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SIMPLE LOCAL TRACE FORMULAS FOR UNRAMIFIED p-ADIC GROUPS

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Let G be a connected unramified semi-simple group over a p-adic field F. In this note, we compute a (Macdonald-)Plancherel-type formula: $\int_{G(F)\times G(F)} f(h)\phi(g^{-1}hg)\,dg\,dh = \int f^\vee(\chi)I(\chi, \phi)\,d\mu(\chi)$. Here f is a spherical function, $f^\vee$ is its Satake transform, and ϕ is a smooth function supported on the elliptic set. For this, we use the Geometrical Lemma of Bernstein and Zelevinsky, Macdonald's Plancherel formula, Macdonald's formula for the spherical function, results of Casselman on intertwining operators of the unramified series, and a combinatorial lemma of Arthur. This derivation follows a procedure of Waldspurger rather closely, where the case of $\mathrm{GL}(n)$ was worked out in detail. We may rewrite this formula as $\int_{G(F)} f(g^{-1}\gamma g)\,dg = \int f^\vee(\chi)I(\chi, \gamma)\,d\mu(\chi)$, for γ elliptic regular in $G(F)$ and f spherical. Here $I(\chi, \gamma)$ is a distribution on the support of the Plancherel measure (regarded as a compact complex analytic variety).

Introduction. Let G be a connected unramified semi-simple group over a p-adic field F, let G_{reg} denote the subset of regular elements of $G(F)$, and let G_{ell} denote the subset of elliptic regular elements of $G(F)$. Let $C_c^\infty(G(F))$ denote the algebra of locally constant compactly supported functions on $G(F)$ and let $\mathscr{H}(G, K)$ denote the commutative subalgebra of spherical functions associated to a hyperspecial, good, maximally bounded subgroup K of $G(F)$.

Let $\Phi \subset C_c^\infty(G(F))$ denote the subspace of functions on $G(F)$ supported in G_{ell}. For each $\phi \in \Phi$, define

$$T_\phi : f \mapsto \int_{G(F)\times G(F)} f(h)\phi(g^{-1}hg)\,dg\,dh. \tag{0.1}$$

It is not hard to show that T_ϕ defines an "elliptic" invariant distribution in $C_c^\infty(G(F))'$ with compactly generated support (that is $\operatorname{supp} T_\phi \subset C^G$, where $C \subset G(F)$ is compact and C^G denotes the set of $G(F)$-conjugacy classes containing an element of C). In this note, we restrict T_ϕ to $\mathscr{H}(G, K)$ and compute, in §§2–3, a (Macdonald-)Plancherel-type formula for T_ϕ:

$$T_\phi(f) = \int f^\vee(\chi)I(\chi, \phi)\,d\mu(\chi) \tag{0.2}$$

(see Theorem 3.10 below). For this, we use the Geometrical Lemma of Bernstein and Zelevinsky, Macdonald's Plancherel formula, Macdonald's formula for the spherical function, results of Casselman on intertwining operators of the unramified series, and a combinatorial lemma of Arthur. This derivation follows the procedure of Waldspurger [**W**] rather closely, where the case of $\mathrm{GL}(n)$ was worked out in detail.

Of course, the distribution T_ϕ also occurs in the context of Arthur's local trace formula [**Art1**]. Let R denote the unitary representation of $G(F) \times G(F)$ on $L^2(G(F))$ given by $(R(x_1, x_2)\psi)(y) := \psi(x_1^{-1} y x_2)$, $\psi \in L^2(G(F))$. Given $f = (f_1, f_2)$ in $C_c^\infty(G(F)) \times C_c^\infty(G(F)) \hookrightarrow C_c^\infty(G(F) \times G(F))$, the kernel of the integral operator $R(f)$ is

$$\begin{aligned} K_f(x_1, x_2) &= \int_{G(F)} f_1(x_1 y) f_2(y x_2)\, dy \\ &= \int_{G(F)} f_1(y) f_2(x_1^{-1} y x_2)\, dy. \end{aligned} \tag{0.3}$$

As in the global trace formula, one wants to find both a "geometric" and "spectral" formula for a truncated version of the integral of $K_f(x, x)$. It should be emphasized that this is done below only for a very restricted class of $f = (f_1, f_2)$.

