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INJECTIVE HILBERT C^*-MODULES

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One difference between Hilbert modules and Hilbert spaces is that Hilbert modules are not "self-dual" in general. Another difference is that Hilbert modules are not orthogonally complementary. Let H be a Hilbert module over a C^*-algebra A. We show that if A is monotone complete then $H^\#$, the "dual" of H, can be made into a self-dual Hilbert A-module. We also show that if H is full and countably generated, then H is orthogonally complementary if and only if every bounded module map in H has an adjoint. It turns out that these results are closely related to the problem of extensions of bounded module maps. Let C_1 be the category whose objects are Hilbert A-modules and morphisms are contractive module maps with adjoints, and C_2 the category whose objects are Hilbert A-modules and morphisms are contractive module maps. We find that injective modules in the category whose objects are Hilbert A-modules and morphisms are contractive module maps. We find that injective modules in the category C_1 are precisely those that are orthogonally complementary. We show that Hilbert modules over a monotone complete C^*-algebra are injective in C_2 if and only if they are self-dual. We also show that if A is not an AW^*-algebra then A itself is not injective A-module in the category C_2. A few related results are also included.

1. Introduction and preliminaries. The general theory of Hilbert modules over a non-commutative C^*-algebra has been studied by many authors (e.g. **[10]**, **[12]**, **[13]**, **[16]**–**[24]**). Its applications vary from the theory of extensions of C^*-algebras and K-theory to non-commutative topology. One of the main differences between Hilbert modules and Hilbert spaces is that Hilbert modules are not "self-dual" in general. Another difference is that Hilbert modules are not orthogonally complementary. Let H be a Hilbert module over a C^*-algebra A and $H^\#$ the A-module of all bounded A-module maps from H into A. It is shown by W. Paschke **[21]** that if A is a W^*-algebra then $H^\#$ can be made into a self-dual Hilbert A-module containing H as a closed submodule. It is then natural to ask if it is true for other C^*-algebras. It turns out that this question is closely related to the following question: Let H_0 be a (closed) submodule of H and φ a bounded module map from H_0 into A. Is there a module map $\tilde{\varphi}$

from H into A such that $\tilde{\varphi}|_{H_0} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$? We show (in §3) that both questions have an affirmative answer for monotone complete C^*-algebras and a negative answer for those C^*-algebras which are not AW^*-algebras.

Suppose that H_1 and H_2 are two Hilbert A-modules and T is an invertible bounded module map from H_1 onto H_2. We find that H_1 may not be unitarily equivalent to H_2. (We also show that H_1 and H_2 are unitarily equivalent if both H_1 and H_2 are assumed to be countably generated.) However, if in addition we assume that T has an adjoint T^* (from H_2 to H_1) then H_1 is unitarily equivalent to H_2. It suggests that we may also consider the category whose objects are Hilbert A-modules and morphisms are contractive module maps with adjoints. We find that injective objects in this category are precisely those Hilbert A-modules which are orthogonally complementary. In particular, we show that A is injective in the category if and only if $LM(A) = M(A)$.

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Recall the definition of a Hilbert module over a C^*-algebra A ([**12**]).

Definition 1.1. Let E be a linear space over the complex field equipped with structure of a right A-module. We suppose that $\lambda(xa) = (\alpha x)a = x(\lambda a)$, where $x \in E$, $a \in A$ and λ is a complex number. The space E is called a pre-Hilbert A-module if there exists an inner product $\langle \cdot , \cdot \rangle : E \times E \to A$ satisfying the following conditions:

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$;
(2) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$;
(3) $\langle x, ya\rangle = \langle x, y\rangle a$;
(4) $\langle x, y\rangle^* = \langle y, x\rangle$, where $x, y, z \in E$, $a \in A$ and λ is a complex number.

Put $\|x\| = \|\langle x, x\rangle\|^{1/2}$. This is a norm on E. If E is complete, E is called a Hilbert module over A. The closure of the span of

$\{\langle x, y\rangle : x, y \in E\}$ is called the support of E, denoted $\langle E, E\rangle$. E is called full if $\langle E, E\rangle = A$.

DEFINITION 1.2. For a Hilbert A-module E, we let $E^\#$ denote the set of bounded A-module maps from E into A. For $x \in E$ we denote a module map $x^\wedge$ in $E^\#$ by $x^\wedge(y) = \langle x, y\rangle$ for $y \in E$. $E^\#$ becomes an A-module if we define $(\tau \cdot a)(x) = a^*\tau(x)$ for $\tau \in E^\#$, $x \in E$ and $a \in A$ or $a \in \mathbb{C}$, and add maps in $E^\#$ pointwise. We call E self-dual if every module map in $E^\#$ arises by taking A-valued inner products with some fixed x in E. (See [**21**]).

DEFINITION 1.3. Let A be a C^*-algebra. We denote by $M(A)$ the idealiser of A in A^{**}, where A^{**} is the enveloping von Neumann algebra of A. We also denote by $LM(A)$ the set $\{x \in A^{**} : xa \in A$ for all $a \in A\}$, by $RM(A)$ the set $\{x \in A^{**} : ax \in A$ for all $a \in A\}$ and by $QM(A)$ the set $\{x \in A^{**} : axb \in A$ for all $a, b \in A\}$.

DEFINITION 1.4. Let E be a Hilbert module over a C^*-algebra A. We denote by $B(E)$ the set of all bounded module maps from E into E and by $L(E)$ the set of all bounded module maps $T \in B(E)$ such that there exists $T^* : E \to E$ satisfying the condition: $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in E$. If $x, y \in E$, let $\theta_{x,y}$ be the module map defined by $\theta_{x,y}(z) = x\langle y, z\rangle$ for z in E. The map $\theta_{x,y}$ is in $L(E)$. The closure of the linear span of $\{\theta_{x,y} : x, y \in E\}$ in $L(E)$ is denoted by $K(E)$ (see [**12**]). We also denote by $B(E, E^\#)$ the set of bounded module maps from E into $E^\#$. With the operator norm, $B(E)$ is a Banach algebra, $L(E)$ and $K(E)$ are C^*-algebras and $B(E, E^\#)$ is a Banach space. (See [**12**] and [**18**].)

We would like to state the following theorems that are used often in this paper

THEOREM 1.5 (Kasparov [**12**, Theorem 1] and Green [**31**, Lemma 16].). *There is an isometric isomorphism ϕ_1 and $L(E)$ onto $M(K(E))$.*

THEOREM 1.6 ([**18**, 1.4]). *There is an isometric isomorphism ϕ_2 from Banach algebra $B(E)$ onto $LM(K(E))$ which is an extension of ϕ_1.*

THEOREM 1.7 ([**18**, 1.5]). *There is an isometric isomorphism ϕ_3 from Banach space $B(E, E^\#)$ onto $QM(K(E))$ which is an extension of ϕ_2.*

DEFINITION 1.8. Let E be a Hilbert A-module and E_1 be the extension of E by A^{**} constructed in [**21**, 4]. We denote by $E^\sim$ the self-dual Hilbert A^{**}-module $E_1^{\#}$ (see [**21**, 4]. Every bounded module map in $B(E, E^{\#})$ can be uniquely extended to a bounded module map in $B(E^{\sim})$. (This easily follows from the construction of $E^{\sim}$ and [**21**, 3.6]. See also [**18**, 1.3].) If E is self-dual, then $B(E) = L(E)$. (See [**21**, 3.5].) Thus $M(K(E)) = LM(K(E)) = QM(K(E))$. If in addition, A is a W^*-algebra, $B(E)$ is also a W^*-algebra. In particular, $B(E^{\sim})$ is a W^*-algebra. Since all maps in $B(E, E^{\#})$ can be uniquely extended to maps in $B(E^{\sim})$, $B(E^{\sim})$ is a W^*-algebra containing $K(E)$, $M(K(E))$, $LM(K(E))$ and $QM(K(E))$.

REMARK 1.9. Finally, throughout this paper, (a) K always denotes the C^*-algebra of all compact operators on a infinite dimensional, separable Hilbert space; (b) if p is an open projection in A^{**} for some C^*-algebra A, $\mathrm{Her}(p)$ denotes the hereditary C^*-algebra corresponding to p; (c) $\overline{p}$ denotes the smallest closed projection in A^{**} majorizing p.

2. Hilbert modules with orthogonal complements.

DEFINITION 2.1. Let H_1 and H_2 be two Hilbert modules over a C^*-algebra A. We denote by $B(H_1, H_2)$ the set of all bounded module maps from H_1 into H_2. We say that H_1 and H_2 are unitarily equivalent or H_1 is H-isomorphic to H_2 and write $H_1 \cong H_2$ if there is a unitary module map U which maps H_1 onto H_2 so that

$$\langle x, y\rangle = \langle Ux, Uy\rangle \quad \text{for all } x, y \in H_1.$$

It is natural to ask whether H_1 is unitarily equivalent to H_2 if there is an invertible map $T \in B(H_1, H_2)$.

THEOREM 2.2 (cf. [**6**, 3.2]). *Let H_1 and H_2 be two countably generated Hilbert modules over a C^*-algebra A. Suppose that there is T in $B(H_1, H_2)$ which is one-to-one and has dense range. Then H_1 and H_2 are unitarily equivalent.*

Proof. By [**20**, 1.5], both $K(H_1)$ and $K(H_2)$ are σ-unital. Suppose that K and L are strictly positive elements in $K(H_1)$ and $K(H_2)$, respectively. Set $H = H_1 \oplus H_2$. We define $\widetilde{T}, \widetilde{K}, \widetilde{L}$ in $B(H)$ as follows: $\widetilde{T}(h_1 \oplus h_2) = 0 \oplus Th_1$, $\widetilde{K}(h_1 \oplus h_2) = Kh_1 \oplus 0$, $\widetilde{L}(h_1 \oplus h_2) = 0 \oplus Lh_2$, where $h_1 \in H_1$, $h_2 \in H_2$. Clearly, $\widetilde{K}, \widetilde{L} \in K(H)$. Then by 1.6, $S = \widetilde{L}\widetilde{T}\widetilde{K} \in K(H)$. Let $S = U|S|$ be the polar decomposition (in $B(H^{\sim})$). We note that S is one-to-one implies that $|S|$ is one-to-one, which implies that $|S|$ is strictly positive in $K(H_1)$. Thus $|S|H_1$

is dense in H_1. Therefore $UH_1 = H_2$ and $\langle x, y\rangle = \langle Ux, Uy\rangle$ if $x, y \in H_1$. This completes the proof.

EXAMPLE 2.3. Now we present a C^*-algebra A and a Hilbert A-module E such that there is an invertible map $\varphi \in B(E, A)$, but E is not unitarily equivalent to A. The example is borrowed from L. G. Brown [**6**, 6.1]. Let $\pi: B(H) \to B(H)/K(H) = Q$ be the quotient map, where H is an infinite dimensional and separable Hilbert space. Let $B \subset Q$ be C^*-subalgebras such that $B \cdot C = 0$ and does not contain $s \in Q$ with $Bs = (1-s)C = 0$ (see [**6**, 6.1] and [**9**]). Let $A = \{[a_{ij}] \in B(H) \otimes M_2: \pi(a_{11}) \in B,\ \pi(a_{22}) \in C,\ a_{12}, a_{21} \in K(H)\}$. $T = [t_{ij}]$ is a quasi-multiplier of A if and only if $A\pi(t_{11})A \subset A$, $B\pi(t_{22})B \subset B$ and $A\pi(t_{12})B = B\pi(t_{21})A = 0$. In particular, any scalar matrix is a quasi-multiplier. Set $T = \left[\begin{smallmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{smallmatrix}\right]$,where ε is a small positive scalar. So T is an invertible positive quasimultiplier. L. G. Brown [**6**, 6.1] showed that $T \notin \mathrm{Span}(RM(A), LM(A))$.

