_P(w, \sigma, \nu) = R(w)A(w^{-1}Pw : P : \sigma : \nu) \tag{3.1}$$

satisfies

$$\pi_{P,w\sigma,w\nu}(\cdot)A_P(w, \sigma, \nu) = A_P(w, \sigma, \nu)\pi_{P,\sigma,\nu}(\cdot).$$

LEMMA 3.3. *Let $P = MAN$ and $P' = MAN'$ be associated parabolic subgroups. Then there exists a scalar-valued function $\gamma(P' : P : \sigma : \nu)$ meromorphic in ν such that*

$$A(P : P' : \sigma : \nu)A(P' : P : \sigma : \nu) = \eta(P' : P : \sigma : \nu)I. \tag{3.2}$$

Let $P = MAN$ and $P' = MAN'$ be as in Lemma 3.3. *A sequence $P_i = MAN_i$ $(0 \le i \le r)$ is called a string from P to P' if there are P-positive reduced $\mathfrak{a}$-roots β_i $(1 \le i \le r)$ such that*

$$V_{i-1} \cap N_i = V^{(\beta_i)} \text{ or } N^{(\beta_i)} \qquad (1 \le i \le r),$$
$$P_0 = P \quad \text{and} \quad P_r = P'.$$

The string P_i from P to P' is called minimal if we have

$$V_{i-1} \cap N_i = V^{(\beta_i)} \qquad (1 \le i \le r),$$
$$P_0 = P \quad \text{and} \quad P_r = P'.$$

LEMMA 3.4. *Suppose that* $P = MAN$ *and* $P' = MAN'$ *are associated parabolic subgroups and* $P_i = MAN_i$ $(0 \leq i \leq r)$ *is a minimal string from* P *to* P', *with associated* P*-positive reduced* $\mathfrak{a}$*-roots* $\{\beta_i\}$. *Then*

(1) *the set* $\{\beta_i\}$ *is characterized as the set of reduced* $\mathfrak{a}$*-roots* α *that are positive for* P *and negative for* P'.

(2) r *is characterized as the number of* $\mathfrak{a}$*-roots described in* (1).

(3) *the intertwining operators satisfy*

$$A(P' : P : \sigma : \nu) = A(P_r : P_{r-1} : \sigma : \nu) \cdots A(P_i : P_0 : \sigma : \nu).$$

LEMMA 3.5. *Let* $P = MAN$ *be a parabolic subgroup, let* σ *be in* $\widehat{M}$ *and* ν *in* $\mathfrak{a}_{\mathbb{C}}^*$ *such that* $\operatorname{Re}\nu$ *is in the open positive Weyl chamber. Then* $\pi_{P,\sigma,\nu}$ *has a unique irreducible quotient* $J(p, \sigma, \nu)$ *and* $J(P, \sigma, \nu)$ *is isomorphic with the image of the intertwining operator* $A(\overline{P} : P : \sigma : \nu)$ *on* $H^{P,\sigma,\nu}$, *where* $\overline{P} = MAV$.

4. The B_γ^σ-functions. In this section we shall work only with minimal parabolic subgroups and omit the subscripts p. Let P, P' be associated minimal parabolic subgroups and let γ be in $\widehat{K}$, σ in $\widehat{M}$ and A in $\operatorname{Hom}_M(V^\gamma, H^\sigma)$, where V^γ denotes the representation space of γ. For ν in $\mathfrak{a}_{\mathbb{C}}^*$, v in V^γ, let

$$L_P(A, v, \nu)(\mathrm{kan}) = e^{-(\nu+\rho_P)(\log a)} A(\pi_\gamma(k^{-1})v)$$

for k in K, a in A, n in N. Then an easy computation shows that $L_P(A, v, \nu)$ is in $H_\gamma^{P,\sigma,\nu}$. Furthermore the map

$$V^\gamma \otimes \operatorname{Hom}_M(V^\gamma, H^\sigma) \to H_\gamma^{P,\sigma,\nu},$$

given by $v \otimes A \to L_P(A, v, \nu)$ is a bijective K-intertwining operator. Set

$$A_\gamma(P' : P : \sigma : \nu) = A(P' : P : \sigma : \nu)|_{H_\gamma^{P,\sigma,\nu}}.$$

Then we have $A_\gamma(P' : P : \sigma : \nu)$ is in $\operatorname{Hom}_K(H_\gamma^{P',\sigma,\nu}, H_\gamma^{P,\sigma,\nu})$.

LEMMA 4.1. (*See* [**4**], [**15**].) *If* ν *is in* $\mathfrak{a}_{\mathbb{C}}^*$ *and* $\langle \operatorname{Re}\nu, \alpha \rangle > 0$ *for all* P*-positive roots* α *then we have*

$$A_\gamma(P' : P : \sigma : \nu) L_P(A, v, \nu) = L_P(A \circ B_\gamma(P' : P : \nu), v, \nu),$$

where

$$B_\gamma(P' : P : \nu) = \int_{V \cap N'} \pi_\gamma(\kappa(v))^{-1} e^{-(\nu+\rho_P)(H(v))}\, dv.$$

Furthermore $B_\gamma(P' : P : \nu)$ *satisfies the following conditions,*

(1) $B_\gamma(P' : P : \nu)$ *is absolutely convergent.*

(2) $B_\gamma(P' : P : \nu)$ *is in* $\mathrm{End}(V^\gamma)$ *and satisfies*

$$B_\gamma(P' : P : \nu)\pi_\gamma(m)B_\gamma(P' : P : \nu) \qquad (m \in M).$$

Now we define B_γ^σ-functions. If σ is in $\widehat{M}$, we denote the σ-component of V^γ by V_σ^γ. Let

$$B_\gamma^\sigma(P' : P : \nu) = B_\gamma(P' : P : \nu)|_{V_\sigma^\gamma}.$$

Then $B_\gamma^\sigma(P' : P : \nu)$ is in $\mathrm{End}(V_\sigma^\gamma)$ and from Lemma 3.1 it has an analytic continuation to a global meromorphic function in ν. Particularly, $B_\gamma(\overline{P} : P : \nu)$ is called Harish-Chandra's C-function.

COROLLARY 4.2. *If* w *is in* M', ν *is in* $\mathfrak{a}_{\mathbb{C}}^*$ *such that* $\langle \mathrm{Re}\, \nu, \alpha\rangle > 0$ *for all* P*-positive roots* α, *then we have*

$$A_P(w, \sigma, \nu)L_P(A, v, \nu) = L_{P'}(A \circ B_\gamma(P, w, \nu) \circ \pi_\gamma(w)^{-1}, v, w\nu),$$

where

$$B_\gamma(P, w, \nu) = B_\gamma(w^{-1}Pw : P : \nu).$$

Let w *be in* M' *such that*

$$w^{-1}Pw = \overline{P} \quad \text{and} \quad w = w_r w_{r-1} \cdots w_1, \tag{4.1}$$

where each w_i $(1 \le i \le r)$ *is the reflection with respect to the* P*-simple* $\mathfrak{a}$*-root* γ_i *and* r *is the length of* w. *Then by the relation*

$$\begin{aligned}(4.2)\quad & A_P(w, \sigma, \nu) \\ &= A_P(w_r, w_{r-1}\cdots w_1\sigma, w_{r-1}\cdots w_1\nu)\cdots A_P(w_1, \sigma, \nu)\end{aligned}$$

and Corollary 4.2, *we have*

$$\begin{aligned}(4.3)\quad & B_\gamma^\sigma(\overline{P} : P : \nu) \\ &= B_\gamma^\sigma(P, w_1, \nu)\pi_\gamma^\sigma(w_1)B_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \\ &\qquad \cdots B_\gamma^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1}\cdots w_1\nu) \\ &\quad \cdot \pi_\gamma^{w_{r-1}\cdots w_1\sigma}(w_r)\pi_\gamma^{w\sigma}(w).\end{aligned}$$

In connection with Lemma 4.1 we have the following proposition.