Thus this paper could be viewed as a special case of Arthur's local trace formula [**Art1**] or as a generalization of part of Waldspurger's work [**W**]. Another way one might interpret these distributions $I(\chi, \phi)$ is as follows. We will see in §3 below that the $\phi \mapsto I(\chi, \phi)$ is G-admissible in the sense of [**HC**]. Then, regarding this invariant distribution as a function (the existence of which is assured by applying [**HC**, Theorem 19]), we may rewrite (0.2) as

$$\int_{G(F)} f(g^{-1}\gamma g)\, dg = \int f^\vee(\chi) I(\chi, \gamma)\, d\mu(\chi), \tag{0.4}$$

for $\gamma \in G_{\mathrm{ell}}$ and f spherical. Here $I(\chi, \gamma)$ is a distribution on the support of the Plancherel measure (regarded as a compact complex analytic variety [**M**]).

A somewhat analogous formula to (0.4), for stable unipotent orbits, has been conjectured in [**A**]. Assem's conjecture is a theorem for $\mathrm{GL}(n)$ and a number of other cases. For $G = \mathrm{GL}(n)$, the germ expansion and Assem's formula yield a relatively explicit expression for $I(\chi, \gamma)$ (this idea can essentially be found in [**W**]). Finally, we remark that in the case of $\mathrm{SL}(n)$ the fundamental lemma of Waldspurger [**Wa**] may be reformulated as a functorial property of the $I(\chi, \gamma)$.

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1. Notation and background.

Root spaces. Let G be a connected unramified reductive group of semi-simple rank l over F which has a splitting defined over an unramified extension E/F. (Recall that a reductive group G over F is unramified if it is quasi-split over F and has a splitting over a finite unramified extension **[Car**, p. 135].) Let T denote a maximal torus of G, B a Borel subgroup defined over F, and A a maximal F-split torus G contained in B. Let $\Gamma := \mathrm{Gal}(E/F)$, let X^* denote the character lattice of T, and let $\Sigma_T \subset X^*$ denote the root system in X^* with respect to T. The Γ-module structure of X^* leaves Σ_T invariant. Let Σ denote the set of reduced roots of G relative to A and Δ the corresponding reduced fundamental system. These are also left invariant by Γ. The character lattice

$$X^*(A) := \mathrm{Hom}_{F\text{-grps}}(A, \mathrm{GL}(1)), \tag{1.1}$$

may be regarded as a quotient of X^* containing Σ. Let

$$X_*(A) := \mathrm{Hom}_{\mathbf{Z}}(X^*(A), \mathbf{Z}) \tag{1.2}$$

denote the co-character lattice. Let $\Sigma^+ \subset \Sigma$ be the subset of positive roots containing Δ. We let $\Delta^\vee$ denote the set of dual roots $\{\varpi_\alpha | \alpha \in \Delta\}$ associated to Δ. For parabolic subgroups P and Q with $A \subset P \subset Q$, let Δ_P^Q denote the set of simple positive roots of $(P \cap M_Q, A_P)$, where $Q = M_Q N_Q$ denotes the Levi decomposition and A_P denotes the center of M_P. As usual, if $Q = G$ then we drop the superscript: $\Delta_P^G = \Delta_P$.

For each subset $\theta \subset \Delta$, we denote by P_θ the parabolic subgroup containing B associated to θ, by $P_\theta = M_\theta N_\theta$ its Levi decomposition (so $P_\phi = B$, $M_\phi = A$), and by $A_M = A_\theta$ the split component of the center of $M = M_\theta$. We abuse language and call a Levi component $M = M_P$ of a parabolic subgroup $P = MN = M_P N_P$ a Levi subgroup of G. Furthermore, by a Levi (parabolic) subgroup we will always mean a Levi (parabolic) subgroup containing the torus A above. We denote by $\mathscr{P}(M)$ the set of parabolic subgroups of G having Levi component M. If $P \subset Q$ are parabolic subgroups there is a surjective map between the Lie algebras $\mathfrak{a}_P \to \mathfrak{a}_Q$ whose kernel will be denoted $\mathfrak{a}_P^Q$. From **[Art3]** we know that there are orthogonal decompositions