Now set $E = \{T^{1/2}a: a \in A\}$. Then E is a right A-module. We define $\langle T^{1/2}a, T^{1/2}b\rangle = a^*Tb$. Then E becomes a Hilbert A-module. There is an one-to-one and bounded module map φ from A onto E defined by $\varphi(a) = T^{1/2}a$. However, A and H are not unitarily equivalent. In fact, if there is a unitary module map U from H onto A, then $U(T^{1/2}e_\alpha)$ converges left strictly to an element s in $LM(A)$, where $\{e_\alpha\}$ is an approximate identity for A. Then

$$a^*s^*sb = \langle U(T^{1/2}a), U(T^{1/2}b)\rangle = a^*Tb$$

for all $a, b \in A$. Therefore $T = S^*S$. This contradicts the fact that $T \notin \mathrm{Span}(RM(A), LM(A))$.

LEMMA 2.4. *Let H be a Hilbert module and $T \in L(H)$. If T has a closed range, then*

$$H = \mathrm{Ker}\, T \oplus |T|H.$$

In particular T has a polar decomposition $T = V|T|$ in $L(H)$.

Proof. Let $T = V|T|$ be the polar decomposition in $B(H^\sim)$. Since TH is closed and V is a partial isometry, $|T|H$ is closed. Notice that $|T| \in L(H)$. Clearly, since $|T|H$ is closed,

$$|T|H = |T|^{1/2}|T|^{1/2}H \subset |T|^{1/2}H \subset |T|H.$$

So $|T|^{1/2}H = |T|H$. Set $B = \{S \in L(H): S|T|H \subset |T|H\}$. So $|T|^{1/2} \in B$. It is obvious that $|T|^{1/2}$ is also one-to-one on $|T|H$. Therefore $|T|^{1/2}$ is invertible in B. Hence either $0 \notin \mathrm{Sp}(|T|^{1/2})$ or zero is an isolated point in $\mathrm{Sp}(|T|^{1/2})$. Let p be the range projection of $|T|$ in $L(H)^{**}$. Then $|T|^{1/n} \to p$ in norm. So $p \in L(H)$. Clearly $pH = |T|H$ and $(1-p)H = \operatorname{Ker} T$, whence V is a bounded module map in $L(H)$.

DEFINITION 2.5. Let H_1 and H_2 be two Hilbert modules and $T \in B(H_1, H_2)$. Define T_1 in $B(H_1 \oplus H_2)$ by

$$T_1(h_1 \oplus h_2) = 0 \oplus Th_1 \quad \text{for } h_1 \in H_1 \text{ and } h_2 \in H_2.$$

We denote by $L(H_1, H_2)$ the set of those $T \in B(H_1, H_2)$ such that $T_1 \in L(H_1 \oplus H_2)$.

PROPOSITION 2.6. *Let H_1 and H_2 be two Hilbert modules. If there is an invertible map $T \in L(H_1, H_2)$ then $H_1 \cong H_2$.*

Proof. It is an immediate consequence of 2.4. In fact, the partial isometry V in the polar decomposition of T lies in $L(H_1, H_2)$.

PROPOSITION 2.7. *Let H_1 and H_2 be two Hilbert modules such that $L(H_1) = B(H_1)$. If there is an invertible map $T \in B(H_1, H_2)$ then $H_1 \cong H_2$.*

Proof. We notice that the adjoint T^* of T always exists, but T^* maps H_2 into $H_1^\#$. Therefore $T^*T \in B(H_1, H_1^\#)$. Since $L(H_1) = B(H_1)$, by 1.5 and 1.6, $M(K(H_1)) = LM(K(H_1))$. It follows from **[6**, 41.8] that $QM(K(H_1)) = M(K(H_1))$. Thus, by 1.7, $B(H_1, H_1^\#) = L(H_1)$. So $T^*T \in L(H_1)$, whence $|T| \in L(H_1)$. Then the argument in 2.4 applies.

DEFINITION 2.8. Let H be a Hilbert module. We say H is orthogonally complementary if any Hilbert module H_1 containing H has an orthogonal decomposition:

$$H_1 = H \oplus H^\perp.$$

Clearly, not all Hilbert modules are orthogonally complementary. It is shown in **[10]** that if A is unital, then any orthogonal direct summand of A^n, the direct sum of n copies of A, is orthogonally complementary.

It is certainly desirable to know which Hilbert modules are orthogonally complementary.

THEOREM 2.9. *Let E be a full Hilbert module over a C^*-algebra A such that $L(E) = B(E)$. Then E is orthogonally complementary. Moreover, if E' is another Hilbert A-module such that there is an invertible map $T \in B(E, E')$, then E' is also orthogonally complementary.*

Proof. By 2.7, we need only to show the first part of the theorem.

Suppose that H is a Hilbert A-module and $E \subset H$. Let P be the bounded module map from H into $E^{\#}$ defined by

$$Px(y) = \langle x, y \rangle \quad \text{for } x \in H, \ y \in E.$$

Fix $x \in H$ and $y \in E$, define

$$T(z) = y[Px(z)] = y\langle x, z\rangle$$

for $z \in E$. Working in $B(E^{\sim})$ if necessary, we see that

$$T^*(z) = Px\langle y, z\rangle \quad \text{for } z \in E.$$

Since $T \in B(E) = L(E)$, $T^* \in L(E)$. Therefore $Px\langle y, y\rangle \in E$ for all $y \in E$. Let $x = u\langle x, x\rangle^{1/2}$ be the polar decomposition of x in $H^{\sim}$. (See [**19**, 3.11].) Then, for $z \in E$,

$$\langle Px\langle y, y\rangle z\rangle = \langle y, y\rangle\langle x, x\rangle^{1/2}\langle u, z\rangle.$$

With $\|z\| \le 1$, we have

$$\begin{aligned} &\|\langle px, z\rangle - \langle px\langle y, y\rangle, z\rangle\| \\ &\quad \le \|(1 - \langle y, y\rangle)\langle x, x\rangle^{1/2}\| \, \|\langle u, z\rangle\| \\ &\quad \le \|(1 - \langle y, y\rangle)\langle x, x\rangle^{1/2}\|. \end{aligned}$$

Since E is full and $Px\langle y, y\rangle \in E$ for all $y \in E$, we conclude from the above inequalities that $Px \in E$ for all $x \in H$. Therefore $P \in B(H)$ and $H = (1-p)H \oplus E$. This completes the proof.

EXAMPLE 2.10. The assumption that E is full in 2.6 cannot be removed. Let H be an infinite dimensional Hilbert space. Then $K(H)$ is a Hilbert $B(H)$-module, where $\langle x, y\rangle = x^*y$ for all $x, y \in K(H)$. Then $L(K(H)) = B(K(H))$. However, it is clear that $K(H)$ is not an orthogonal direct summand of $B(H)$. If we regard $K(H)$ as $K(H)$-module, then $K(H)$ is an orthogonal direct summand of any Hilbert $K(H)$-module containing it. The point is that if E is a Hilbert A-module and $\langle E, E\rangle = I$, an ideal of A, we should regard E as an I-module.

One may compare the following corollary to Proposition 1 in [**10**]. The condition $LM(A) = M(A)$ is actually necessary (see 2.15).

COROLLARY 2.11. *Let A be a C^*-algebra such that $LM(A) = M(A)$. Then orthogonal direct summands of A^n are orthogonally complementary, where n is a positive integer.*

Proof. By 2.9, A is an orthogonally complementary Hilbert A-module. Consequently, A^n is orthogonally complementary. Now we suppose that E is an orthogonal direct summand of A^n, for some positive integer, and H is a Hilbert A-module such that $E \subset H$. We have $A^n = E \oplus E_1$. So $H \oplus E_1 \supset A^n$. Therefore

$$H \oplus E_1 = E_2 \oplus E \oplus E_1 \quad \text{and} \quad H = E_2 \oplus E.$$

This completes the proof.

DEFINITION 2.12. Let H_0 be a (closed) submodule of a Hilbert module H over a C^*-algebra A, and H_1 is another Hilbert A-module. Suppose that there is a bounded module map $T: H_0 \to H_1$. Does there exist a module map $\widetilde{T}: H \to H_1$ such that $\widetilde{T}|_{H_0} = T$ and $\|\widetilde{T}\| = \|T\|$? Fix a C^*-algebra A. We denote by C_1 the category whose objects are Hilbert A-modules and morphisms are contractive module maps with adjoints (i.e. those module maps with norms no more than 1 in $L(H_1, H_2)$, for some Hilbert A-modules H_1 and H_2). Theorem 2.14 shows that the injective Hilbert modules in C_1 are precisely those Hilbert modules with orthogonal complements.

LEMMA 2.13. *Let H be a Hilbert module over a C^*-algebra A and H_0 a closed submodule of H. Suppose that $T \in K(H_0)$; then there is $\widetilde{T} \in K(H)$ such that $\|\widetilde{T}\| = \|T\|$ and $\widetilde{T}|_{H_0} = T$. Consequently, $K(H_0)$ may be regarded as a hereditary C^*-subalgebra of $K(H)$.*

Proof. Let $x_i, y_i \in H_0$, $i = 1, 2, \ldots, n$. Clearly $\sum_{i=1}^n \theta_{x_i, y_i}$ extends to a map in $K(H)$. We first show that

$$\left\| \sum_{i=1}^n \theta_{x_i, y_i} \Big|_{H_0} \right\| = \left\| \sum_{i=1}^n \theta_{x_i, y_i} \right\|.$$

Suppose that $\| \sum_{i=1}^n \theta_{x_i, y_i} \| = 1$. Then

$$\left\| \left(\sum_{i=1}^n \theta_{x_i, y_i} \right) \left(\sum_{i=1}^n \theta_{y_i, x_i} \right) \right\| = 1.$$

For any $\varepsilon > 0$, there is $\xi \in H$ with $\|\xi\| = 1$ such that

$$\left\| \left(\sum_{i=1}^{n} \theta_{x_i, y_i} \right) \left(\sum_{i=1}^{n} \theta_{y_i, x_i} \right) (\xi) \right\| > 1 - \varepsilon.$$

But $\|(\sum_{i=1}^{n} \theta_{y_i, x_i})(\xi)\| \leq 1$ and $(\sum_{i=1}^{n} \theta_{y_i, x_i})(\xi) \in H_0$. So

$$\left\| \sum_{i=1}^{n} \theta_{x_i, y_i} \Big|_{H_0} \right\| = \left\| \sum_{i=1}^{n} \theta_{x_i, y_i} \right\|.$$

Now we assume that $T \in K(H_0)$. Then there are $\{x_i^{(m)}\}$, $\{y_i^{(m)}\} \subset H_0$ such that

$$\left\| \sum_{i} \theta_{x_i^{(m)}, y_i^{(m)}} - T \right\| \to 0.$$

By the first part of the proof, $\sum_i \theta_{x_i^{(m)}, y_i^{(m)}}$ is also norm convergent as elements in $K(H)$. Let $\widetilde{T}$ be the limit. So $\widetilde{T} \in K(H)$ and $\|\widetilde{T}\| = \|T\|$. Moreover, it is easy to see that $\widetilde{T}|_{H_0} = T$. Set

$$B = \{S \in K(H) : SH_0 \subset H_0\}.$$

Clearly B is a hereditary C^*-subalgebra of $K(H)$. We have just proved that $B \cong K(H_0)$.