PROPOSITION 4.3. *Let* w *be as above. We set*

$$P_i = (w_i w_{i-1}\cdots w_1)^{-1}P(w_i w_{i-1}\cdots w_1) \qquad (0 \le i \le r)$$

and

$$\beta_i = (w_{i-1} \cdots w_1)^{-1} \gamma_i \qquad (1 \le i \le r).$$

Then P_i $(0 \le i \le r)$ *is a minimal string* P *to* $\overline{P}$, *with associated reduced* P*-positive* $\mathfrak{a}$*-roots* $\{\beta_i\}$ *and we have*

$$\begin{aligned} &A(\overline{P} : P : \sigma : \nu) \\ &\quad = A(P_r : P_{r-1} : \sigma : \nu) A(P_{r-1} : P_{r-2} : \sigma : \nu) \cdots A(P_1 : P_0 : \sigma : \nu). \end{aligned}$$

Proof. By an easy computation, we have

$$V_{i-1} \cap N_i = V^{(\beta_i)} \qquad (1 \le i \le r). \tag{4.4}$$

We shall prove reduced $\mathfrak{a}$-roots β_i $(1 \le i \le r)$ are P-positive. For an integer i such that $1 \le i \le r$ we set

$$[N_i] = \{\alpha | \alpha \text{ is a } P\text{-positive and } P_i\text{-positive reduced } \mathfrak{a}\text{-root}\}$$

and denote the cardinality of $[N_i]$ by n_i. Since r is n_0, we have

$$n_{i-1} - n_i = 1 \qquad (1 \le i \le r). \tag{4.5}$$

From (4.4) and (4.5), β_i $(1 \le i \le r)$ are P-positive. Therefore P_i $(1 \le i \le r)$ is the minimal string with associated P-positive reduced $\mathfrak{a}$-roots $\{\beta_i\}$. The other assertion follows from Lemma 3.4(3).

5. The B_γ-function in the $\mathrm{SL}(n, \mathbb{R})$ case. We shall specialize to $\mathrm{SL}(n, R)$ the notation described in the previous sections. Our notation is as follows. Let G be in $\mathrm{SL}(n, R)$, the group of n-by-n real matrices g of determinant one. Let

$$\begin{aligned} \theta &= -\text{ transpose}, \\ K &= \mathrm{SO}(n), \\ \mathfrak{a} &= \text{the vector space of the diagonal matrices of trace } 0, \\ M &= \{m \in G \mid m = \mathrm{diag}(m_1, \dots, m_n) \text{ and } m_i = \pm 1 \ (1 \le i \le n)\}, \\ A &= \exp \mathfrak{a}, \\ N &= \{n \in G \mid n \text{ is the sum of the identity and strictly upper} \\ &\qquad\qquad \text{triangular matrices}\}, \\ P &= MAN. \end{aligned}$$

Then P is a minimal parabolic subgroup of G. Let e_j $(1 \le j \le n)$ be the linear functional on $\mathfrak{a}_{\mathbb{C}}$ that picks out the jth diagonal entry and set $\alpha_j = e_j - e_{j+1}$ $(1 \le j \le n-1)$. Then simple $\mathfrak{a}$-roots are α_j $(1 \le j \le n-1)$. We denote the simple reflection with respect to α_j by s_{α_j}.

LEMMA 5.1. *If ν is in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \operatorname{Re}\nu, \alpha\rangle > 0$ for all P-positive $\mathfrak{a}$-roots α, then for each integer j such that $1 \le i \le n-1$ we have*

$$B_\gamma(P, s_{\alpha_j}, \nu) = \text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)} \pi_\gamma(f(x)^{-1} k_j(x))^{-1}\, dx,$$

where

$$f(x) = (1+x^2)^{1/2}, \qquad \nu_j = 2\langle \nu, \alpha_j\rangle \cdot \langle \alpha_j, \alpha_j\rangle^{-1}$$

and

$$k_j(x) = \begin{array}{cl} \overbrace{\hphantom{f(x)\ f(x)\ \ddots\ f(x)}}^{j-1} & \\ \left(\begin{array}{cccc|cc|ccc} f(x) & & & & & & & & \\ & f(x) & & & & & & & \\ & & \ddots & & & & & & \\ & & & f(x) & & & & & \\ \hline & & & & 1 & -x & & & \\ & & & & x & 1 & & & \\ \hline & & & & & & f(x) & & \\ & & & & & & & \ddots & \\ & & & & & & & & f(x) \end{array}\right) & \begin{array}{l} \left.\vphantom{\begin{array}{c} f \\ f \\ f \\ f \end{array}}\right\} j-1 \\ \vphantom{\begin{array}{c} f \\ f \\ f \\ f \\ f \end{array}} \end{array} \end{array}$$

Since the results are obtained by an easy computation, we omit the proof.

Let E_{ij} $(1 \le i, j \le n)$ be the matrix that is 1 in the $i-j$th entry and 0 elsewhere. Set

$$\mathfrak{h} = \sum_{1 \le l \le [n/2]} \mathbb{R} \cdot H_l,$$

where $H_l = E_{2l-1,2l} - E_{2l,2l-1}$ $(1 \le l \le [n/2])$ and $[t]$ $(t \in \mathbb{R})$ is the integer satisfying $[t] \le t < [t]+1$. Then $\exp\mathfrak{h}$ is a maximal torus of K.

LEMMA 5.2. *Let γ be in $\widehat{K}$, μ a weight of V^γ and ν in $\mathfrak{a}_{\mathbb{C}}^*$. If v_μ is a μ-weight vector of V^γ, then for each integer j such that $0 \le j \le n-1$ and $j \equiv 1 \pmod 2$, we have*

$$B_\gamma(P, s_{\alpha_j}, \nu) v_\mu = \text{Const} \cdot \alpha(\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]})) v_\mu,$$

and

$$B_\gamma(\overline{P}, s_{\alpha_j}, \nu)v_\mu = \text{Const} \cdot \alpha(-\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))v_\mu,$$

where

$$\alpha(s, n) = \frac{\gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+1-n}{2})\Gamma(\frac{s+1+n}{2})} \qquad (s \in \mathbb{C}, n \in \mathbb{Z}).$$

Proof. From Lemma 5.1, we have

$$\begin{aligned}(5.1)\quad & B_\gamma(P, s_{\alpha_j}, \nu)v_\mu \\ & = \text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)}\pi_\gamma(f(x)^{-1}k_j(x))^{-1}v_\mu \, dx.\end{aligned}$$

We note that

$$\pi_\gamma(\exp tH_{[(j+1)/2]})v_\mu = e^{t\mu(H_{[(j+1)/2]})}v_\mu \qquad (t \in \mathbb{R}).$$

Putting $\cos t = f(x)^{-1}$, $\sin t = x/f(x)$, we obtain that

$$\pi_\gamma(f(x)^{-1}k_j(x))^{-1}v_\mu = \left(\frac{1+\sqrt{-1}x}{f(x)}\right)^{-\sqrt{-1}\mu(H_{[(j+1)/2]})} v_\mu.$$

Thus (5.1) is equal to

$$\text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)} \left(\frac{1+\sqrt{-1}x}{f(x)}\right)^{-\sqrt{-1}\mu(H_{[(j+1)/2]})} dx v_\mu.$$

Therefore, the assertion of the lemma follows from the next proposition.