$\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q$ and $\mathfrak{a}_P^* = \mathfrak{a}_Q^* \oplus (\mathfrak{a}_P^Q)^*$. For $P \in \mathscr{P}(M)$ and $X \in \mathfrak{a}_B$, let $X \mapsto X_M$ denote the projection $\mathfrak{a}_B \to \mathfrak{a}_P$ and let $X \mapsto X^M$ denote the projection $\mathfrak{a}_B \to \mathfrak{a}_B^P$. Furthermore, $(\Delta_P^Q)^\vee$ forms a basis for $\mathfrak{a}_P^Q$ and $(\Delta_P^Q)^\wedge$ forms a basis for $(\mathfrak{a}_P^Q)^*$. Let τ_P^Q denote the characteristic function on $\mathfrak{a}_B$ of the set

$$\{X \in \mathfrak{a}_P^Q | \alpha(X) > 0,\ \alpha \in \Delta_P^Q\},$$

and let $\hat{\tau}_P^Q$ denote the characteristic function on $\mathfrak{a}_B$ of the set

$$\{X \in \mathfrak{a}_P^Q | \alpha(X) > 0,\ \alpha \in \Delta_P^Q\}.$$

The kernel of the map $H_M\colon M(F) \to a_P$ defined in [**Art3**] will be denoted by $M(F)^1$. This may also be described as the intersection of all the kernels of the absolute values of the rational characters of $M(F)$. The Haar measure on $M(F)^1$ will be that measure determined by those on $M(F)$, $\mathfrak{a}_P$, and the pull-back by the map H_M.

The natural pairing $\langle \cdot, \cdot \rangle\colon X_*(A) \times X^*(A) \to \mathbf{Z}$ allows us to identify $X^*(A)$ with the dual lattice of $X_*(A)$. Using this, we may define an isomorphism

$$\text{(1.3)} \qquad \mathrm{Hom}_{F\text{-grps}}(\mathrm{GL}(1), A) \cong X_*(A).$$

We fix a uniformizing parameter π of F, $|\pi|_F = q^{-1}$, and let $A_\alpha \in A(F)$ denote the image $\alpha^\vee(\pi)$ of π, regarding the coroot $\alpha^\vee$ as an element of $\mathrm{Hom}_{F\text{-grps}}(\mathrm{GL}(1), A)$. If G is split over F then it satisfies (a) $\delta_B(a_\alpha) = q^{-2}$, (b) $\{a_\alpha | \alpha \in \Delta\}$ generates the abelian group $A(F)/(A(F) \cap K)$ freely, and (c) $w_\alpha^{-1} a w_\alpha = a a_\alpha^{-\langle \nu_A(a), \alpha \rangle}$, where ν_A is as in (1.4) below ([**Car**, pp. 141–142], [**M**, pp. 42–43]). To each $\beta \in \Sigma \cup \frac{1}{2}\Sigma$, we associate as in [**Car**, (24)] a real number $q_\beta > 0$. If G is split and $\alpha \in \Sigma$ then $q_\alpha = q$, $q_{\alpha/2} = 1$, where q denotes the order of the residue field.

Let

$$\mathscr{A}_{A,\mathbf{R}} := X_*(A)_{\mathbf{R}} := X_*(A) \otimes_{\mathbf{Z}} \mathbf{R} = \mathfrak{a}_B, \quad X^*(A)_{\mathbf{R}} := X^*(A) \otimes_{\mathbf{Z}} \mathbf{R},$$

and extend $\langle \cdot, \cdot \rangle$ to $X_*(A)_{\mathbf{R}} \times X^*(A)_{\mathbf{R}}$. We use this pairing to identify $X^*(A)_{\mathbf{R}}$ and its $\mathbf{R}$-vector space dual with $\mathscr{A}_{A,\mathbf{R}}$. Thus we have two bases $\Delta \subset X^*(A)_{\mathbf{R}}$ and $\Delta^\vee \subset X_*(A)_{\mathbf{R}}$ of $\mathscr{A}_{A,\mathbf{R}}$ such that $\langle \alpha^\vee, \beta \rangle = 2\delta_{\alpha\beta}$ for all $\alpha, \beta \in \Delta$. Let $X_*(A)_{\mathbf{C}} := X_*(A) \otimes_{\mathbf{Z}} \mathbf{C}$.