THEOREM 2.14. *A Hilbert A-module H is injective in the category C_1 if and only if H is orthogonally complementary.*

Proof. We first assume that H is orthogonally complementary. Let H_0 be a closed submodule of a Hilbert A-module H_1 and T a bounded module map in $L(H_0, H)$. Set $H_2 = H_0 \oplus H$ and define

$$T_\lambda(h_0 \oplus h) = 0 \oplus T(h_0) + \lambda h \quad \text{for } h_0 \in H_0,\ h \in H,$$

where $0 < \lambda \leq 1$. Clearly $T_\lambda \in L(H_2)$ and

$$\|T_\lambda\| \leq (\|T\|^2 + \lambda^2)^{1/2}.$$

Moreover, T_λ is surjective. It follows from 2.4 that

$$H_2 = \operatorname{Ker} T_\lambda \oplus |T_\lambda| H_2.$$

Furthermore, T_λ is one-to-one on $|T_\lambda| H_2$ and maps $|T_\lambda| H_2$ onto $0 \oplus H$. By 2.5, $|T_\lambda| H_2 \cong H$. So $|T_\lambda| H_2$ is orthogonally complementary. Set $H_3 = H_1 \oplus H$; then

$$H_3 \supset H_2 \supset |T_\lambda| H_2.$$

Therefore, we may write

$$H_3 = H_4 \oplus |T_\lambda| H_2$$

for some closed submodule H_4. We define $\widetilde{T}_\lambda$ in $L(H_3)$ by

$$\widetilde{T}_\lambda(h_4 \oplus h) = T_\lambda h \quad \text{for } h_4 \in H_4 \text{ and } h \in |T_\lambda| H_2.$$

Clearly $\widetilde{T}_\lambda|_{H_2} = T_\lambda$ and $\|\widetilde{T}_\lambda\| = \|T_\lambda\|$. By 1.5, we have $\widetilde{T}_\lambda \in M(K(H_3))$. It follows from 2.13 that $K(H_2)$ is a hereditary C^*-subalgebra of $K(H_3)$. Let p be the open projection in $K(H_3)^{**}$ corresponding to $K(H_2)$. If $h \in H_2^\perp = \{h \in H_3 : \langle h, x\rangle = 0 \text{ for } x \in H_2\}$, then $\widetilde{T}_\lambda h = 0$. Therefore $\widetilde{T}_\lambda(1 - \overline{p}) = 0$. For any $k \in K(H_3)$,

$$k\widetilde{T}_\lambda(1 - \overline{p}) = 0$$

since $\widetilde{T}_\lambda \in M(K(H_3))$, $k\widetilde{T}_\lambda \in K(H_3)$. Thus

$$k\widetilde{T}_\lambda(\overline{1 - \overline{p}}) = 0, \quad \text{i.e. } k\widetilde{T}_\lambda(1 - p) = 0$$

for all $k \in K(H_3)$. Therefore $\widetilde{T}_\lambda(1 - p) = 0$.

For any $K \in K(H_2)$, $h \in H_2$, $Kh \in H_2$ and

$$\|(\widetilde{T}_\lambda - \widetilde{T}_{\lambda'})Kh\| \le |\lambda - \lambda'| \, \|Kh\|.$$

Therefore

$$\|(\widetilde{T}_\lambda - \widetilde{T}_{\lambda'})K\| \le |\lambda - \lambda'| \, \|K\|$$

for any $K \in K(H_2)$. Thus

$$\|(\widetilde{T}_\lambda - \widetilde{T}_{\lambda'})p\| \le |\lambda - \lambda'|.$$

Since $\widetilde{T}_\lambda(1 - p) = 0$, we obtain that

$$\|\widetilde{T}_\lambda - \widetilde{T}_{\lambda'}\| \le |\lambda - \lambda'|.$$

Set $\widetilde{T} = \lim_{\lambda\to 0} \widetilde{T}_\lambda$. So $\widetilde{T} \in L(H_3)$ and $\|\widetilde{T}\| = \lim_{\lambda \to 0} \|\widetilde{T}_\lambda\| = \|T\|$. Since $\widetilde{T}_\lambda|_{H_0} = T$ (if we identify H with $0 \oplus H$). We conclude that $\widetilde{T}|_{H_0} = T$ and $\|\widetilde{T}|_{H_1}\| = \|T\|$. This shows that H is injective in the category C_1.

For the converse, we assume that H is injective in the category C_1. Suppose that E is a Hilbert A-module containing H as a closed submodule. Let $i: H \to H$ be the identity map. Since H is injective in C_1 there is $\tilde{i} \in L(E, H)$ such that $\tilde{i}|_H = i$ and $\|\tilde{i}\| = \|i\|$. It is then easily checked that $(\tilde{i})^*(\tilde{i})$ is a projection in $L(E)$ and $(\tilde{i})^*(\tilde{i})|_H = i$. This implies that H is an orthogonal direct summand of E. This completes the proof.

THEOREM 2.15. *Let A be a σ-unital C^*-algebra. Then the following are equivalent*:

(1) $LM(A) = M(A)$;

(2) *A is orthogonally complementary as a Hilbert A-module*;

(3) *A is injective as a Hilbert A-module in the category C_1*;

(4) *For any closed right ideal R of A and $T \in L(R, A)$ there is $\widetilde{T} \in M(A)$ such that $\widetilde{T}|_R = T$ and $\|\widetilde{T}\| = \|T\|$.*

Proof. (1) $\Rightarrow$ (2) follows from 2.9. (2) $\Leftrightarrow$ (3) follows from 2.14 and (3) $\Rightarrow$ (4) is trivial.

It remains to show that (4) implies (1). Suppose that $S \in RM(A)$ and set

$$R = \{r \in A : sr \in A\}.$$

Then R is a closed right ideal of A. Let p be the open projection corresponding to R.

Case (I): $\overline{p} = 1$. For $r \in R$ define

$$Tr = Sr.$$

Since $S \in RM(A)$, $S^* \in LM(A)$. So $T \in L(R, A)$. Therefore there is $\widetilde{T} \in M(A)$ such that $\widetilde{T}|_R = T$ and $\|\widetilde{T}\| = \|T\|$. For any $k \in \mathrm{Her}(p)$ and $a \in A$,

$$\begin{aligned} k[(\widetilde{T})^* - S^*]a &= [(\widetilde{T}k^*)^* - kS^*]a \\ &= [(Tk^*)^* - kS^*]a = [(Sk^*)^* - kS^*]a = 0. \end{aligned}$$

Therefore, for any $a \in A$

$$\|p[(\widetilde{T})^* - S^*]a\| = 0$$

since p is dense and $[(\widetilde{T})^* - S^*]a \in A$, $((\widetilde{T})^* - S^*)a = 0$. So $(\widetilde{T})^* = S^*$, whence $S \in M(A)$.

Case (II): $\overline{p} \neq 1$ So $S \notin M(A)$. Let $q = 1 - \overline{p}$ and $B = \mathrm{Her}(q)$. Then, for any $b \in B$, $b \neq 0$, $Sb \notin A$. It is obvious that for any $b \in B$, $b^*S^*Sb \in B^{**}$. If B is of finite dimension, then $B^{**} = B$. So $b^*S^*Sb \in B \subset A$. Since $Sb \in QM(A)$, by [**5**, 2.63], $Sb \in M(A)$ for all $b \in B$. But then $Sb \in A$ for all $b \in B$. So we now assume that B is of infinite dimension. Take a sequence $\{b_n\}_{n=0}^{\infty} \subset B_+$ such that $b_0 b_n = b_n \neq 0$ for $n = 1, 2, \dots$ and $b_n b_m = 0$ if $n \neq m$. Let $\{e_n\}$ be an approximate identity for A satisfying

$$e_n e_m = e_m e_n = e_n \quad \text{if } m > n.$$

Since $b_nS^* \notin A$ for all n, by passing to a subsequence and changing notations, we may assume that

$$b_nS^*(e_{2n} - e_{2n-1}) \neq 0$$

for all n. Set

$$c_n = b_nS^*(e_{2n} - e_{2n-1})/\|b_nS^*(e_{2n} - e_{2n-1})\|,$$

$n = 1, 2, \ldots$. It is routine to check that $\{\|\sum_{n=1}^{k} c_n\|\}$ is bounded. It is then easy to check that $\sum_{n=1}^{k} c_n$ converges strictly to an element $c \in LM(A)$, as $k \to \infty$. Since $\|c_n\| = 1$ for each n, $c \notin A$. Let B_1 be the closure of $\bigcup_{n=1}^{\infty}(b_nAb_n)$. Then B_1 is a hereditary C^*-subalgebra of A. Let p_1 be the open projection corresponding to B_1 and $R_1 = pA^{**} \cap A$. For any $b \in B_1$ and $\varepsilon > 0$, there is n and k such that

$$\left\| b\left(\sum_{i=1}^{n}(b_i)^{1/k}\right) - b \right\| < \varepsilon.$$

Since $\sum_{i=1}^{n}(b_i)^{1/k}c = \sum_{i=1}^{n}(b_i)^{1/k}\sum_{i=1}^{n} c_i$, we conclude that $bc \in A$ for all $b \in B_1$. Hence $c^*r \in A$ for all $r \in R_1$. So $c^* \in L(R_1, A)$, since $c \in LM(A)$. Let $p_2 = p_1 + (1 - \overline{p}_1)$ and $R_2 = p_2A^{**} \cap A$. Define L in $L(R_2, A)$ by

$$Lr = c^*r \quad \text{for } r \in R_2.$$

By (4), there is $\widetilde{L} \in M(A)$ such that $\widetilde{L}|_{R_2} = L$ and $\|\widetilde{L}\| = \|L\|$. Since $\overline{p}_2 = 1$, an argument used in Case (I) shows that $c \in M(A)$. However, we know that $b_0c = c \notin A$. We reach a contradiction for Case (II). This completes the proof.

REMARK 2.16. It should be noted that for the implications $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$ we do not need to assume that A is σ-unital.

EXAMPLES 2.17. (a) Every unital C^*-algebra satisfies the conditions (1)–(4).

(b) Every commutative C^*-algebra satisfies the conditions (1)–(4).

(c) Let B be a C^*-algebra such that $LM(B) = M(B)$ and c_0 be the C^*-algebra of sequences of complex numbers which converge to zero. Then $c_0 \otimes B$ satisfies the conditions (1)–(4).

(d) Let B be a unital C^*-algebra and X a locally compact Hausdorff space. Then $C_0(X) \otimes B$ satisfies the conditions (1)–(4).

(e) We will show in 3.21 that every ideal of a monotone complete C^*-algebra satisfies the conditions (1)–(4).

(f) We will see that if $LM(B) = M(B)$, then $A = M_n(B)$, the C^*-algebra of $n \times n$ matrices over B, satisfies the conditions (1)–(4).

(g) The only stable C^*-algebra (C^*-algebras with the form $B \otimes K$) satisfying the conditions (1)–(4) are those dual C^*-algebras.

(h) The only σ-unital simple C^*-algebra satisfying the conditions (1)–(4) are those elementary ones (and unital ones). (See **[14]**.)

EXAMPLE 2.18. Let A be a σ-unital C^*-algebra such that $LM(A) \neq M(A)$. From 2.15 we know that there is a Hilbert A-module $H \supset A$ such that A is not an orthogonal direct summand of H. However, the proof of the implication (2) $\Rightarrow$ (1) in 2.15 depends on 2.14 and the implication (4) $\Rightarrow$ (1). It does not tell us how to construct such a Hilbert A-module H. The following is an example how one may construct such H. Take $A = c \otimes K$, the C^*-algebra of norm convergent sequences in K. An element x in A^{**} may be identified with a bounded collection $\{x_n : 1 \leq n \leq \infty,\ x_n \in B(l^2)\}$. Let S be in A^{**} given by $S_n = \theta_{e_n, e_1}$, $0 \leq n < \infty$ and $S_\infty = 0$, where $\{e_n\}$ is an orthonormal basis for l^2. One can check that $s \in RM(A)$. Let x be the element in A with $x_n = \theta_{e_1, e_1}$ for $1 \leq n \leq \infty$. Notice that $S_n^* S_n = \theta_{e_1, e_1}$ for $0 \leq n < \infty$ and $S_\infty^* S_\infty = 0$. If $a, b \in A$ such that $aS^*Sb \in A$. Then $a_n \theta_{e_1, e_1} b_n \to 0$ in norm as $n \to \infty$. So $aS^*Sb = axb$. Now set $E = \{a + Sb : a, b \in A\}$ and define

$$\langle a + sb, a' + sb' \rangle = a^*a' + a^*sb' + b^*sa' + b^*xb'$$

for $a, b, a', b' \in A$. It is now clear that with this inner product E is a pre-Hilbert A-module containing A. Let H be the completion of E. Clearly, A is not an orthogonal direct summand of H.