PROPOSITION 5.3 (*cf*. A.3 *in* **[3]**). *Suppose that* s *is a complex number and* n *an integer. Then we have*

$$\int_{-\infty}^{\infty}(1+x^2)^{-(s+1)/2}\left(\frac{1-\sqrt{-1}x}{(1+x^2)^{1/2}}\right)^n dx = \frac{\sqrt{-1}\Gamma(\frac{s}{2})\Gamma(-\frac{s+1}{2})}{\Gamma(\frac{s+1-n}{2})\Gamma(\frac{s+1+n}{2})}.$$

Let C_l $(1 \le l \le [(n+1)/2]-1)$ *be the* n-*by*-n *matrix defined by*

$$\begin{array}{l} \qquad\quad \overbrace{\hphantom{1\quad 1\quad \ddots\quad 1}}^{2l-2} \\ 2l-2\left\{\left(\begin{array}{cccc|ccc|cccc} 1 & & & & & & & & & & \\ & 1 & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & 1 & & & & & & & \\ \hline & & & & & & 1 & & & & \\ & & & & & -1 & & & & & \\ & & & & 1 & & & & & & \\ \hline & & & & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{array}\right)\right. \in M'. \end{array}$$

Then C_l^2 *is equal to identity and we have*

$$C_l \cdot k_{2l}(x) \cdot C_l^{-1} = k_{2l-1}(x), \tag{5.2}$$

whenever $1 \le l \le [(n+1)/2]-1$ *and* $x \in \mathbb{R}$.

Lemma 5.4. *Suppose that* γ *is in* $\widehat{K}$, μ *a weight of* V^γ *and* ν *in* $\mathfrak{a}_{\mathbb{C}}^*$. *If* v_μ *is a* μ-*weight vector of* V^γ, *then for each integer* j *such that* $0 \le j \le n-1$ *and* $j \equiv 0 \pmod 2$, *we have*

$$\pi_\gamma(C_{j/2})B_\gamma(P, s_{\alpha_j}, \nu)\pi_\gamma(C_{j/2}) = B_\gamma(P, s_{\alpha_{j-1}}, -(C_{j/2}\cdot\nu)),$$

where $C_{j/2}\cdot\nu$ *is in* $\mathfrak{a}_{\mathbb{C}}^*$ *defined by*

$$C_{j/2}\cdot\nu(H) = \nu(C_{j/2}^{-1}HC_{j/2}) \qquad (H \in \mathfrak{a}_{\mathbb{C}}).$$

Proof. By Lemma 5.1 and (5.2), we have

$$\begin{aligned} &\pi_\gamma(C_{j/2})B_\gamma(P, s_{\alpha_j}, \nu)\pi_\gamma(C_{j/2}) \\ &\qquad = \text{Const}\cdot\int_{-\infty}^{\infty} f(x)^{-(\nu_j+1)}\pi_\gamma(f(x)^{-1}k_{j-1}(x))^{-1}\,dx. \end{aligned} \tag{5.3}$$

Since the bilinear form $\langle\cdot,\cdot\rangle$ is invariant under the action of $C_{j/2}$, we have

$$\begin{aligned} &\langle -(C_{j/2}\cdot\nu), \alpha_{j-1}\rangle\cdot\langle\alpha_{j-1}, \alpha_{j-1}\rangle^{-1} \\ &\qquad = -\langle\nu, C_{j/2}\cdot\alpha_{j-1}\rangle\cdot\langle\alpha_j, \alpha_j\rangle^{-1} = \langle\nu, \alpha_j\rangle\cdot\langle\alpha_j, \alpha_j\rangle^{-1}. \end{aligned}$$

Therefore (5.3) is equal to

$$\begin{aligned}&= \text{Const} \cdot \int_{-\infty}^{\infty} f(x)^{-((-(C_{j/2}\cdot\nu))_{j-1}+1)} \pi_\gamma(f(x)^{-1}k_{j-1}(x))^{-1}\,dx\\ &= B_\gamma(P, s_{\alpha_{j-1}}, -(C_{j/2}\cdot\nu)).\end{aligned}$$

This proves the lemma.

6. ***M*-isotypic components of γ.** In this section we shall describe the M-isotypic components of γ in $\widehat{K}$. We fix γ in $\widehat{K}$. Let σ be in $\widehat{M}$ and denote the σ-isotypic component by V_σ^γ. Then we have

$$V_\gamma = \sum_{\sigma\in\widehat{M}} V_\sigma^\gamma \qquad \text{(direct sum)}.$$

Let P_σ be the projection map $V^\gamma \to V_\sigma^\gamma$. From Lemma 4.1(2), for P, P' in $\mathscr{P}(A)$ and ν in $\mathfrak{a}_{\mathbb{C}}^*$ we have

$$B_\gamma(P' : P : \nu)P_\sigma = P_\sigma B_\gamma(P' : P : \nu). \tag{6.1}$$

Let μ be a weight of V^γ and let $[\mu]$ denote the equivalence class of μ, which is defined as follows; μ' is in $[\mu]$ if and only if $\mu(H_l)$ is equal to $\pm\mu'(H_l)$ for any integer l such that $1 \le l \le [n/2]$. Let $\check{\gamma}$ be the set of the equivalence classes $[\mu]$ and $V^{\gamma,\mu}$ the μ-weight space of V^γ. Set

$$V_\sigma^{\gamma,\mu} = P^\sigma(V^{\gamma,\mu}) \quad\text{and}\quad V_\sigma^{\gamma,[\mu]} = \sum_{\mu'\in[\mu]} V_\sigma^{\gamma,\mu'}.$$

LEMMA 6.1. *In the above situation we have*

$$V_\sigma^\gamma = \sum_{[\mu]\in\check{\gamma}} V_\sigma^{\gamma,[\mu]} \qquad (\textit{direct sum}).$$

Proof. Let m be a positive integer and μ_k $(1 \le k \le m)$ a weight of V^γ such that μ_k is not equivalent to μ_k, if $k \neq k'$. Suppose $v_{[\mu_k]}$ $(1 \le k \le m)$ are in $V_\sigma^{\gamma,[\mu_k]}$ which satisfy the following relation,

$$\sum_{k=1}^m v_{[\mu_k]} = 0.$$

To prove the lemma, it is enough to show that

$$v_{[\mu_k]} = 0 \qquad (1 \le k \le m).$$

We shall prove by induction on m. If $m = 1$ it is clear. Suppose the assertion is true for $1 \leq m < t$. We check the case that $m = t$. Suppose that

$$\sum_{k=1}^{t} v_{[\mu_k]} = 0. \tag{6.2}$$

Then for an integer i such that $1 \leq i \leq l$ we have

$$0 = (B_\gamma(P, w_{2i-1}, \nu) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i))) \left(\sum_{k=1}^{t} v_{[\mu_k]}\right),$$

by Lemma 5.2 and (6.1)

$$= \sum_{k=2}^{t} (\alpha(\nu_{2i-1}, \sqrt{-1}\mu_k(H_i)) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i)))v_{[\mu_k]}.$$

Applying the inductive hypothesis, we have

$$(\alpha(\nu_{2i-1}, \sqrt{-1}\mu_k(H_i)) - \alpha(\nu_{2i-1}, \sqrt{-1}\mu_1(H_i)))v_{[\mu_k]} = 0 \qquad (2 \leq k \leq t).$$

Since $[\mu_k] \neq [\mu_1]$ $(2 \leq k \leq t)$, we obtain

$$v_{[\mu_k]} = 0 \qquad (2 \leq k \leq t).$$

From (6.2) we have

$$v_{[\mu_k]} = 0 \qquad (1 \leq k \leq t).$$

This proves the lemma.