We fix a special, good, maximally bounded subgroup K of $G(F)$. Each $w \in W_G := N_{G(F)}(A)/C_{G(F)}(A)$ has a representation in K by means of the identification

$$N_{G(F)}(A)/C_{G(F)}(A) \cong (K \cap N_{G(F)}(A))/(K \cap C_{G(F)}(A))$$

[**Car**, p. 140] (here C_G denotes the centralizer and N_G denotes the normalizer).

There is a surjection $\nu_A\colon A(F) \to X_*(A)$ characterized by

$$(1.4) \qquad \langle \nu_A(a), \lambda^* \rangle = \nu_F(\lambda^*(a)), \qquad \forall \lambda^* \in X^*(A),\ a \in A(F),$$

where $\nu_F\colon F^\times \to \mathbf{Z}$ denotes the normalized valuation. Thus we obtain an isomorphism

$$(1.5) \qquad \nu_A^{-1}\colon X_*(A) \to A(F)/(A(F) \cap K).$$

Thus every unramified character of $A(F)$ may be identified with a character of the discrete group $X_*(A)$. More generally, to each Levi M of G we have

$$X_*(A_M) = \{X \in X_*(A) | \nu_A^{-1}(X) \in A_M(F)/(A_M(F) \cap K)\}.$$

Denote

$$\mathscr{A}_{M,\mathbf{R}} := \mathfrak{a}_P = X_*(A_M) \otimes_{\mathbf{Z}} \mathbf{R}.$$

Let

$$\mathscr{A}_{\mathbf{Z}}^M := \{X \in X_*(A) | \nu_A^{-1}(X) \in M^1(F)/(A(F) \cap K)\}$$

and let $\mathscr{A}_{\mathbf{R}}^M := \mathscr{A}_{\mathbf{Z}}^M \otimes_{\mathbf{Z}} \mathbf{R}$. Recall the projection $\mathscr{A}_{A,\mathbf{R}} \to \mathscr{A}_{\mathbf{R}}^M$ is $X \mapsto X^M$.

The complex dual of $X_*(A)$ is

$$(1.6) \qquad X_*(A)^\wedge = X^*(A)_{\mathbf{C}}/X_*(A)^\perp,$$

where $X_*(A)^\perp$ denotes the lattice of all $\lambda^* \in X^*(A)_{\mathbf{C}}$ such that, for all $\lambda_* \in X_*(A)$, $\langle \lambda_*, \lambda^* \rangle \in 2\pi i \mathbf{Z}$. Let $\mathbf{C}^{\times 1}$ denote the unit circle in $\mathbf{C}$. We will use the notation $\mathscr{A}_{A,\mathbf{C}}^*$ to denote the complex dual of $\mathscr{A}_{A,\mathbf{R}}$, so that

$$(1.7) \qquad \mathrm{Hom}_{\mathrm{unr}}(A(F), \mathbf{C}^{\times 1}) \cong \mathscr{A}_{A,\mathbf{C}}^*/L,$$

as complex varieties, where

$$L := \{\lambda \in \mathscr{A}_{A,\mathbf{C}}^* | \lambda(\alpha^\vee) \in \mathbf{Z},\ \forall \alpha \in \delta\}.$$

In fact, once we fix an ordering of the roots Δ this isomorphism is canonical. Here $\mathrm{Hom}_{\mathrm{unr}}$ is defined as follows. If $H(F)$ is any closed subgroup of $G(F)$, with the inherited compact-open topology and if V is any (complex) Hilbert space, with the discrete topology, then $\mathrm{Hom}_{\mathrm{unr}}(H(F), \mathrm{End}\, V)$ is the set of continuous homomorphisms $H(F) \to \mathrm{End}\, V$ with a non-zero $H(F) \cap K$-fixed vector.

Intertwining operators. For the unramified principal series representations $(\nu_\chi, I(\chi))$ of $G(F)$, associated to a character χ of $A(F)$, we refer to **[Car]**. We remark that the pairing $\langle \cdot, \cdot \rangle$ on $I(\chi) \times I(\chi^{-1})$ defined in **[Car]** allows us to identify the contragredient representation $(\nu_\chi^\sim, I(\chi)^\sim)$ with $(\nu_{\chi^{-1}}, I(\chi^{-1}))$.