THEOREM 2.19. *Let H be a countably generated Hilbert A-module. If H is orthogonally complementary or equivalently, H is injective in the category C_1, then $L(H) = B(H)$.*

Proof. It follows from **[20**, 1.5] that $k(H)$ is σ-unital. By 1.5, 1.6 and 2.15, it suffices to show that $K(H)$ satisfies the condition (4) in 2.15. Let R be a closed right ideal of $K(H)$ and $T \in L(R, K(H))$. Let p be the open projection in $K(H)^{**}$ corresponding to R and $B = \mathrm{Her}(p)$. Set

$$H_{00} = \{bh : b \in B,\ h \in H\}.$$

Let H_0 be the closure of H_{00}. It follows from 2.13 that $B = K(H_0)$. For any $x \in H_0$ define

$$T(x) = \lim_{n \to \infty} (T\theta_{x,x})(x)[\langle x, x \rangle + \tfrac{1}{n}]^{-1}.$$

Exactly as in **[18]**, one shows that T defines a module map from H_0 into H with the same norm. Since $T \in L(R, K(H))$,

$T^* \in LM(K(H))$ (by 1.6). By 1.6, this implies $T \in L(H_0, H)$. Since H is injective in the category C_1, there is $\widetilde{T} \in L(H)$ such that $\widetilde{T}|_{H_0} = T$ and $\|\widetilde{T}\| = \|T\|$. By 1.6, $\widetilde{T} \in M(K(H))$. Clearly, since $\widetilde{T}|_{H_0} = T$, for any $K \in K(H)$, $\widetilde{T}K = TK$. So $K(H)$ does satisfy the condition (4). This completes the proof.

REMARK 2.20. One may notice that the converse of 2.19 is true if H is full, without the assumption that H is countably generated.

COROLLARY 2.21. *Let B be a σ-unital C^*-algebra with the property that $LM(B) = M(B)$ and $A = M_n(B)$, the C^*-algebra of $n \times n$ matrices over B. Then $LM(A) = M(A)$.*

Proof. Let $H = B^n$, then 2.21 follows immediately from 2.19.

3. Extensions of bounded module maps. Let H be a Hilbert module over a C^*-algebra A. In general, the A-module $H^\#$ is not equal to H, (see 1.2). In **[21]**, W. Paschke shows that if A is a W^*-algebra, the A-valued inner product $\langle \cdot, \cdot \rangle$ extends to $H^\# \times H^\#$ in such a way as to make $H^\#$ into a self-dual Hilbert A-module. It is certainly desirable to know if it is also true for other C^*-algebras. It turns out that the problem is closely related to the following extension problem: Let H_0 be a (closed) submodule of a Hilbert A-module H and φ a bounded module map from H_0 into A. Does there exist a module map $\widetilde{\varphi}$ from H into A such that $\widetilde{\varphi}|_{H_0} = \varphi$ and $\|\widetilde{\varphi}\| = \|\varphi\|$?

DEFINITON 3.1. Let A be a C^*-algebra. We denote by C_2 the category whose objects are Hilbert A-modules and morphisms are contractive module maps. The extension problem mentioned above is equivalent to ask if A is injective in the category C_2. We say a Hilbert A-module H is C_2-injective if it is injective in the category C_2. In particular, if A is C_2-injective as an A-module, we say A is a C_2-injective C^*-algebra.

PROPOSITION 3.2. *Closed ideals and unital hereditary C^*-subalgebras of a C_2-injective C^*-algebra are C_2-injective.*

Proof. Let A be a C_2-injective C^*-algebra and B a hereditary C^*-subalgebra of A. Suppose that H is a Hilbert B-module, H_0 a (closed) B-submodule of H and φ a bounded B-module map from H_0 into B. Consider the algebraic tensor product $H \otimes \widetilde{A}$, which becomes a right A-module when we set $(x \otimes b) \cdot a = x \otimes ba$ for $x \in H$, $a \in A$ and $b \in \widetilde{A}$. Define $[\cdot, \cdot]: H \otimes \widetilde{A} \times H \otimes \widetilde{A} \to A$ by

$$\left[\sum_{j=1}^{n} x_j \otimes a_j, \sum_{i=1}^{m} y_i \otimes b_i\right] = \sum_{i,j} a_j^* \langle x_j, y_i \rangle b_i.$$

Let $N = \{z \in H \otimes \widetilde{A}: [z, z] = 0\}$. By **[21**, 5.1], $E_0 = H \otimes \widetilde{A}/N$ is a pre-Hilbert A-module and H (by identifying with $H \times 1 + N$) is a closed B-submodule of E_0. Denote by E the completion of E_0. So E is a Hilbert A-module. Let E_1 be the closed A-submodule of E generated by H_0. It is clear that φ extends an A-module map φ_1 from E_1 into A. For any $x \in E_1$, we may write $x = ya$ where $y \in H_0$ $(= H_0 \otimes 1 + N)$. Notice that

$$\varphi(y)^* \varphi(y) \leq \|\varphi\|^2 \langle y, y \rangle \qquad \text{see [21, 2.8 (ii)].}$$

We have

$$\begin{aligned} \|\varphi_1(x)\|^2 &= \|\varphi(y)a\|^2 = \|a^* \varphi(y)^* \varphi(y) a\| \\ &\leq \|\varphi\|^2 \|a^* \langle y, y \rangle a\| = \|\varphi\|^2 \|xa\|^2. \end{aligned}$$

So $\|\varphi_1\| = \|\varphi\|$. Since A is C_2-injective, there is $\widetilde{\varphi}_1 \in E^\#$ such that $\|\varphi_1\| = \|\varphi_1\|$ and $\widetilde{\varphi}_1|_{E_1} = \varphi_1$.

For any $x \in H$, let $x = u\langle x, x \rangle^{1/2}$ be the polar decomposition of x in $H^\sim$. Then $z = u\langle x, x \rangle^{1/4} \in H$. We have

$$\tilde{\varphi}_1(x) = \tilde{\varphi}_1(z) \langle x, x \rangle^{1/4}.$$

If B is an ideal, $\tilde{\varphi}_1(z)\langle x, x \rangle^{1/4} \in B$, since $\langle x, x \rangle^{1/4} \in B$. Thus $\tilde{\varphi}_1|_H$ is a B-module map from H into B such that $\tilde{\varphi}_1|_{H_0} = \varphi$ and $\|\tilde{\varphi}_1|_H\| = \|\varphi_1\| = \|\varphi\|$.

If B is a unital hereditary C^*-subalgebra of A, set $\psi = e\tilde{\varphi}_1$, where e is the unit of B. Then for $x \in H$

$$\psi(x) = e\tilde{\varphi}_1(z)\langle x, x \rangle^{1/4} \in B.$$

Clearly $e\varphi = \varphi$. So ψ extends φ and $\|\psi\| = \|\varphi\|$. This completes the proof.

THEOREM 3.3. *Every self-dual Hilbert module over a C_2-injective C^*-algebra is C_2-injective.*

Proof. Let H be a self-dual Hilbert module over a C_2-injective C^*-algebra A. Suppose that H_1 is a Hilbert A-module, H_0 a (closed) submodule of H_1 and T a bounded module map from H_0 into H. For fixed $x \in H$, define $\varphi_x \in H_0^\#$ by

$$\varphi_x(h) = \langle x, Th \rangle \quad \text{for } h \in H_0.$$

Since A is C_2-injective, there is $\tilde{\varphi}_x \in H_1^\#$ with $\|\tilde{\varphi}_x\| = \|\varphi_x\|$ such that $\tilde{\varphi}_x(h) = \varphi_x(h)$ for all $h \in H_0$. Define a map $\widetilde{T}: H_1 \to H^\#$ $(= H)$ by

$$\widetilde{T}h(x) = [\tilde{\varphi}_x(h)]^* \quad \text{for } x \in H, \ h \in H_1.$$

Clearly $\widetilde{T}$ is a module map, $\widetilde{T}h = Th$ if $h \in H_0$ and

$$\|\widetilde{T}h(x)\| \le \|\tilde{\varphi}_x\| \, \|h\| = \|\varphi_x\| \, \|h\| \le \|T\| \, \|x\| \, \|h\|$$

for $x \in H$ and $h \in H_1$. So $\|\widetilde{T}\| = \|T\|$. This completes the proof.

REMARK 3.4. It should be noted that if A is not C_2-injective then any Hilbert A-module containing A as a submodule is not C_2-injective. Proposition 3.11 gives a partial converse of 3.3.

LEMMA 3.5. *Let H and E be two Hilbert modules over a C^*-algebra A, and T a bounded module map from H into E. If there is a bounded module extension $\widetilde{T}$ of T from $H^\#$ into $E^\#$, then $\widetilde{T}$ is unique.*

Proof. Suppose that L is a bounded module map from $H^\#$ into $E^\#$ such that $L|_H = T$. Set $F = H \oplus E$ and define $\widetilde{T}_1$ and L_1 in $B(F^\#)$ by

$$\widetilde{T}_1(h \oplus e) = 0 \oplus \widetilde{T}h \quad \text{and}$$
$$L_1(h \oplus e) = 0 \oplus Lh \quad \text{for } h \in H^\# \text{ and } e \in E^\#.$$

By **[21, 4]**, $F^\sim \cong M(F, A^{**})$, where $M(F, A^{**})$ is the set of all bounded A-module maps from F into A^{**}. It is then clear $F^\# \subset F^\sim$. Let F_0 be the closed Hilbert A^{**}-submodule (of $\widetilde{F}$) generated by $F^\#$. So both $\widetilde{T}_1$ and L_1 can be extended to maps in $B(F_0)$. Since $\widetilde{F}$ is self-dual W^*-module, by 3.4, $F^\sim$ is C_2-injective A^{**}-module. Therefore both $\widetilde{T}_1$ and L_1 can be further extended to module maps in $B(F^\sim)$. However, by 1.8, $\widetilde{T}_1|_F$ has only one extension in $B(F^\sim)$. This implies that $\widetilde{T}_1|_{F_0} = L_1|_{F_0}$. So $\widetilde{T}$ is unique.

DEFINITION 3.6. Let A be a monotone complete C^*-algebra. Then A is always unital. If $\{x_\lambda\}$ is a bounded, monotone increasing net in $A_{s.a.}$, then it has a least upper bound x in $A_{s.a.}$. We write $x_\lambda \nearrow x$

to describe this relation. For any net $\{x_\lambda\}$ in A, R. V. Kadison and G. K. Pedersen in [11] write $x_\lambda \to x$ if there are four increasing nets $\{x_\lambda^{(k)}\}$ in $A_{s.a.}$, $k = 0, 1, 2, 3$, such that (with $i = \sqrt{(-1)}$)

$$x_\lambda^{(k)} \nearrow x^{(k)}, \quad \sum_{k=0}^{3} i^k x_\lambda^{(k)} = x_\lambda \quad \text{and} \quad \sum_{k=0}^{3} i^k x^{(k)} = x.$$

This Kadison-Pedersen arrow "$\to$" plays an important role in the following lemma.

LEMMA 3.7. *Let H be a Hilbert module over a monotone complete C^*-algebra A. Then the A-valued inner product $\langle\cdot, \cdot\rangle$ extends to $H^\# \times H^\#$ in such a way as to make $H^\#$ into a self-dual Hilbert A-module, $\langle\tau, x\rangle = \tau(x)$ and*

$$\|\langle\tau, \tau\rangle\|^{1/2} = \sup\{\|\tau(x)\|: \|x\| = 1, \ x \in H\}$$

for $\tau \in H^\#$ and $x \in H$.