LEMMA 6.2. *Suppose ν is in $\mathfrak{a}_{\mathbb{C}}^*$ and j an integer such that $1 \leq j \leq n-1$. Then $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$ are diagonalizable and*

(1) *if $j \equiv 1 \pmod 2$, we have*

$$\deg(B_\gamma^\sigma(P, \alpha_j, \nu)) = \text{Const} \cdot \prod_{[\mu] \in \check{\gamma}} \alpha(\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))^{d(\gamma, \sigma, [\mu])},$$

(2) *if $j \equiv 0 \pmod 2$, we have*

$$\det(B_\gamma^\sigma(P, \alpha_j, \nu)) = \text{Const} \cdot \prod_{[\mu] \in \check{\gamma}} \alpha(\nu_j, \sqrt{-1}\mu(H_{[(j+1)/2]}))^{d(\gamma, C_{[(j+1)/2]} \cdot \sigma, [\mu])},$$

where $d(\gamma, \sigma, [\mu])$ *is the dimension of the space* $V_\sigma^{\gamma, [\mu]}$ *and* $C_{j/2} \cdot \sigma$ $(1 \le j \le n-1, j \equiv 0 \pmod 2)$ *are defined by*

$$C_{j/2} \cdot \sigma(m) = \sigma(C_{j/2}^{-1} \cdot m \cdot C_{j/2}) \qquad (m \in M).$$

Proof. The relation (1) follows immediately from Lemma 5.2, Lemma 6.1 and (6.2). The relation (2) follows from Lemma 5.4 and (1). The first assertion is obvious.

7. The determinant of the C**-function.** Let w be in W and satisfy that

$$w^{-1} P w = \overline{P} \quad \text{and} \quad w = w_r w_{r-1} \cdots w_1,$$

where each w_i $(1 \le i \le r)$ is the reflection with respect to the simple $\mathfrak{a}$-root α_{j_i} and r is the length of w. Then we have

$$A(\overline{P} : P : \sigma : \nu) = R(w) A_P(w, \sigma, \nu).$$

By the relation

$$(7.1) \qquad A_P(w, \sigma, \nu) = A_P(w_r, w_{r-1} \cdots w_1 \sigma, w_{r-1} \cdots w_1 \nu) \cdots A_P(w_2, w_1 \sigma, w_1 \nu) \cdot A_P(w_1, \sigma, \nu)$$

and by Corollary 4.2, we have for γ in $\widehat{K}$

$$(7.2) \qquad B_\gamma(\overline{P} : P : \nu) = B_\gamma(P, w_1, \nu) \pi_\gamma(w_1) B_\gamma(P, w_2, w_1 \nu) \cdots B_\gamma(P, w_r, w_{r-1} \cdots w_1 \nu) \cdot \pi_\gamma(w_r) \pi_\gamma(w).$$

For each integer j such that $1 \le j \le n-1$, we define $\widetilde{C} \cdot \sigma$ $(\in \widehat{M})$ as follows:

if $j \equiv 0 \pmod 2$,

$$\widetilde{C}_j \cdot \sigma = C_j \cdot (w_{j-1} \cdots w_1 \sigma),$$

if $j \equiv 1 \pmod 2$,

$$\widetilde{C}_j \cdot \sigma = w_{j-1} \cdots w_1 \sigma.$$

THEOREM 7.1. *Suppose* ν *is in* $\mathfrak{a}_{\mathbb{C}}^*$, γ *in* $\widehat{K}$ *and* σ *in* $\widehat{M}$. *Then we have*

$$\det(B_\gamma^\sigma(\overline{P} : P : \nu)) = \text{Const} \cdot \prod_{i=1}^{r} \prod_{[\mu] \in \check{\gamma}} \alpha(2 \cdot \langle \nu, \beta_i \rangle \cdot \langle \beta_i, \beta_i \rangle^{-1}, \sqrt{-1}\mu(H_{[(j_i+1)/2]}))^{d_{i, [\mu]}}$$

where β_i $(1 \le i \le r)$ *are as in Corollary* 3.3 *and*

$$d_{i,[\mu]} = d(\gamma, \widetilde{C}_{j_i} \cdot \sigma, [\mu]).$$

Proof. From (7.2), we have

$$B_\gamma^\sigma(\overline{P} : P : \nu) = B_\gamma^\sigma(P, w_1, \nu)\pi_\gamma^\sigma(w_1)B_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \cdots B_\gamma^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1}\cdots w_1\nu) \cdot \pi_\gamma^{w_{r-1}\cdots w_1\sigma}(w_r)\pi_\gamma^{w\sigma}(w),$$

where $\rho_\gamma^\sigma(w')$ $(w' \in W)$ is $\pi_\gamma(w')|_{V_\sigma^\gamma}$.

Let i be an integer such that $0 \le i \le n-1$ and σ' in $\widehat{M}$ such that $V_\sigma^\gamma \ne \{0\}$. We extend $B_\gamma^{\sigma'}(w_i, \cdot)$ to an operator $\widetilde{B}_\gamma^{\sigma'}(w_i, \cdot)$ of V^γ by

$$\widetilde{B}_\gamma^{\sigma'}(w_i, \cdot) = \begin{cases} B_\gamma^{\sigma'}(w_i, \cdot) & \text{on } V_\sigma^\gamma, \\ \text{identity} & \text{on } V_{\sigma''}^\gamma\ (\sigma'' \ne \sigma') \end{cases} \tag{7.3}$$

and define

$$\widetilde{B}_\gamma^\sigma(\overline{P} : P : \nu) = \widetilde{B}_\gamma^\sigma(P, w_1, \nu)\pi_\gamma^\sigma(w_1)\widetilde{B}_\gamma^{w_1\sigma}(P, w_2, w_1\nu) \cdots \widetilde{B}_\gamma^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1}\cdots w_1\nu) \cdot \pi_\gamma^{w_{r-1}\cdots w_1\sigma}(w_r)\pi_\gamma^{w\sigma}(w). \tag{7.4}$$

Then we have

$$\widetilde{B}_\gamma^\sigma(\overline{P} : P : \nu)|_{V_\sigma^\gamma} = B_\gamma^\sigma(\overline{P} : P : \nu) \tag{7.5}$$

and

$$\det(\widetilde{B}_\gamma^\sigma(\overline{P} : P : \nu)) = d_1 \cdot \det(B_\gamma^\sigma(\overline{P} : P : \nu)), \tag{7.6}$$

where d_1 is a nonzero constant which is independent of ν. On the other hand, from (7.3) and (7.4) we have

$$\det(\widetilde{B}_\gamma^\sigma(\overline{P} : P : \nu)) = d_2 \cdot \det(B_\gamma^\sigma(P, w_1, \nu)) \cdots \det(B_\gamma^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1}\cdots w_1\nu)), \tag{7.7}$$

where d_2 is a constant such that $|d_2| = 1$. Therefore, from (7.6) and (7.7) we have

$$\det(B_\gamma^\sigma(\overline{P} : P : \nu)) = \text{Const} \cdot \det(B_\gamma^\sigma(P, w_1, \nu)) \cdots \det(B^{w_{r-1}\cdots w_1\sigma}(P, w_r, w_{r-1}\cdots w_1\nu))$$

by Lemma 6.2

$$= \text{Const} \cdot \prod_{i=1}^{r} \prod_{[\mu]\in\check{\gamma}} \alpha((w_{i-1}\cdots w_1\nu)_{j_i}, \sqrt{-1}\mu(H_{[(j+1)/2]}))^{d_{i,[\mu]}}$$

by Proposition 4.3

$$= \text{Const} \cdot \prod_{i=1}^{r} \prod_{[\mu]\in\check{\gamma}} \alpha(2\cdot\langle\nu, \beta_i\rangle\cdot\langle\beta_i, \beta_i\rangle^{-1}, \sqrt{-1}\mu(H_{[(J+1)/2]}))^{d_{i,[\mu]}}.$$

This proves the theorem.