Let χ be a regular unramified character of $A(F)$ (so $w\chi$, $w \in W_G$, are all distinct), and let $T_w \colon I(\chi) \to I(w\chi)$ denote the intertwining operator of **[Car]**. If $\Phi_{K,\chi} \in I(\chi)^K$ denotes the unique K-fixed vector satisfying $\Phi_{K,\chi}(1) = 1$ then Casselman **[Cas1]**, **[Car**, Theorem 3.9**]** has shown that

$$T_w(\Phi_{K,\chi}) = c_w(\chi)\Phi_{K,w\chi}, \tag{1.8}$$

where

$$c_w(\chi) := \prod_{\alpha\in\Sigma^+,\, w\alpha<0} c_\alpha(\chi),$$

$$c_\alpha(\chi) := \frac{(1 - q_{\alpha/2}^{1/2} q_\alpha^{-1}\chi(a_\alpha))(1 - q_{\alpha/2}^{-1/2}\chi(a_\alpha))}{1 - \chi(a_\alpha)^2}.$$

It is also known that $T_{w_1 w_2} = T_{w_1} T_{w_2}$, provided $l(w_1 w_2) = l(w_1) + l(w_2)$ (here $l(w)$ denotes the length of $w \in W$).

Plancherel's formula and Macdonald's formula. Let $\mathscr{H}(G, K)$ denote the subalgebra of $C_c^\infty(G)$ consisting of bi-K-invariant functions and $C^\infty(G, K)$ the analogous subalgebra of $C^\infty(G)$—the space of locally constant functions on $G(F)$. Let $G(F)^1$ denote the kernel of the map $H_G \colon G(F) \to \mathfrak{a}_G$.

For $f \in I(\chi)$, and χ unramified, define

$$f^\#(g) := \int_K f(kg)\, dk,$$

where the measure on K has total volume 1, and let $\Gamma_\chi := \Phi_{K,\chi}^\#$. Macdonald's formula states that

LEMMA 1.9. *If χ is an unramified regular character of $A(F)$ then*

$$\Gamma_\chi(a) = Q^{-1}\delta_B(a)^{1/2} \sum_{w\in W} c(w\chi)(w\chi)(a), \qquad a \in A(F),$$

where

$$Q := \sum_{w\in W} (IwI : I),$$

I denoting the Iwahori subgroup of G and $c(\chi) := c_1(\chi^{-1})$ (see (1.8)).

For $f \in \mathscr{H}(G, K)$ and χ unramified, define the Fourier transform of f at χ by

$$f^\vee(\chi) := \int_{G(F)^1} f(g)\Gamma_\chi(g^{-1})\,dg. \tag{1.10}$$

Let $\Omega_K(G)$ denote the set of all zonal spherical functions of $G(F)^1$ relative to K:

$$\begin{aligned}(1.11)\quad \Omega_K(G) := \{\omega \in C^\infty(G, K) \mid \omega(1) = 1 \text{ and}, \\ \forall f \in \mathscr{H}(G, K),\ f * \omega = \lambda_f \omega,\ \text{some } \lambda_f \in \mathbf{C}\}.\end{aligned}$$

Let $\Omega_K^+(G)$ denote the subset of all positive definite zonal spherical functions. It is known that, if χ is unitary, then $\Gamma_\chi \in \Omega_K^+(G)$ [**M**, Theorem 3.3.12]. We define, more generally, the Fourier transform of $f \in \mathscr{H}(G, K)$ at $\omega \in \Omega_K^+(G)$ by

$$f^\vee(\omega) := \int_{G(F)^1} f(g)\omega(g^{-1})\,dg. \tag{1.12}$$

The relation between the Fourier transform and the Satake transform is given on [**M**, p. 47]. The Plancherel measure $d\mu(\omega)$ is a positive measure on $\Omega_K^+(G)$ such that, for all $f \in \mathscr{H}(G, K)$,

$$f^\vee(\omega) \in L^2(\Omega_K^+(G), d\mu) \tag{1.13}$$

and

$$\int_{G(F)^1} |f(g)|^2\,dg = \int_{\Omega_K^+(G)} |f^\vee(\omega)|^2\,d\mu(\omega). \tag{1.14}$$

By a theorem of Godement, such a measure exists and is unique.