Proof. Let $\varphi \in H^\#$. Set $H_1 = H \oplus A$ and define $\varphi_1: H_1 \to H_1$ by

$$\varphi_1(h \oplus a) = 0 \oplus \varphi_1(h) \quad \text{for } h \in H \text{ and } a \in A.$$

So $\|\varphi\| = \|\varphi\|$ and $\varphi_1 \in B(H_1)$. By 1.6, $\varphi_1 \in LM(K(H_1))$. Let $\{U_\lambda\}$ be an approximate identity for $k(H_1)$, $e = 0 \oplus 1$ and $p = \theta_{e,e}$. Then $\varphi_1 U_\lambda \in K(H_1)$ and $p\varphi_1 U_\lambda = \varphi_1 U_\lambda$ for each λ. Thus, there is $K_\lambda \in K(H_1)$ such that $\varphi_1 U_\lambda = pK_\lambda$, whence $\varphi_1 U_\lambda = \theta_{e, K_\lambda^* e}$ for each λ. Therefore $\varphi_1 U_\lambda(\varphi_1 U_\lambda)^* \in pK(H_1)p$ $(\cong A)$ and $\varphi_1 U_\lambda(\varphi_1 U_\lambda)^*$ is a bounded increasing net in $pK(H_1)p$. We identify $pK(H_1)p$ with A and denote by $\langle\varphi, \varphi\rangle$ the least upper bound of $\varphi_1 U_\lambda(\varphi_1 U_\lambda)^*$ in A. If $\tau \in H^\#$, then

$$(\varphi_1 U_\lambda)(\tau_1 U_\lambda)^* = \frac{1}{4}\sum_{k=0}^{3} i^k(\tau_1 U_\lambda - i^k \varphi_1 U_\lambda)(\tau_1 U_\lambda - i^k \varphi_1 U_\lambda)^*.$$

Therefore $(\varphi_1 U_\lambda)(\tau_1 U_\lambda)^* \to \langle\varphi, \tau\rangle$ for some $\langle\varphi, \tau\rangle$ in A with the Kadison-Pedersen arrow. Notice that if $\varphi \in H^\#$, $a \in A$, $(\varphi \cdot a)(z) = \alpha^*\varphi(z)$ and if $\tau, \psi \in H^\#$,

$$(\varphi_1 U_\lambda)[(\tau_1 + \psi_1)U_\lambda]^* = (\varphi_1 U_\lambda)(\tau_1 U_\lambda)^* + (\varphi_1 U_\lambda)(\psi_1 U_\lambda)^*.$$

By [11, 2.1], we have

$$\langle\varphi \cdot \alpha, \tau\beta\rangle = \alpha^*\langle\varphi, \tau\rangle\beta \quad \text{and}$$
$$\langle\varphi_1 \tau + \psi\rangle = \langle\varphi, \tau\rangle + \langle\varphi, \psi\rangle$$

where $\alpha, \beta \in A$, $\varphi, \tau, \psi \in H^{\#}$. Since

$$[(\varphi_1 U_\lambda)(\tau_1 U_\lambda)^*]^* = (\tau_1 U_\lambda)(\varphi_1 U_\lambda)^*,$$

we also have $\langle \varphi, \tau \rangle^* = \langle \tau, \varphi \rangle$. Moreover, $\langle \varphi, \varphi \rangle \geq 0$ and $\langle \varphi, \varphi \rangle = 0$ if and only if $\varphi = 0$. Thus we have defined an A-valued inner product on $H^{\#}$ such that $H^{\#}$ becomes a pre-Hilbert A-module. If $x, y \in H$, the $(x^\wedge)_1 = \theta_{e,x}$ and $(y^\wedge)_1 = \theta_{e,y}$. So $[(x^\wedge)_1 U_\lambda][(y^\wedge)_1 U_\lambda]^* = \theta_{e, \langle U_\lambda^2 x, y \rangle e}$. By identifying $pK(H_1)p$ with A, we have

$$[(x^\wedge)_1 U_\lambda][(y^\wedge)_1 U_\lambda]^* = \langle U_\lambda^2 x, y \rangle$$

so $[(x^\wedge)_1 U_\lambda][(y^\wedge)_1 U_\lambda]^*$ converges to $\langle x, y \rangle$ in norm. It follows from [**11**, Lemma 2.2] that $[(x^\wedge)_1 U_\lambda][(y^\wedge)_1 U_\lambda]^* \to \langle x, y \rangle$ with the Kadison-Pedersen arrow.

If $\tau \in H^{\#}$, $x \in H$, then we have

$$[\tau_1 U_\lambda][(x^\wedge)_1 U_\lambda]^* = \tau(U_\lambda^2 x),$$

by identifying $pK(H_1)p$ with A. So $\langle \tau, x \rangle = \tau(x)$. Since $\varphi_1 U_\lambda(\varphi_1 U_\lambda)^* \leq \|\varphi_1\|^2 p$, $\|\langle \varphi, \varphi \rangle\| \leq \|\varphi\|^2$. By Cauchy-Schwarz inequality for A-valued inner products, we conclude that

$$\|\langle \tau, \tau \rangle\|^{1/2} = \sup\{\|\tau(x)\| : \|x\| = 1,\ x \in H\}.$$

Since every self-dual pre-Hilbert module is complete (see [**21**, 3]), it remains to show that $H^{\#}$ with newly defined inner product is self-dual. Suppose that ψ is a bounded module map from $H^{\#}$ into A. Therefore there is $\varphi \in H^{\#}$ such that $\psi(x) = \varphi(x)$ for all $x \in H$. By 3.5, $\varphi = \psi$. This completes the proof.

THEOREM 3.8. *Let A be a monotone complete C^*-algebra. Suppose that H is a Hilbert A-module, H_0 a (closed) submodule of H and φ a bounded module map from H_0 into A. Then there is a module map $\tilde{\varphi}: H \to A$ such that $\|\overline{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi}(h) = \varphi(y)$ for all $h \in H_0$.*

Proof. By Lemma 3.8 for any $\tau \in H_0^{\#}$ define

$$\varphi'(\tau) = \langle \varphi, \tau \rangle.$$

By Lemma 3.7, φ' is a module map from $H_0^{\#}$ into A and $\|\varphi'\| = \|\varphi\|$. Let P be the module map from H into $H_0^{\#}$ defined by

$$Px(h) = \langle x, h \rangle \quad \text{for } x \in H,\ h \in H_0.$$

Set $\tilde{\varphi} = \varphi' \circ P$. It is easy to verify that $\|\tilde{\varphi}\| = \|\varphi'\| = \|\varphi\|$ and $\tilde{\varphi}$ extends φ as desired.

COROLLARY 3.9. *Every closed ideal of a monotone complete C^*-algebra is C_2-injective.*

PROPOSITION 3.10. *Let H be a Hilbert module over a monotone C^*-algebra A. Then H is C_2-injective if and only if H is self-dual.*

Proof. By 3.3 and 3.9 we only need to show the "only if" part. Let H be a C_2-injective Hilbert A-module. It follows from 3.7 that H is a submodule of $H^\#$. Let $i: H \to H$ be the identity map. Then there is $i^\#: H^\# \to H$ such that $\|i^\#\| = 1$ and $i^\#(h) = h$ for $h \in H$. Let $\bar{i}$ be the identity map from $H^\#$ into itself. Then $i^\# - i|_H = 0$. It follows from Lemma 3.5 that $i^\# = \bar{i}$. But this is impossible, since $i^\#(H^\#) \subset H$, unless $H = H^\#$. This completes the proof.

DEFINITION 3.11. Let A be a C^*-algebra. We denote by C_3 the category whose objects are closed right ideals and morphisms are contractive A-module maps. We say that A is C_3-injective if it is injective in the category C_3, i.e. for any closed right ideal R of A and $\varphi \in R^\#$, there is $\tilde{\varphi} \in LM(A)$ such that $\tilde{\varphi}(r) = \varphi(r)$ for all $r \in R$ and $\|\tilde{\varphi}\| = \|\varphi\|$. Clearly, every C_2-injective C^*-algebra is C_3-injective.

DEFINITION 3.12. Let A be a C^*-algebra, p an open projection in A^{**}. Let $R_p = A \cap pA^{**}$; then R is a closed right ideal of A. So R_p is a Hilbert A-module. Let $S \in R_p^\#$ and $\{e_\alpha\}$ be an approximate identity for $\mathrm{Her}(p)$. Then for any $r \in R$,

$$S(r) = \lim_\alpha S(e_\alpha \cdot r).$$

Suppose that S_1 is a weak limit of $\{S(e_\alpha)\}$ in A^{**}. Then

$$S(r) = S_1 r \quad \text{for all } r \in R_p.$$

We see that S_1 is uniquely determined. We denote by $LM(R_p, A)$ the set of elements S in $A^{**}p$ such that $Sr \in A$. It can be shown (as in [25, 3.2.3]) that there is a linear isometry from $R_p^\#$ onto $LM(R_p, A)$. We will identify these two sets.

PROPOSITION 3.1.3. *Every closed ideal or unital hereditary C^*-subalgebra of a C_3-injective C^*-algebra is C_3-injective.*

Proof. Let A be a C_3-injective C^*-algebra and B a hereditary C^*-algebra of A. Suppose that R is a closed right ideal of B and $S \in LM(R, B)$. Let R_1 be the closure of $R \cdot A$. Then R_1 is a closed right ideal of A. Clearly $S \in LM(R_1, A)$. Therefore there is

$\overline{S} \in LM(A)$ such that $\overline{S}r = Sr$ for $r \in R_1$ and $\|\overline{S}\| = \|S\|$. For any $x \in B$, by [25, 1.4.5], $x = ua$, for some $u, a \in B$. So $\overline{S}x = (\overline{S}u)a$. If B is an ideal, $\overline{S}x \in B$ for all $x \in B$. Let $S_1 = \overline{S}_p$, where p is the open projection corresponding to B, then $S_1 \in LM(B)$ and $S_1 r = Sr$ for all $r \in R$ and $\|S_1\| = \|S\|$. If B has a unit e, we can take $S_1 = e\overline{S}e$. This completes the proof.

Recall that a projection p in A^{**} is called regular (Tomita [28], see [1, II.12] and [26, 19] also) if $\|xp\| = \|x\overline{p}\|$ for every x in A. A projection p in A^{**} is called dense if $\overline{p} = 1$.

THEOREM 3.14. *Let A be a unital C_3-injective C^*-algebra. Then*
(a) *every open projection in A^{**} is regular;*
(b) *for every open projection p in A^{**}, $\overline{p} \in A$.*
(c) *A is an AW^*-algebra.*

Proof. We first show that every dense open projection q in A^{**} is regular. Put $R = qA^{**} \cap A$. So R is a closed right ideal of A, whence a (closed) submodule of A. For any $x \in A$, define a map $\varphi \in R^{\#}$ by

$$\varphi(r) = xqr = xr \quad \text{for } r \in R.$$

Since A is C_3-injective, there is $\tilde{\varphi} \in A^{\#}$ $(= A)$ which extends φ and $\|\tilde{\varphi}\| = \|\varphi\|$. Therefore there is $y \in A$ such that

$$(y - x)r = 0 \quad \text{for all } x \in R$$

and $\|y\| = \|\varphi\|$. Hence $(y - x)q = 0$. Since q is dense, $y = x$. In other words, $\tilde{\varphi}$ is unique. Thus

$$\|x\| = \|\tilde{\varphi}\| = \|\varphi\| = \|xq\|.$$

Therefore q is regular.