8. The reducibility of $\pi_{P,\sigma,\nu}$ in the nonsingular case. Let ν be in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \operatorname{Re}\nu, \alpha\rangle \neq 0$ for all P-positive roots. In this section we shall describe a necessary and sufficient condition for that $\pi_{P,\sigma,\nu}$ is reducible.

Let β be a reduced P-positive $\mathfrak{a}$-root and $G^{(\beta)}$ as in §1. In this case $G^{(\beta)}$ is isomorphic to $SL(2, \mathbb{R})$ and we can put

$$M \cap G^{(\beta)} = \{e, m_\beta\},$$

where e is the identity matrix. Let σ be in $\widehat{M}$. Since M is abelian and any element of M is of order two, $\sigma(m)$ $(m \in M)$ is a scalar operator and the scalar is ± 1. We define integers σ_β such that $0 \le \sigma_\beta \le 1$ by

$$\sigma(m_\beta) = (-1)^{\sigma_\beta}\cdot I,$$

where I is the identity operator.

LEMMA 8.1. *Let σ be in $\widehat{M}$, γ in $\widehat{K}$ and μ a weight of V^γ. Let j be an integer such that $0 \le j \le n-1$ and $j \equiv 1 \pmod 2$. Suppose that*

$$\sqrt{-1}\mu(H_{[(j+1)/2]}) - \sigma_{\alpha_j} \equiv 1 \pmod 2. \tag{8.1}$$

Then we have

$$V_\sigma^{\gamma,[\mu]} = \{0\}.$$

Proof. Let v be in $V_\sigma^{\gamma,[\mu]}$. By an easy computation, we have

$$\pi_\gamma(m_{\alpha_j})v = \sqrt{-1}\mu(H_{[(j+1)/2]})v.$$

On the other hand, we have

$$\pi_\gamma(m_{\alpha_j})v = \sigma_{\alpha_j}v.$$

Therefore, from (8.1) the element v must be zero. This proves the lemma.

LEMMA 8.2. *Let γ be in $\widehat{K}$, σ in $\widehat{M}$ and let j be an integer such that $1 \leq j \leq n-1$ and $j \equiv 1 \pmod 2$. If ν is in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \mathrm{Re}\, \nu, \alpha_j \rangle > 0$, then the operator $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$ has a nontrivial kernel if and only if*

(c1) *ν_j is an integer and $\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod 2$.*

(c2) *there exists a weight μ of V^γ such that*

$$|\sqrt{-1}\mu(H_{[(j+1)/2]})| \geq \nu_j + 1 \quad \text{and} \quad V_\sigma^{\gamma,[\mu]} \neq \{0\},$$

(c3) *there exists a weight μ' of V^γ such that*

$$|\sqrt{-1}\mu'(H_{[(j+1)/2]})| < \nu_j + 1 \quad \text{and} \quad V_\sigma^{\gamma,[\mu']} \neq \{0\},$$

where ν_j are as in §5.

Proof. Suppose that $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$ has the nontrivial kernel. By Lemma 5.4, the conditions (c2), (c3) are obvious and ν_j is an integral. Moreover, we have

$$(8.2) \qquad \nu_j + 1 + \sqrt{-1}\mu(H_{[(j+1)/2]}) \equiv 0 \pmod 2.$$

Therefore, by Lemma 8.1, we have

$$\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod 2.$$

Conversely, suppose that (c1), (c2) and (c3) are satisfied. Then from Lemma 8.1 and (c1), it follows that any weight μ of V^γ such that $V_\sigma^{\gamma,[\mu]} \neq \{0\}$ satisfies (8.2). Therefore, from Lemma 5.1, (c2) and (c3) it follows that $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$ has the nontrivial kernel.

COROLLARY 8.3. *Let γ be in $\widehat{K}$, σ in $\widehat{M}$ and let j be an integer such that $1 \leq j \leq n-1$. If ν is in $\mathfrak{a}_{\mathbb{C}}^*$, such that $\langle \mathrm{Re}\, \nu, \alpha_j \rangle > 0$ then the operator $B_\gamma^\sigma(P, s_{\alpha_j}, \nu)$ has the nontrivial kernel if and only if*

(c1) *ν_j is an integer and $\nu_j + 1 \equiv \sigma_{\alpha_j} \pmod 2$,*

(c2) *there exists a weight μ of V^γ such that*

$$|\sqrt{-1}\mu(H_{[(j+1)/2]})| \geq \nu_j + 1 \quad \text{and} \quad V_\sigma^{\gamma,[\mu]} \neq \{0\},$$

(c3) *there exists a weight μ' of V^γ such that*

$$|\sqrt{-1}\mu'(H_{(j+1/2)})| < \nu_j + 1 \quad \text{and} \quad V_\sigma^{\gamma,[\mu']} \neq \{0\},$$

where ν_j $(1 \leq j \leq n-1)$ are as in §5.

Proof. If the integer j is odd, then the assertion is that of Lemma 6.2. Thus we may assume that j is even. By Lemma 5.4, the operator $B_\gamma^\infty(P_p, s_{\alpha_j}, \nu)$ has the nontrivial kernel if and only if the operator $B_\gamma^{C_{j/2}\cdot\sigma}(P, s_{\alpha_{j-1}}, -(C_{j/2}\cdot\nu))$ does also. Since

$$\langle \mathrm{Re}(-(C_{j/2}\cdot\nu)), \alpha_j\rangle = \langle \mathrm{Re}\,\nu, \alpha_{j-1}\rangle > 0,$$

we can apply Lemma 8.2 to the operator $B_\gamma^{C_{j/2}\cdot\sigma}(P, s_{\alpha_{j-1}}, -(C_{j/2}\cdot\nu))$.

We note that

$$(8.3) \qquad (C_{j/2}\cdot\sigma)_{\alpha_j} = \sigma_{\alpha_{j-1}} \quad \text{and} \quad (-(C_{j/2}\cdot\nu))_j = \nu_{j-1}.$$

Combining Lemma 8.2 and the relations (8.3) we have the assertion of the corollary.

LEMMA 8.4. *Let ν be in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \mathrm{Re}\,\nu, \alpha\rangle > 0$ for all P-positive roots α and σ in $\widehat{M}$. Then $A(\overline{P} : P : \sigma : \nu)$ has the nontrivial kernel if and only if there exists a reduced P-positive $\mathfrak{a}$-root β satisfying the following conditions*:

(*) $2\langle\nu, \beta\rangle\cdot\langle\beta, \beta\rangle^{-1}$ *is an integer and* $2\langle\nu, \beta\rangle\cdot\langle\beta, \beta\rangle^{-1} + 1 \equiv \sigma_\beta$, $(\mathrm{mod}\,2)$.