LEMMA 1.5 (Macdonald [**M**, Theorem 5.1.2]). *The support of the Plancherel measure is the complex torus* (1.7). *Let* $s = (s_1, \dots, s_l) \in \mathscr{A}_{A,\mathbf{C}}^*$, *let* ds *denote the Haar measure on* $\mathscr{A}_{A,\mathbf{C}}^*/L$ *having total volume* 1, *and let* $d\chi$ *be the corresponding Haar measure on* $\mathrm{Hom}_{\mathrm{cont}}(A(F), \mathbf{C}^{\times 1})$ *obtained by transport of measure by* (1.7). *The Plancherel measure of* $G(F)$ *with respect to* K *is*

$$d\mu(\chi) = \frac{Q}{|W|}|c(\chi)|^{-2}\,d\chi.$$

COROLLARY 1.16. *For all* $f \in \mathscr{H}(G, K)$, *we have*

$$f(g) = \int_{\Omega_K^+(G)} f^\vee(\omega)\overline{\omega(g^{-1})}\,d\mu(\omega)$$
$$= \int_{\mathscr{A}_{A,\mathbf{c}}^*/L} f^\vee(\chi)\Gamma_\chi(g)\,d\mu(\chi).$$

Also, for all $f_1, f_2 \in \mathscr{H}(G, K)$, *we have*

$$\int_{G(F)^1} f_1(g)\overline{f_2(g)}\,dg = \int_{\Omega_K^+(G)} f_1^\vee(\omega)\overline{f_1^\vee(\omega)}\,d\mu(\omega)$$
$$= \int_{\mathscr{A}_{A,\mathbf{c}}^*/L} f_1^\vee(\chi)\overline{f_1^\vee(\chi)}\,d\mu(\chi).$$

The Jacquet functor. The maximal compact subgroup K has the property that for any parabolic subgroup $P = MN$ of G, $G(F) = P(F)K$, and for each Levi M of G, and every parabolic P^M of M, we also have $M(F) = P^M(F)(K \cap M(F))$ [**Car**, p. 140]. In this case, the notion of "compactly induced" representations [**BZ**, §1.8] agrees with the usual notion of "unitarily induced" representations. Let

$$i_{G,M}\colon \operatorname{Alg} M \to \operatorname{Alg} G, \tag{1.17}$$

denote unitary induction, in the notation of [**BZ**], and let

$$r_{M,G}\colon \operatorname{Alg} G \to \operatorname{Alg} M \tag{1.18}$$

denote the Jacquet functor [**BZ**, §2.3] (called the first Jacquet functor in [**Car**, §2.2]). We shall sometimes write $\pi_N = r_{M,G}(\pi)$ and $V_P(\tau) = i_{G,M}(\tau)$.

Let W_M denote the F-rational Weyl group of M. The special case of the "Geometrical lemma" of Bernstein-Zelevinsky which we need is the following

LEMMA 1.19 [**BZ**, §2.12]. *There is an enumeration* $w_1, \dots, w_k$ *of* W/W_M *(which we regard as a subgroup of* W *as in* [**Cas2**, §1]) *such that, for each* $\chi \in \operatorname{Alg} A$, *we have the following decomposition of* M-*modules*

$$r_{M,G} \circ i_{G,A}(\chi) = V_k \supset V_{k-1} \supset \cdots \supset V_1 \supset V_0 = \{0\},$$

where each

$$V_{j+1}/V_j \cong i_{M,A}(w_j\chi),$$

is an irreducible M-*module.*

Let $V_{(s)}$ denote the semi-simplification of an M-module V.

COROLLARY 1.20. *For each* $\chi \in \operatorname{Alg} A$, *we have the following decomposition of* M*-modules*

$$r_{M,G} \circ i_{G,A}(\chi)_{(s)} \cong \bigoplus_{w \in W/W_M} i_{M,A}(w\chi),$$

where the coset representations w *are chosen as in Lemma* 1.19.

REMARK. If χ is regular, so $w\chi \neq \chi$ for all $w \in W/W_M$, then a result of Casselman [**Cas2**, §§3, 6] implies that $r_{M,G} \circ i_{G,A}(\chi)$ is a semi-simple M-module.