Now let p be any open projection in A^{**}. Put $q = p + (1 - \overline{p})$ and $R = qA^{**} \cap A$, $R_1 = pA^{**} \cap A$ and $R_2 = (1 - \overline{p})A^{**} \cap A$. R, R_1 and R_2 are closed right ideals of A, whence they are submodules of A. Moreover, we have $R = R_1 \oplus R_2$ (as an orthonormal direct sum of two Hilbert A-modules). Define a map $\psi \in R^{\#}$ by

$$\psi(r_1 \oplus r_2) = r_1 \quad \text{for all } r_1 \in R_1 \text{ and } r_2 \in R_2.$$

We have $\tilde{\psi} \in A^{\#}$ $(= A)$ such that $\tilde{\psi}|_R = \psi$ and $\|\tilde{\psi}\| = \|\psi\|$. Thus there is $e \in A$ such that $er_1 = r_1$ and $er_2 = 0$ for all $r_1 \in R_1$ and $r_2 \in R_2$. Let $B = \text{Her}(q)$, $B_1 = \text{Her}(p)$ and $B_2 = \text{Her}(1 - \overline{p})$. For any $a_1, b_1 \in B_1$ and $a_2, b_2 \in B_2$,

$$(a_1 + a_2)e(b_1 + b_2) = (a_1 + a_2)b_1 = a_1 b_1.$$

So

$$(a_1 + a_2)e^*(b_1 + b_2) = (a_1^* + a_2^*)^* e^*(b_1^* + b_2^*)^*$$
$$= [(b_1^* + b_2^*)e(a_1^* + a_2^*)]^* = [b_1^* a_1^*]^* = a_1 b_1.$$

Thus for any $a, b \in B$, $a(e - e^*)b = 0$. This implies

$$q(e - e^*)q = 0$$

since q is a regular dense open projection, $e = e^*$ (see [7, 4.1 (c)] for example). For any $b \in B$ with $b = b_1 + b_2$, where $b_1 \in B_1$, $b_2 \in B_2$, we have

$$e^2 b = e(e(b_1 + b_2)) = eb_1 = eb,$$

so $(e^2 - e)q = 0$. By the density of q, $e^2 = e$. Hence e is a projection in A. Since $e \geq p$, $e \geq \overline{p}$. But $e(1 - \overline{p}) = 0$, so $e = \overline{p}$. It follows from Proposition 3.14 that eAe is a C_2-injective. Since p is a dense open projection in $[eAe]^{**}$, from the first part of the proof, p is regular.

It remains to show that A is an AW^*-algebra. In fact, we have already shown it. If B_1 and B_2 are two orthogonal hereditary C^*-subalgebras and p_1 and p_2 are open projections corresponding to B_1 and B_2, respectively, then $p_1 p_2 = 0$. Since $\overline{p}_1 \in A$, $\overline{p}_1 p_2 = 0$. It follows from [**26**, 1] that A is an AW^*-algebra.

COROLLARY 3.15. *Every unital C_2-injective C^*-algebra is an AW^*-algebra.*

THEOREM 3.16. *Let A be a C_3-injective C^*-algebra. Then $M(A)$ is C_2-injective if and only if $M(A) = LM(A)$.*

Proof. Let p be an open projection in $M(A)^{**}$, $R_p = pM(A)^{**} \cap M(A)$ and $\mathrm{Her}(p) = pM(A)^{**} \cap M(A)$. Set $R_0 = R_p \cap A$ and $B_0 = \mathrm{Her}(p) \cap A$. Then R_0 is a closed ideal of A and B_0 is a hereditary C^*-subalgebra of A. Let p_0 be the open projection in A^{**} corresponding to R. Suppose that $x \in LM(R_p, M(A))$. Let y be the element in $LM(R_0, A)$ such that $yr = xr$ for $r \in R_0$. Clearly $\|y\| \leq \|x\|$. Since A is C_3-injective, there is $\overline{x} \in LM(A) = M(A)$ such that $\|\overline{x}\| = \|y\|$ and $\overline{x}r = xr$ for all $r \in R_0$. It is obvious that $p(1 - \overline{p}_0) = 0$. Put $q_0 = p_0 + (1 - \overline{p}_0)$. So q_0 is a dense open projection in A^{**}. For any $a \in \mathrm{Her}(q_0)$ (the hereditary C^*-subalgebra of A corresponding to q_0) and $b \in \mathrm{Her}(p)$,

$$\overline{x}ba = xba$$

since $ba \in \mathrm{Her}(q_0)$. Thus

$$\|(\overline{x}b - xb)q_0\| = 0 \quad \text{for } b \in \mathrm{Her}(p).$$

Let $\{e_\lambda\}$ be an approximate identity for A; then

$$\|e_\lambda(\overline{x} - x)bq_0\| = 0 \quad \text{for each } \lambda.$$

Since q_0 is a dense open projection in A^{**} and $e_\lambda(\overline{x} - x)b \in A$,

$$\|e_\lambda(\overline{x} - x)b\| = 0$$

for each λ. This implies $\overline{x}b = xb$ for all $b \in \mathrm{Her}(p)$. Therefore $\overline{x}r = xr$ for all $r \in R_p$. Moreover, $\|\overline{x}\| = \|x\|$.

For the converse, take $x \in LM(A)\backslash M(A)$. Since A is a closed ideal of $M(A)$, if $M(A)$ were C_3-injective, there would be a $\overline{x} \in M(A)$ such that $\|\overline{x}\| = \|x\|$ and $\overline{x}a = xa$ for all $a \in A$. This is impossible.

THEOREM 3.17. *Let A be a C^*-algebra. Consider the following conditions:*

(i) *A is C_3-injective;*
(ii) *For every hereditary C^*-subalgebra B of A and $x \in QM(B)$ there is $\overline{x} \in QM(A)$ such that $a\overline{x}b = axb$ and $\|\overline{x}\| = \|x\|$.*

Then

(a) *if every dense open projection in A^{**} is regular, then* (i) $\Rightarrow$ (ii).
(b) *if $LM(A) = M(A)$, then* (i) $\Leftrightarrow$ (ii).

Proof. (a) Let A be a C_3-injective C^*-algebra and p be an open projection in A^{**}. Set $B = \mathrm{Her}(p)$ and $q = p + (1 - \overline{p})$. Suppose that $x \in QM(B)$ and $\{e_\lambda\}$ is an approximate identity for B. Then for each λ, $e_\lambda \in LM(R_q, A)$. Thus there is $x_\lambda \in LM(A)$ such that $x_\lambda r = e_\lambda xr$ for all $r \in R_q$ and $\|x_\lambda\| = \|e_\lambda x\|$. For any $r \in R_1$, $b \in B$,

$$\|bx_\lambda r - bx_{\lambda'}r\| \le \|be_\lambda - be_{\lambda'}\| \, \|xr\|.$$

Thus

$$\|(bx_\lambda - bx_{\lambda'})q\| \le \|be_\lambda - be_{\lambda'}\| \, \|xq\|.$$

Since q is a dense open projection, by the assumption, q is regular. Since be_λ converges to b in norm, q is dense and regular and bx_λ, $bx_{\lambda'} \in QM(A)$, by [7, 4.3 (a)],

$$\|bx_\lambda - bx_{\lambda'}\| \to 0.$$

Suppose that x_∞ is a weak limit of $\{x_\lambda\}$ in A^{**}; then bx_λ converges to bx_∞ in norm and $\|x_\infty\| \le \|x\|$. For any $a \in A$, $bx_\lambda a$ converges to bx_∞ in norm. Because $bx_\lambda a \in A$ for each λ, $bx_\infty a \in A$. Let $\{u_\alpha\}$ be an approximate identity for A. Then

$$(x_\infty u_\alpha)^* \in LM(R_q, A).$$

Therefore there is $\overline{x}_\alpha \in RM(A)$ with $\|\overline{x}_\alpha\| = \|x_\infty u_\alpha\|$ such that for any $t \in (R_q)^*$, $t\overline{x}_\alpha = tx_\infty u_\alpha$ for all α. Thus, for any $t \in (R_q)^*$ and $a \in A$,

$$\|t\overline{x}_\alpha a - t\overline{x}_{\alpha'} a\| \leq \|tx_\infty\| \, \|u_\alpha a - u_{\alpha'} a\|.$$

Notice that $t\overline{x}_\alpha$, $t\overline{x}_{\alpha'} \in QM(A)$. Repeating the previous arguments, we conclude that $\overline{x}_\alpha a$ converges in norm for every $a \in A$. Let $\overline{x}$ be a weak limit of $\{\overline{x}_\alpha\}$ in A^{**}, then $\overline{x}a = \lim \overline{x}_\alpha a$ for all $a \in A$ and $\|\overline{x}\| = \|\overline{x}_\alpha\|$. For any $a, c \in A$, $c\overline{x}_\alpha a$ converges to $c\overline{x}a$ in norm. Since $c\overline{x}_\alpha a \in A$ for each α, $c\overline{x}a \in A$. Therefore $\overline{x} \in QM(A)$. Clearly, $a\overline{x}b = axb$ for all $a, b \in B$ and $\|\overline{x}\| = \|x\|$.

(b) We first show (ii) $\Rightarrow$ (i). We assume that R is a closed right ideal of A and $s \in R^\#$ $(= LM(R, A))$. Let p be the open projection corresponding to R, set $q = p + (1 - \overline{p})$ and $B = \mathrm{Her}(q)$. Then $qs \in QM(B)$. Therefore there is $\overline{s} \in M(A) = QM(A)$ (see [**6**, 4.18]) with $\|\overline{s}\| = \|qs\|$ such that $a\overline{s}b = asb$ for all a and $b \in B$. So $\|\overline{s}\| = \|s\|$. Moreover, for all $b \in B$,

$$q(\overline{s} - s)b = 0$$

since $(\overline{s} = s)b \in A$, q is dense, $(\overline{s} - s)b = 0$. Hence $\overline{s}r = sr$ for $r \in R$. This shows that A is C_3-injective. For (i) $\Rightarrow$ (ii), we notice from 3.16 that $M(A)$ is C_3-injective. It follows from 3.14 that every open projection in $M(A)$ is regular. Since A is an ideal of $M(A)$, every open projection in A is regular. By (a), (i) implies (ii).

REMARK 3.18. We do not know if (i) $\Rightarrow$ (ii) is true in general. However, we do know that (ii) does not imply (i) in general. See 3.26 (c) for an example.

COROLLARY 3.19. *Every hereditary C^*-subalgebra of a unital C_3-injective C^*-algebra satisfies the condition* (ii) *in* 3.17.

COROLLARY 3.20. *Let A be a unital C_3-injective C^*-algebra and p an open projection in A^{**}.*

(1) *Suppose that $s \in R_p^\#$, then there is a unique $\overline{s} \in A\overline{p}$ such that $s(r) = \overline{s}r$ for all $r \in R_p$ and $\|\overline{s}\| = \|s\|$.*

(2) *Suppose that* $s \in QM(\mathrm{Her}(p))$, *then there is a unique* $\bar{s} \in \bar{p}A\bar{p}$ *such that* $a\bar{s}b = asb$ *for all* $a, b \in \mathrm{Her}(p)$ *and* $\|\bar{s}\| = \|s\|$.

Proof. The uniqueness of $\bar{s}$ follows from the regularity of p.

COROLLARY 3.21. *Let* A *be a closed ideal of a unital* C_3*-injective* C^**-algebra. Then* $LM(A) = M(A)$.

Proof. Let B be a unital C_3-injective C^*-algebra containing A as a closed ideal. Suppose that p is the open projection in B^{**} corresponding to A. By 3.14 (b), we may assume that $\bar{p} = 1$. Suppose that $s \in LM(A)$. By 3.20 (1), there is $\bar{s} \in B$ such that $\bar{s}a = sa$ for all $a \in A$ and $\|\bar{s}\| = \|s\|$. So for all $a, b \in A$, $a\bar{s}b = asb$. Thus

$$a\bar{s}p = asp = as \quad \text{for all } a \in A.$$

Since A is an ideal of B, $a\bar{s} \in A$. Therefore $a\bar{s}p = a\bar{s}$, whence $a\bar{s} = as$. This implies that $s \in M(A)$.