Proof. Let w be in M' such that

$$w^{-1}Pw = \overline{P} \quad \text{and} \quad w = w_r w_{r-1}\cdots w_1,$$

where each w_i $(1 \le i \le r)$ is the reflection with respect to the P-simple $\mathfrak{a}$-root α_{k_i} $(1 \le k_i \ne n-1)$ and r is the length of w. Let P_i $(1 \le i \le r)$ be the minimal string P to $\overline{P}$, which is described in Proposition 4.3. From Lemma 4.1 it follows that $A(\overline{P} : P : \sigma : \nu)$ has the nontrivial kernel if and only if

(c1) there exists γ in $\widehat{K}$ such that $B_\gamma^\sigma(\overline{P} : P : \nu)$ has the nontrivial kernel.

Moreover, the condition (c1) is equivalent to

(c2) there exist γ in $\widehat{K}$ and an integer j $(1 \le j \le r)$ such that $B_\gamma^{w_{j-1}\cdots w_1\sigma}(P, w_j, w_{j-1}\cdots w_1\nu)$ has the nontrivial kernel.

Since we have

$$\langle w_{j-1}\cdots w_1\nu, \alpha_j\rangle = \langle\nu, \beta_j\rangle > 0,$$

from Corollary 6.3 the condition (c2) is equivalent to

(c3) there exist γ in $\widehat{K}$, weights of $V^\gamma\mu$, μ' and an integer j $(1 \le j \le r)$ satisfying the following relations:

$$(8.4)\qquad \begin{gathered} 2\cdot\langle \nu , \beta_j\rangle\cdot\langle\beta_j , \beta_j\rangle^{-1}\in\mathbb{Z}, \\ V_\sigma^{\gamma,[\mu]}\neq\{0\}, \qquad V_\sigma^{\gamma,[\mu']}\neq\{0\}, \\ 2\cdot\langle \nu , \beta_j\rangle\cdot\langle\beta_j , \beta_j\rangle^{-1}+1\equiv\sigma_{\alpha_{k_j}} \pmod 2, \\ |\sqrt{-1}\mu(H_{k_j})|\geq 1+2\cdot\langle \nu , \beta_j\rangle\cdot\langle\beta_j , \beta_j\rangle^{-1}, \\ |\sqrt{-1}\mu'(H_{k_j})|< 1+2\cdot\langle \nu , \beta_j\rangle\cdot\langle\beta_j , \beta_j\rangle^{-1}. \end{gathered}$$

From Proposition 8.5, the condition (c3) is equivalent to

(c3′) there exists an integer j $(1\leq j\leq r)$ such that $2\cdot\langle \nu , \beta_j\rangle\cdot\langle\beta_j , \beta_j\rangle^{-1}$ is an integer and satisfies the relation (8.4).

Since $\beta_j=\alpha_{k_j}$, the assertion of the lemma follows from the condition (c3′).

PROPOSITION 8.5. *Let σ be in $\widehat{M}$ and k an integer such that $1\leq k\leq n-1$ and $k\equiv 1 \pmod 2$. Then for any positive integer l which satisfies* (6.1), *there exists γ in $\widehat{K}$ such that*

$$V_\sigma^{\gamma,[\overline{\mu}]}\neq\{0\}\quad\textit{and}\quad \overline{\mu}(H_{[(k+1)/2]})=l,$$

where $\overline{\mu}$ is the highest weight of V^γ.

Proof. Let γ be an element in $\widehat{K}$ such that the highest weight of V^γ is $\overline{\mu}$. We put

$$n_j=\sqrt{-1}\overline{\mu}(H_{[(j+1)/2]})\qquad (1\leq j\leq n-1\,, j\equiv 1 \pmod 2).$$

Then each n_j is an integer. By the representation theory of compact groups, we can choose γ in $\widehat{K}$ satisfying the following conditions;

$$\begin{gathered} n_k=n, \\ n_j\neq 0\quad\text{and}\quad n_j-\sigma_j\equiv 0\qquad (1\leq j\leq n-1,\ j\equiv 1 \pmod 2). \end{gathered}$$

Let $v_{\overline{\mu}}$ be a $\overline{\mu}$-weight vector. We shall prove that $P_\sigma(v_{\overline{\mu}})\neq 0$. We can easily see that

$$P_\sigma(v_{\overline{\mu}})=\prod_{\substack{1\leq i\leq n-1\\ i\equiv 0 \pmod 2}}\frac{1}{2}(I+\sigma_{\alpha_i}\cdot\pi_\gamma(m_{\alpha_i}))(v_{\overline{\mu}}),$$

where I is the identity operator on V^γ. On the other hand, for integers i, j such that $1\leq i, j\leq n-1$, $1\equiv 0 \pmod 2$ and $j\equiv 1 \pmod 2$ we have

$$\sqrt{-1}m_{\alpha_i}\cdot\overline{\mu}(H_{[(j+1)/2]})=\begin{cases} -n_j & (i\leq 1\leq j\leq i+1), \\ n_j & \text{otherwise}. \end{cases}$$

Therefore, $P_\sigma(v_{\overline{\mu}}) \neq 0$. This proves the assertion of the lemma.

THEOREM 8.7. *Let ν be an element in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \operatorname{Re}\nu, \alpha\rangle \neq 0$ for all P-positive roots α and σ in $\widehat{M}$. Then $\pi_{P,\sigma,\nu}$ is reducible if and only if there exists a reduced P-positive $\mathfrak{a}$-root β satisfying the following conditions*:

(*) $2\langle \nu, \beta\rangle \cdot \langle \beta, \beta\rangle^{-1}$ *is an integer and* $2\langle \nu, \beta\rangle \cdot \langle \beta, \beta\rangle^{-1} + 1 \equiv \sigma_\beta \pmod 2$.

Proof. Suppose that $\langle \operatorname{Re}\nu, \alpha\rangle > 0$ for all P-positive $\mathfrak{a}$-roots α. Then by Lemma 3.5 $\pi_{P,\sigma,\nu}$ is reducible if and only if $A(\overline{P} : P : \sigma : \nu)$ has the nontrivial kernel. Thus in this case, the assertion of the theorem follows from Lemma 8.4. In general, there exists w in $W(\mathfrak{a})$ such that $\langle \operatorname{Re} w\nu, \alpha\rangle > 0$ for all P-positive $\mathfrak{a}$-roots. Since $\pi_{P,\sigma,\nu}$ and $\pi_{P,w\sigma,w\nu}$ have equivalent composition series, $\pi_{P,\sigma,\nu}$ is reducible if and only if there exists a reduced P-positive $\mathfrak{a}$-root β such that $w\beta$ satisfies the condition (*). Since the inner product $\langle \cdot, \cdot\rangle$ is $W(\mathfrak{a})$-invariant and $\sigma_{w\beta} = \sigma_\beta$, Theorem 8.6 is proved.