COROLLARY 3.22. *Let* A *be a hereditary* C^**-subalgebra of a* C_3*-injective* C^**-algebra* B. *Consider the following conditions*:

(1) A *is* C_3*-injective*;

(2) $LM(A) = M(A)$;

(3) A *is an ideal of a* C_3*-injective* C^**-algebra*;

(4) A *is an ideal of a unital* C_3*-injective* C^**-algebra.*

Then

(a) *In general, we have*

$$(2) \Leftarrow (4) \Rightarrow (3) \Rightarrow (1).$$

(b) *If* B *is unital, then*

$$(1) \Leftrightarrow (4) \Rightarrow (3) \Rightarrow (1).$$

(c) *If* A *is* σ*-unital and every dense open projection of* A *is regular, then*

$$(4) \Rightarrow (3) \Rightarrow (1) \Rightarrow (2).$$

(d) *If* B *is unital and* A *is* σ*-unital, then*

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4).$$

Proof. Both (a) and (b) are now known and (d) follows from (b) and (c). It remains to show (c) and it suffices to show $(1) \Rightarrow (2)$. Let R be a closed right ideal of A and $s \in L(R, A)$. Suppose that p is the open projection in A^{**} corresponding to R. Set $q = p + (1 - \overline{p})$ and define $s_1 \in L(R_q, A)$ by

$$s_1(r) = spr \quad \text{for } r \in R_q.$$

(Notice that R_p is orthogonal to $R_{(1-\overline{p})}$.) Since A is C_3-injective, and $s_1 \in L(R_q, A) \subset LM(R_q, A)$, there is $\overline{s}_1 \in LM(A)$ such that $\overline{s}_1 r = s_1 r$ for all $r \in R_q$ and $\|\overline{s}_1\| = \|s_1\|$. Since $s_1 \in L(R_q, A)$, $s_1^* \in LM(A)$. For any $r \in R_q$ and $a \in A$,

$$r^*(\overline{s}_1^* - s_1^*)a = 0.$$

Thus $\|q(\overline{s}_1^* - s_1^*)a\| = 0$ for all $a \in A$. Since $(\overline{s}_1^* - s_1^*)a \in QM(A)$, and q is dense and regular, it follows from [7, 4.3 (a)] that

$$(\overline{s}_1^* - s_1^*)a = 0 \quad \text{for all } a \in A.$$

Hence $\overline{s}_1 \in M(A)$. So A satisfies the condition (4) in 2.15. Since A is σ-unital, by 2.15, $LM(A) = M(A)$. This completes the proof.

THEOREM 3.23. *Let H be a self-dual Hilbert module over a C_2-injective C^*-algebra A. Then $B(H)$ is a unital C_3-injective C^*-algebra. Consequently, $K(H)$ is a C_3-injective C^*-algebra.*

Proof. Let p be an open projection in $K(H)^{**}$ and $B = \mathrm{Her}(p)$. Suppose that $T \in QM(B)$. Set $H_{00} = \{bh : b \in B,\ h \in H\}$ and H_0 is the closure of H_{00}. By 2.13, $K(H_0) = B$. It follows from 1.7 that $T \in B(H_0, H_0^{\#})$. Since H is a self-dual Hilbert module over a C_2-injective C^*-algebra A, $H_0^{\#} \subset H$ $(= H^{\#})$. So $T \in B(H_0, H)$. It follows from 3.3 that there is $\widetilde{T} \in B(H)$ such that $\widetilde{T}|_{H_0} = T$ and $\|\widetilde{T}\| = \|T\|$. By 1.6, $\widetilde{T} \in LM(K(H))$. So $K(H)$ satisfies the condition (ii) in 3.17. Since H is self-dual, by [**21**, 3.5] and 1.6, $B(H)$ is unital and $LM(K(H)) = M(K(H)) = B(H)$. By 3.17 (b) and 3.16 both $K(H)$ and $B(H)$ are C_3-injective C^*-algebras.

COROLLARY 3.24. *Let A be a unital C_2-injective C^*-algebra and let $M_n(A)$ be the $n \times n$ matrix algebra over A. Then every hereditary C^*-subalgebra of $M_n(A)$ is C_3-injective. In particular, $M_n(A)$ is an AW^*-algebra.*

Proof. Let $H = A^{(n)}$. Then H is self-dual.

REMARK 3.25. It is known that $M_n(A)$ is an AW^*-algebra if A is an AW^*-algebra. (See [**3**, §62].) It is definitely a deep theorem. It is shown by Gert K. Pedersen [27] that $M_n(A)$ is a monotone complete C^*-algebra if A is. Corollary 3.24 is somehow related to these results. For a better result, see 4.11.

THEOREM 3.26. *Let A be an infinite dimensional monotone complete C^*-algebra. Then*
(a) *$M(A\otimes K)$ is not C_3-injective;*
(b) *$QM(Q\otimes K)$ becomes a monotone complete C^*-algebra;*
(c) *Every hereditary C^*-subalgebra of $A\otimes K$ satisfies the condition* (ii) *in* 3.17. *However, $A\otimes K$ is not C_3-injective.*

Proof. Let H_A be the Hilbert A-module

$$\left\{\{a_n\}: a_n\in A\,,\ \sum_n a_n^*a_n \text{ norm convergent}\right\}.$$

It follows from 3.7 that $H_A^\#$ is a self-dual A-module. By 3.8 and 3.5, every map in $B(H_A\,,\,H_A^\#)$ extends uniquely to a map in $B(H_A^\#)$ with the same norm. It follows from [21, 3.5] that $B(H_A^\#)=L(H_A^\#)$, whence $B(H_A^\#)$ is a C^*-algebra. For every map $T\in B(H_A^\#)$, $T|_{H_A}\in B(H_A\,,\,H_A^\#)$. Therefore we may identify $B(H_A\,,\,H_A^\#)$ with $B(H_A^\#)$. By 1.8,

$$QM(A\otimes K)\cong B(H_A\,,\,H_A^\#).$$

By identifying $QM(A\otimes K)$ with $B(H_A^\#)$, $QM(A\otimes K)$ becomes a C^*-algebra. Suppose that $\{x_\alpha\}\subset QM(A\otimes K)_{s.a.}$ is a bounded increasing net. Let $\{e_{ij}\}$ be a matrix unit for K and set

$$e_k=\sum_{i=1}^k 1\otimes e_{ii}.$$

It follows from [21] that for each n, $M_n(A)$ is monotone complete. So $e_kQM(A\otimes K)e_k$ ($\cong M_k(A)$) is monotone complete. Let $x^{(k)}$ be the least upper bound of the net $\{e_kx_\alpha e_k\}$. Since for any $m>0$,

$$e_k(e_{k+m}x_\alpha e_{k+m})e_k=e_kx_\alpha e_k\,,$$

we conclude that $e_kx^{(k+m)}e_k=x^{(k)}$ for all k (e.g. [11, Lemma 2.1]). For any $a\,,\,b\in\bigcup_k e_k(A\otimes K)e_k$, we define

$$axb=ax^{(k)}b\quad\text{for some large } k$$

such that both a and b are in $e_k(A\otimes K)e_k$. This is well defined. Since $\{x^{(k)}\}$ is bounded, x defines a quasi-multiplier of $A\otimes K$. We are now ready to check that x is a least upper bound of the net $\{x_\alpha\}$. This proves (b). Since $A \times K$ and its hereditary C^*-subalgebras are hereditary C^*-subalgebras of $QM(A \otimes K)$, by 3.8 and 3.19, the first part of (c) follows. It is well known that if A is a unital, infinite dimensional C^*-algebra, $LM(A \otimes K) \neq M(A \otimes K)$ (e.g. [**19**, part II, Remarks]). Since $A \otimes K$ is σ-unital, by 3.22 (d), $A \otimes K$ is not C_3-injective. Since $A \otimes K$ is an ideal of $M(A \otimes K)$, it follows from 3.2 that $M(A \otimes K)$ is not C_3-injective.

REMARK 3.27. From [**27**] we know that $M_n(A)$ are monotone complete for all n if A is a monotone complete C^*-algebra. One may suspect that $M(A \otimes K)$ is also monotone complete. However, 3.26 tells us that $QM(A \otimes K)$ is a monotone complete C^*-algebra, $M(A \otimes K)$ is not even C_3-injective. On the other hand, $A \otimes K$ does have a nice extension property.

One should notice that $QM(A \otimes K)$ is not a subalgebra of $(A \otimes K)^{**}$. It is shown by L. G. Brown that for general C^*-algebra B, if $x \in QM(B)_+$ with $x^2 \in QM(B)$ then $x \in M(B)$ ([**5**, 2.61]). However, this by no means contradicts 3.26 (c). If one examines carefully, one may actually see how the multiplication is defined in $QM(A \otimes K)$ in 3.2 (b). In fact, if $x \in QM(A \otimes K)$, x is represented by an infinite matrix (a_{ij}) with $a_{ij} \in A$ such that (a_{ij}) is bounded. Moreover, $\|\sum_j a_{ij} a_{ij}^*\|$ is bounded. Therefore if (b_{ij}) is also in $QM(A \otimes K)$, $\sum_k a_{ik} b_{kj} \to c_{ij}$ for some $c_{ij} \in A$ with the Kadison-Pedersen arrow. And (c_{ij}) is in fact in $QM(A \otimes K)$. The product of (a_{ij}) with (b_{ij}) in 3.25 (b) is in fact (c_{ij}). On the other hand $\sum_k a_{ik} b_{kj}$ does converge weakly to an element c'_{ij} in A^{**}. This is why in $(A \otimes K)^{**}$, $(a_{ij}) \notin QM(M(A \otimes K))$, in general. Let π be a faithful representation of $A \otimes K$. Then π can be extended to a faithful isomorphism of $M(A \otimes K)$ and if we extend π further, $\pi(QM(A \otimes K))$ is faithful. (See [**25**, 3.12.5] and [**6**, 4.15].) Let Q be the C^*-subalgebra of $(A \otimes K)^{**}$ generated by $QM(A \otimes K)$. By [**6**, 4.15], the atomic representation π_a is faithful on Q. But the above shows that in general $\pi(Q)$ is not faithful.

COROLLARY 3.28. *Let H be a countably generated Hilbert module over a monotone complete C^*-algebra A. Then $B(H^\#)$ is a monotone complete C^*-algebra.*

Proof. By Kasparov's stabilization theorem [**12**], $H \cong pH_A$ for some projection p in $M(K(H_A))$. It is clear then $H^\# \cong pH_A^\#$ and $B(H^\#) \cong B(pH_A^\#) \cong pQM(A \otimes K)p$.

REMARK 3.29. Let A be a unital C^*-algebra. Consider the following conditions:

(1) A is monotone complete;
(2) A is C_2-injective;
(3) A is C_3-injective;
(4) A is an AW^*-algebra.

In general we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If, in addition, A is commutative, then they all are equivalent. There is no AW^*-algebra known not to be monotone complete. The first three types of C^*-algebras have a common property that all open projections in their second duals are regular. We ask the following questions:

(a) Is every open projection in the second dual of an AW^*-algebra regular (cf. [**26**, 21])?

(b) Is every AW^*-algebra with the property that every open projection is regular in its second dual monotone complete?

(c) Any implication in reverse order among (1), (2), (3), (4)?

4. Extensions of bounded module maps, continued. In this section we consider countably generated Hilbert modules. However, we are not going to give countable versions of 3.1 and 3.11. In fact, there are several ways to put countable conditions. We begin with a few easy consequences of the last section.

COROLLARY 4.1. *Let A be a monotone sequentially complete C^*-algebra and H a countably generated Hilbert A-module. Then the A-valued inner product $\langle \cdot, \cdot \rangle$ extends to $H^\# \times H^\#$ in such a way as to make $H^\#$ into a self-dual Hilbert A-module. Moreover, the extended inner product satisfies $\langle \tau, x \rangle = \tau(x)$ and*

$$\|\langle \tau, \tau \rangle\|^{1/2} = \sup\{\|\tau(x)\| : \|x\| = 1,\ x \in H\}$$

for $t \in H^\#$ and $x \in H$.

Proof. By [**20**, 1.5], $K(H)$ is σ-unital.