9. The reducibility of $\pi_{P,\sigma,\nu}$ in the singular cases. Let ν_0 be in $\mathfrak{a}_{\mathbb{C}}^*$ such that $\langle \operatorname{Re}\nu_0, \alpha\rangle \geq 0$ for all P-positive $\mathfrak{a}$-roots. Set

$$\Delta_{\nu_0}^+(P) = \{i \in \mathbb{N} \mid 1 \leq i \leq n-1 \text{ and } \langle \operatorname{Re}\nu_0, \alpha_i\rangle \neq 0\}.$$

Then we have

$$\operatorname{Re}\nu_0 = \sum_{j \in \Delta_{\nu_0}^+(P)} b_j \omega_j,$$

where b_j $(j \in \Delta_{\nu_0}^+(P))$ are positive real numbers and ω_j $(1 \leq j \leq n-1)$ in $\mathfrak{a}_{\mathbb{C}}^*$ are defined by

$$\langle \alpha_i, \omega_j\rangle = \delta_{ij} \qquad (1 \leq i, j \leq n-1).$$

We take

$$\mathfrak{a}_1 = \sum_{j \in \Delta_{\nu_0}^+(P)} \mathbb{R}\cdot H_{\omega_j}, \qquad \mathfrak{a}_2 = \sum_{j \in \Delta_{\nu_0}^+(P)} \mathbb{R}\cdot H_{\alpha_j},$$

$$\mathfrak{n}_1 = \sum_{\substack{\beta \in \Sigma^+ \\ \beta|_{\mathfrak{a}} \neq 0}} \mathfrak{g}_\beta, \qquad \mathfrak{n}_2 = \sum_{\substack{\beta \in \Sigma^+ \\ \beta|_{\mathfrak{a}} = 0}} \mathfrak{g}_\beta,$$

$$\mathfrak{m}_1 = \mathfrak{m} \oplus \mathfrak{a}_2 \oplus \mathfrak{n}_2 \oplus \mathfrak{v}_2, \qquad M_1 = Z_K(\mathfrak{a})(M_1)_0,$$

$$P_1 = M_1 A_1 N_1, \qquad P_2 = M A_2 N_2,$$

where Σ^+ is the set of P-positive $\mathfrak{a}$-roots. Then P_1 is a parabolic subgroup of G and P_2 is a minimal parabolic subgroup of M_1. Let us write $\nu_0 = \nu_0^1 + \nu_0^2$ correspondingly, with $\nu_0^1 = \nu_0|_{\mathfrak{a}_1}$ and $\nu_0^2 = \nu_0|_{\mathfrak{a}_2}$. From the double induction formula (see **[8]**, p. 170), $\mathrm{ind}_P^G\, \sigma \otimes \nu_0 \otimes 1$ and $\mathrm{ind}_{P_1}^G\ (\mathrm{ind}_{P_2}^{M_1}\, \sigma \otimes \nu_0^2 \otimes 1) \otimes \nu_0^1 \otimes 1$ are infinitesimally equivalent. $\mathrm{ind}_{P_2}^{M_1}\, \sigma \otimes \nu_0^2 \otimes 1$ is a tempered unitary representation of M_1 and we denote it by ξ.

Set $P' = MA\overline{N}_2 N_1$ and let w', w'' be elements in $W(\mathfrak{a})$ such that

$$(w')^{-1} P w' = P', \qquad (w'')^{-1} P' w'' = \overline{P},$$

respectively. Suppose that $w' = w_s' \cdot w_{s-1}' \cdots w_1'$ and $w'' = w_t'' \cdot w_{t-1}'' \cdots w_1''$ are the minimal expressions, respectively. Let $w = w'' \cdot w'$. Then we have

$$w^{-1} P w = \overline{P}.$$

By Lemma 3.4, the length of w is equal to $r + s$ and

$$w = w_t'' \cdot w_{t-1}'' \cdots w_1'' \cdot w_s' \cdot w_{s-1}' \cdots w_1'$$

is the minimal expression. Let P_i $(1 \le i \le s+t)$ be the minimal string P to $\overline{P}$ with associated reduced P-positive $\mathfrak{a}$-roots $\{\beta_i\}$, which are described in Proposition 4.3.

LEMMA 9.1. *Let β_i $(1 \le i \le s+t)$ be defined as above. We have*

$$\mathfrak{n}_2 = \sum_{\substack{1 \le i \le s \\ c > 0}} \mathfrak{g}_{c\beta_i}.$$

Therefore, we have

$$\langle \mathrm{Re}\, \nu_0, \beta_i \rangle = 0 \qquad (1 \le i \le s), \tag{9.2}$$

$$\langle \mathrm{Re}\, \nu_0, \beta_j \rangle = 0 \qquad (s+1 \le j \le s+t). \tag{9.3}$$

Since the proof is easy, it is left to the reader.

For σ in $\widehat{M}$ and γ in $\widehat{K}$, we set

$$F_{\sigma,\gamma,\nu_0} = \{ i \in \mathbb{N} \mid 1 \le i \le s \text{ and } B_\gamma^{w_{i-1}' \cdots w_1' \sigma}(P, w_i', w_{i-1}' \cdots w_1' \nu) \text{ has a singularity at } \nu_0 \}.$$

LEMMA 9.2. *Set $F_{\sigma,\nu_0} = F_{\sigma,\gamma,\nu_0}$. Then we have*

$$F_{\sigma,\nu_0} = F_{\sigma,\gamma,\nu_0}.$$

Proof. The assertion of the lemma follows from Lemma 6.2 and Lemma 8.1.

LEMMA 9.3. *Let* ν *be in* $\mathfrak{a}_{\mathbb{C}}^*$, σ *in* $\widehat{M}$ *and* γ *in* $\widehat{K}$. *Then the function*

$$\prod_{i\in F_{\sigma,\nu_0}} -\langle\nu, \beta_i\rangle^2 B_\gamma^\sigma(\overline{P}:P:\nu)B_\gamma^\sigma((\overline{P'}):\overline{P}:\nu)$$

has no singularity at ν_0.

Proof. For any u in W, we define $\pi_\gamma^\sigma(u)$ by $\pi_\gamma(u)|_{V_\gamma^\sigma}$. By the relation (4.3), we have

$$\begin{aligned}
&B_\gamma^\sigma(\overline{P}:P:\nu)\\
&= B_\gamma^\sigma(P, w_1', \nu)\rho_\gamma^{w_1'\sigma}(w_1')\\
&\qquad\cdots B_\gamma^{w_{s-1}'\cdots w_1'\sigma}(P, w_s', w_{s-1}'\cdots w_1'\nu)\pi_\gamma^{w'\sigma}(w_s')\\
&\quad\cdot B_\gamma^{w'\sigma}(P, w_1'', w'\nu)\pi_\gamma^{w_1''w'\sigma}(w_1'')\\
&\qquad\cdots B_\gamma^{w_{t-1}''\cdots w_1''w'\sigma}(P, w_t'', w_{t-1}''\cdots w_1''w'\nu)\\
&\quad\cdot \pi_\gamma^{w\sigma}(w_t'')\pi_\gamma^\sigma(w),\\
&= B_\gamma^\sigma(P, w', \nu)\pi_\gamma^{w'\sigma}(w')B_\gamma^{w'\sigma}(P, w_1'', w'\nu)\pi_\gamma^{w_1''w'\sigma}(w_1'')\\
&\qquad\cdots B_\gamma^{w_{t-1}''\cdots w_1''w'\sigma}(P, w_t'', w_{t-1}''\cdots w_1''w'\nu)\pi_\gamma^{w\sigma}(w_t'')\pi_\gamma(w).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&B_\gamma^\sigma(\overline{P}:P:\nu)\\
&= B_\gamma^\sigma(P, w, \nu)\pi_\gamma^{w'\sigma}(w')B_\gamma^{w'}(P, w_1'', w'\nu)\pi_\gamma^{w_1''w'\sigma}(w_1'')\\
&\qquad\cdots B_\gamma^{w_{t-1}''\cdots w_1''w'\sigma}(P, w_t'', w_{t-1}''\cdots w_1''w'\nu)\pi_\gamma^{w\sigma}(w_t'')\pi_\gamma(w)\\
&\quad\cdot B_\gamma^\sigma((\overline{P'}):\overline{P}:\nu).
\end{aligned}$$