COROLLARY 4.2. *Let A be a monotone sequentially complete C^*-algebra. Suppose that H is a Hilbert A-module, H_0 is a countably generated, closed submodule of H and $\varphi \in H_0^\#$. Then there is $\tilde{\varphi} \in H^\#$ such that $\tilde{\varphi}|_{H_0} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

COROLLARY 4.3. *Let A be a monotone sequentially complete, unital C^*-algebra. Then $QM(A \otimes K)$ becomes a monotone sequentially complete C^*-algebra.*

An open projection is called σ-unital, if $\operatorname{Her}(p)$ is σ-unital ([**26**, 21]).

LEMMA 4.4. *Let A be a C^*-algebra with real rank zero and cancellation of projections. Suppose that p is a σ-unital open projection in $M_n(A)^{**}$. There is a partial isometry $u \in M_n(A)^{**}$ such that*

$$u^*u = p \quad \text{and} \quad uu^* = \sum_{i=1}^{n} q_i \otimes e_{ii},$$

where q_i are open projections in A^{**} and $\{e_{ij}\}$ is a matrix unit for M_n. Moreover, for any $x \in \operatorname{Her}(p)$,

$$uxu^* \in \operatorname{Her}(uu^*).$$

Proof. Since A has real rank zero, there are projections $\{e_n\}$ in $\operatorname{Her}(p)$ such that $\{e_n\}$ forms an approximate identity for $\operatorname{Her}(p)$. Let $p_1 = e_1$ and $p_{n+1} = e_{n+1} - e_n$, $n = 1, 2, \ldots$. By [**29**], there is a partial isometry $u_1 \in M_n(A)$ such that

$$u_1^*u_1 = p \quad \text{and} \quad u_1u_1^* = \sum_{i=1}^{n} q_i^{(1)} \otimes e_{ii},$$

where $q_i^{(1)}$ are projections in A. Since $M_n(A)$ has cancellation of projections (see [**4**, III.2.4]),

$$p_2 \leq \sum_{i=1}^{n} (1 - q_i^{(1)}) \otimes e_{ii}.$$

Applying [**29**] again, there is $u_2 \in M_n(A)$ such that

$$u_2^*u_2 = p_2 \quad \text{and} \quad u_2u_2^* = \sum_{i=1}^{n} q_i^{(2)} \otimes e_{ii},$$

where $q_i^{(2)}$ are projections in A such that $q_i^{(2)} \leq 1 - q_i^{(1)}$. By induction, there are a sequence of partial isometries $u_k \in M_n(A)$ and a sequence of $\{q_i^{(k)}\}_{i=1}^n$ in A such that

$$u_k^*u_k = p_k \quad \text{and} \quad u_ku_k^* = \sum_{i=1}^{n} q_k^{(k)} \otimes e_{ii}$$

and $q_i^{(k)} \perp q_i^{(k')}$ if $k \neq k'$, $i = 1, 2, \dots, n$. Then $u \in M_n(A)^{**}$. By the construction, we have

$$u^*u = p \quad \text{and} \quad uu^* = \sum_{i=1}^{n} q_i \otimes e_{ii},$$

where $q_i = \sum_{k=1}^{\infty} q_i^{(k)}$ is an open projection in A^{**}. Clearly, if $x \in \operatorname{Her}(p)$, $uxu^* \in \operatorname{Her}(uu^*)$. This completes the proof.

THEOREM 4.5. *Let A be a unital C_3-injective C^*-algebra. Suppose that H_0 is a countably generated Hilbert A-submodule of a Hilbert A-module H and φ is a bounded module map from H_0 into A. If H_0 is a closed submodule of A^n, then there is a module map $\tilde{\varphi}$ from H into A such that $\tilde{\varphi}|_{H_0} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. By 3.14, A is an AW^*-algebra. It follows form [**3**, §15] that we may write $A = A_1 \oplus A_2$, where A_1 is properly infinite and A_2 is finite. For any Hilbert A-module H, then $H = H' \oplus H''$, where H' is an A_1-module and H'' an A_2-module. It follows from [**30**] that A_1 is monotone sequentially complete. By 4.2, we may assume that A is finite.

Since $H_0 \subset A^n$, by 2.13, $K(H_0)$ is a hereditary C^*-subalgebra of $M_n(A)$. Let p be the open projection in $M_n(A)^{**}$ corresponding to $K(H_0)$. Since A is a finite AW^*-algebra, A has real rank zero and $M_n(A)$ is finite for each n. So Lemma 4.4 applies. Thus

$$H_0 \cong \bigoplus_{i=1}^{n} R_{p_i},$$

where the p_i's are open projections in A^{**}. So we may write

$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_n,$$

where each φ_i is in $R_{p_i}^{\#}$. It follows from 3.20, that $R_{p_i}^{\#} = R_{\overline{p}_i}$. So

$$H_0^{\#} \cong \bigoplus_{i=1}^{n} R_{\overline{p}_i}.$$

Therefore, φ extends to a module map on $H_0^{\#}$ (with the same norm). Let P be the projection from H into $H_0^{\#}$ defined by

$$Px(h) = \langle x, h \rangle \quad \text{for } x \in H,\ h \in H_0.$$

Set $\tilde{\varphi} = \varphi \circ P$. Then $\tilde{\varphi} \in H^{\#}$, $\tilde{\varphi}|_{H_0} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$. This completes the proof.

COROLLARY 4.6. *Let A be a unital C_3-injective C^*-algebra and H a countably generated Hilbert A-module. If H is a closed submodule of A^n, then the A-valued inner product $\langle \cdot, \cdot \rangle$ extends to $H^\# \times H^\#$ such that $H^\#$ becomes a self-dual Hilbert A-module with this inner product,*

$$\langle \tau, x\rangle = \tau(x) \quad \textit{and} \quad \|\langle \tau, \tau\rangle\|^{1/2} = \sup\{\|\tau(x)\| : \|x\| = 1, \ x \in H\}$$

for $\tau \in H^\#$ and $x \in H$.

Proof. It is a combination of 4.1 and the proof of 4.5.

COROLLARY 4.7. *Let A be a unital C_3-injective C^*-algebra with a faithful representation on a separable Hilbert space. Then $M_n(A)$ is also a C_3-injective C^*-algebra for all n.*

Proof. Since $M_n(A)$ is unital, by 3.17 (b), it suffices to show that $M_n(A)$ satisfies the condition (ii) in 3.17. Let B be a hereditary C^*-subalgebra of $M_n(A)$ and $T \in QM(B)$. Since A has a faithful representation on a separable Hilbert space, so does $M_n(A)$. Therefore B is σ-unital. Let H_0 be the closure of the set

$$\{bh : b \in B, \ h \in A^n\}.$$

Then, by 2.13, $K(H_0) = B$. By [**20**, 1.5], H_0 is countably generated. By 1.7, $T \in B(H_0, H_0^\#)$. For fixed $x \in A^n$, define $T_x \in H_0^\#$ by

$$T_x(h) = \langle x, Th\rangle \quad \text{for } h \in H_0.$$

It follows from 4.5 that there is $\widetilde{T}_x \in A^n$ (A^n is a self-dual) with $\|\widetilde{T}_x\| = \|T_x\|$ such that $\widetilde{T}_x(h) = T_x(h)$ for all $h \in H_0$. Define a map $\widetilde{T} : A^n \to H_0^\#$ by

$$\widetilde{T}h(x) = [\widetilde{T}_x(h)]^* \quad \text{for } x \text{ and } h \in A^n.$$

Clearly $\widetilde{T}$ is a module map, $\widetilde{T}h = Th$ for all $h \in H_0$ and

$$\|\widetilde{T}h(x)\| \le \|\widetilde{T}_x\|\,\|h\| = \|T_x\|\,\|h\| \le \|T\|\,\|x\|\,\|h\|$$

for $x, y \in A^n$. So $\|\widetilde{T}\| = \|T\|$. By 4.6, $H_0^\# \subset A^n$. Therefore $\widetilde{T} \in B(A^n) = M_n(A)$. Since $\widetilde{T}|H_0 = T$, $\widetilde{T}|_B = T$. This completes the proof.

COROLLARY 4.8. *Let A be a unital C_3-injective C^*-algebra with a faithful representation on a separable Hilbert space. Then every open projection in $M_n(A)^{**}$ is regular.*

THEOREM 4.9. *Let A be a C^*-algebra. If the A-valued inner product $\langle\cdot\,,\,\cdot\rangle$ extends to $H_A^\# \times H_A^\#$ so that $H_A^\#$ becomes a self-dual Hilbert A-module and $\langle\tau\,,\,x\rangle = \tau(x)$,*

$$\|\langle\tau\,,\,\tau\rangle\|^{1/2} = \sup\{\|\tau(x)\|: \|x\| = 1\,,\ x \in H\}$$

for $\tau \in H_A^\#$ and $x \in H_A$, then A is monotone sequentially complete.

Proof. It suffices to show that for any $\{x_n\} \subset A_+$ such that $\{\|\sum_{k=1}^n x_k\|\}$ is bounded, there is a least upper bound for $\{\sum_{k=1}^n x_k\}$ in A.

Set $\tau = \{x_k^{1/2}\}$; then τ defines an element in $H_A^\#$ (see **[14]**). We claim that $\langle\tau\,,\,\tau\rangle$ is a least upper bound for $\{\sum_{k=1}^n x_k\}$. Let p_n be the projection in $K(H_A)$ such that

$$p_n(\{a_k\}) = \{b_k\}\,,$$

where $b_k = a_k$ if $0 \le k \le n$ and $b_k = 0$ if $k > n$. Then $\{p_n\}$ forms an approximate identity for $K(H_A)$. Clearly,

$$\langle\tau\,,\,\tau\rangle \ge \langle p_n\tau\,,\,\tau\rangle = \sum_{k=1}^n x_k.$$

Suppose that $y \in A$ and $y \ge \sum_{k=1}^n x_k$ for all n. We need to show that $y \ge \langle\tau\,,\,\tau\rangle$.

Let $0 < \alpha < \frac{1}{2}$. For each k, set

$$u_k^{(n)} = x_k^{1/2}(\tfrac{1}{n} + y)^{-1/2}y^{1/2-\alpha}.$$

It is known (e.g. **[25**, 1.4.4]) that $u_k^{(n)}$ converges in norm (as $n \to \infty$). Set

$$u_k = \lim_{n\to\infty} u_k^{(n)}.$$

Since

$$\begin{aligned}\sum_{k=1}^m (u_k^{(n)})^*(u_k^{(n)}) &= \sum_{k=1}^m (\tfrac{1}{n} + y)^{-1/2}y^{1/2-\alpha}x_k(\tfrac{1}{n} + y)^{-1/2}y^{1/2-\alpha}\\ &\le (\tfrac{1}{n} + y)^{-1/2}y^{1/2-\alpha}y(\tfrac{1}{n} + y)^{-1/2}y^{1/2-\alpha},\end{aligned}$$

as $n \to \infty$,

$$\sum_{k=1}^m u_k^* u_k \le y^{1-2\alpha} \le \|y^{1-2\alpha}\|,$$

for all m. Set $\xi = \{u_k\}$, then $\xi \in H_A^\#$. Clearly, $\tau = \xi\cdot y^\alpha$. Therefore

$$\langle\tau\,,\,\tau\rangle = y^\alpha\langle\xi\,,\,\xi\rangle y^\alpha \quad \text{for } 0 < \alpha < \tfrac{1}{2}.$$

By [14],

$$\|\xi\| = \left\| \sum_{k=1}^{\infty} u_k^* u_k \right\|^{1/2},$$

(where $\sum_{k=1}^{\infty} u_k^* u_k$ is the strong limit of $\{\sum_{k=1}^{m} u_k^* u_k\}$ in A^{**}). Thus $\|\xi\|^2 \leq \|y^{1-2\alpha}\|^2$. Hence

$$\langle \tau, \tau \rangle \leq \|y^{1-2\alpha}\|^2 y^{2\alpha}$$

for all $0 < \alpha < \frac{1}{2}$. Let $\alpha \to \frac{1}{2}$, we have

$$\langle \tau, \tau \rangle \leq y.$$

This completes the proof.

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