From Lemma 6.2 and Lemma 9.1, the functions

$$\begin{aligned}
&B_\gamma^{w_1''w'\sigma}(P, w_1'', w'\nu)\\
&\qquad\cdots B_\gamma^{w_{t-1}''\cdots w_1''w'\sigma}(P, w_t'', w_{t-1}''\cdots w_1''w'\nu)\pi_\gamma^{w\sigma}(w_t'')\pi_\gamma(w)
\end{aligned}$$

and

$$\prod_{i\in F_{\sigma,\nu_0}} \langle\nu, \beta_i\rangle B_\gamma^\sigma(P, w', \nu)$$

have no singularity at ν_0. On the other hand, we have

$$\begin{aligned} B_\gamma^\sigma((\overline{P'}) : P' : \nu) &= B_\gamma(\overline{P}, w', \nu) \\ &= B_\gamma^\sigma(\overline{P}, w_1', \nu)\pi_\gamma^\sigma(w_1') \cdots B_\gamma^{w_{s-1}' \cdots w_1' \sigma}(\overline{P}, w_s', w_{s-1}' \cdots w_1'^{\nu}) \\ &\quad \cdot \pi_\gamma^{w' \sigma}(w_s')\pi_\gamma^\sigma(w') \end{aligned}$$

by Lemma 5.2,

$$\begin{aligned} (9.6) &= B_\gamma^\sigma(\overline{P}, w_1', -\nu)\pi_\gamma^\sigma(w_1') \cdots B_\gamma^{w_{s-1}' \cdots w_1'^{\sigma}}(\overline{P}, w_s', -w_{s-1}' \cdots w_1'\nu) \\ &\quad \cdot \pi_\gamma^{w' \sigma}(w_s')\pi_\gamma^\sigma(w') \\ &= B_\gamma^\sigma(P, w', -\nu). \end{aligned}$$

Then the function $\prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle B_\gamma^\sigma((\overline{P'}) : \overline{P} : \nu)$ also has no singularity at ν_0. Therefore, from the relation (9.5), the function

$$\prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 B_\gamma^\sigma(\overline{P} : P : \nu) B_\gamma^\sigma((\overline{P'}) : \overline{P} : \nu)$$

has no singularity at ν_0.

Corollary 9.4. *Let ν be in $\mathfrak{a}_\mathbb{C}^*$ and σ in $\widehat{M}$. Then the operator*

$$\prod_{i \in F_{\sigma, \nu_0}} -\langle \nu, \beta_i \rangle^2 A((\overline{P'}) : \overline{P} : \sigma : \nu) A(\overline{P} : P : \sigma : \nu)$$

has no singularity at ν_0.

Lemma 9.5. *Let ν be in $\mathfrak{a}_\mathbb{C}^*$ and σ in $\widehat{M}$. Then the kernel of the operator*

$$\lim_{\nu \to \nu_0} \prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle A((\overline{P'}) : \overline{P} : \sigma : \nu)$$

is equal to $\{0\}$.

Proof. It is enough to show that for any γ in $\widehat{K}$, the kernel of the operator

$$\lim_{\nu \to \nu_0} \prod_{i \in F_{\sigma, \nu_0}} \langle \nu, \beta_i \rangle^2 B_\gamma^\sigma((\overline{P'}) : \overline{P} : \nu)$$

is equal to $\{0\}$. The assertion of the lemma follows from Lemma 6.2 and (9.6).

THEOREM 9.6. *Let ν be in $\mathfrak{a}_{\mathbb{C}}^*$, σ in $\widehat{M}$. Then we have*

$$\operatorname{Im}\left(\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}\langle -\nu\,,\,\beta_i\rangle A(\overline{P}:P:\sigma:\nu)\right)\simeq \operatorname{Im}(A(\overline{P}_1:P_1:\xi:\nu_0^1))\,,$$

(*infinitesimally equivalent*).

Proof. We have

$$\begin{aligned}&\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle A((\overline{P'}):\overline{P}:\sigma:\nu)\lim_{\nu'\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu'\,,\,\beta_i\rangle\\ &\quad\cdot A(\overline{P}:P:\sigma:\nu')\\ &=\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle^2 A((\overline{P'}):\overline{P}:\sigma:\nu)A(\overline{P}:P:\sigma:\nu)\\ &=\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle^2\eta(\overline{P}:(\overline{P'}):\sigma:\nu)A((\overline{P'}):P:\sigma:\nu_0)\,.\end{aligned}$$

Thus, from Lemma 9.5 we have

$$\begin{aligned}(9.7)\quad &\operatorname{Im}\left(\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle^2 A(\overline{P}:P:\sigma:\nu)\right)\\ &\simeq\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle^2\eta(\overline{P}:(\overline{P'}):\sigma:\nu)A((\overline{P'}):P:\sigma:\nu_0)\,.\end{aligned}$$

Since we have for any γ in $\widehat{K}$

$$\eta(\overline{P}:(\overline{P'}):\sigma:\nu)=B_\gamma^\sigma(\overline{P}:(\overline{P'}):\nu)B_\gamma^\sigma((\overline{P'}):\overline{P}:\nu)$$

and

$$B_\gamma^\sigma(\overline{P}:(\overline{P'}):\nu)=B_\gamma^\sigma(P':P:\nu)\,,$$

we obtain

$$\eta(\overline{P}:(\overline{P'}):\sigma:\nu)=B_\gamma^\sigma(P\,,\,w'\,,\,\nu)B_\gamma^\sigma(\overline{P}'\,,\,w'\,,\,\nu)\,.$$

Thus by Lemma 5.2, we have

$$\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}}-\langle\nu\,,\,\beta_i\rangle^2\eta(\overline{P}:(\overline{P'}):\sigma:\nu)\neq 0\,,$$

and (9.7) is infinitesimally equivalent to $\operatorname{Im}(A(\overline{P'}):P:\sigma:\nu_0)$. From the double induction formula we have

$$\operatorname{Im}(A((\overline{P'}):P:\sigma:\nu))\simeq\operatorname{Im}(A(\overline{P}:P:\xi:\nu^1))\,.$$

Therefore, we have

$$\operatorname{Im}\left(\lim_{\nu\to\nu_0}\prod_{i\in F_{\sigma,\nu_0}} -\langle\nu\,,\,\beta_i\rangle A(\overline{P}:P:\sigma:\nu)\right)\simeq \operatorname{Im}(A(\overline{P}:P:\xi:\nu^1))\,.$$

THEOREM 9.7. *The representation π_{P,σ,ν_0} is reducible if and only if the tempered unitary representation ξ of M is reducible or there exists a P-positive reduced $\mathfrak{a}$-root β satisfying the following conditions*:

(*) $2\langle\nu\,,\,\beta\rangle\cdot\langle\beta\,,\,\beta\rangle^{-1}$ *is an integer and* $2\langle\nu\,,\,\beta\rangle\cdot\langle\beta\,,\,\beta\rangle^{-1}+1\equiv\sigma_\beta$ (mod 2),

(**) $\beta|_{\mathfrak{a}_1}\neq 0$.

Proof. According to Lemma 3.4, π_{P,σ,ν_0} is reducible if and only if $A(\overline{P}:P:\xi:\nu_0^1)$ has the nontrivial kernel or ξ is reducible. By Theorem 9.6 or the double induction formula, $A(\overline{P}:P:\xi:\nu_0^1)$ has the nontrivial kernel if and only if $A((\overline{P'}):P:\nu_0)$ does so. Thus by similar argument to that in §8, we can prove the assertion of the theorem.

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