Proof. Follows from Lemma 1.19 and the definition of the semi-simplification. □

2. Inner products of some matrix coefficients.

Matrix coefficients. Assume as before that G is connected, unramified, and reductive. Let $P = MN$ denote a standard parabolic subgroup of G. We often write G in place of $G(F)$ when there is no confusion.

We choose measures da, dn, dg so that $\operatorname{meas}(A(F) \cap K) = \operatorname{meas}(N(F) \cap K) = \operatorname{meas}(K) = 1$, let χ denote an unramified regular character of $A(F)$, and let $V_B(\chi)$ denote the space of the full principal series representation induced unitarily from χ. The elements of $V_B(\chi)$ may be regarded as functions on G determined by their restriction to K. Let R_M denote the restriction map sending locally constant functions on K to functions on $M(F) \cap K$. From Casselman [**Cas2**, §4], there is a canonical pairing $\langle \cdot, \cdot \rangle_N$ on $V_B(\chi)_N \times V_B(\chi^{-1})_N$ such that, for all $f, f' \in V_B(\chi)$, there is an $\varepsilon > 0$ (depending on f, f' but independent of χ) satisfying

$$\langle i_{G,A}(\chi)(a)f, f' \rangle_G = \langle r_{A,G} \circ i_{G,A}(\chi) f_N, f'_N \rangle_N, \tag{2.1}$$

for all $a \in A^-(\varepsilon)$, where

$$A^-(\varepsilon) := \{a \in A(F)/Z(G(F)) \mid |\alpha(a)|_F \leq \varepsilon, \forall \alpha \in \Delta\},$$

and where $\langle \cdot, \cdot \rangle_G$ is as in [**Car**]. When $\varepsilon = 1$ we denote this by A^-. This pairing allows us to identify $V_B(\chi^{-1})_N$ with the contragredient of $V_B(\chi)_N$. The fact that ε is independent of (unramified) χ follows from [**Car**, §3]. The dependence of f, f' (once a basis $\{K_n\}$ of subgroup neighborhoods of the identity with respect to B has been fixed [**Cas2**, §1.4]) can be seen from [**Cas2**, §§4.1–4.2].

LEMMA 2.2. *For χ unramified regular, and any $a \in A^-$,*

$$r_{M,G}(i_{G,A}(\chi)(a)f) = \sum_{w \in W/W_M} i_{G,A}(w\chi)(a) f_w ,$$

where f_w is defined by $r_{M,G}(f) = f_N = \bigoplus_w f_w$ for $f \in V_B(\chi)$. Here the decomposition is by Corollary 1.20 above and the coset representatives w are chosen (without further mention) as in Lemma 1.19.

Proof. This is an immediate consequence of the fact that

$$f_N = \sum_w R_M \circ T_w f \tag{2.3}$$

and hence

$$\begin{aligned}
(i_{G,A}(\chi)(a)f)_N &= \sum_w R_M \circ T_w i_{G,A}(\chi)(a) f \\
&= \sum_w \delta^{1/2}(a)(w\chi)(a) R_M \circ T_w f \\
&= \sum_w i_{G,A}(\chi)_N(a) R_M \circ T_w v \\
&= i_{G,A}(\chi)_N(a) f_N .
\end{aligned}$$

LEMMA 2.4. *If the image of $f \in V_B(\chi)$ under the Jacquet functor is $f_N = \bigoplus_{w \in W/W_M} f_w \in V_B(\chi)_N$ and the image of $f' \in V_B(\chi^{-1})$ under the Jacquet functor is $f'_N = \bigoplus_{w \in W/W_M} f'_w \in V_B(\chi^{-1})_N$ then*

$$\langle f_N , f'_N \rangle_N = \sum_{w \in W/W_M} c(w , M , \chi) \langle f_w , f'_w \rangle_M ,$$

for some constants $c(w , M , \chi)$ (to be determined later) and where

$$\langle u , u' \rangle_M := \int_{M(F) \cap K} u(k) u'(k)\, dk .$$

REMARK. We remark that for our choice of Haar measures, if $u , u' \in \mathscr{H}(M , M \cap K)$ then $\langle u , u' \rangle_M = u(1) u'(1)$.

Proof. Identify $V_B(\chi)_N$ with the M-module $\bigoplus_{w \in W/W_M} i_{A,M}(w\chi^{-1})$. As vector spaces, any bilinear pairing on

$$V_B(\chi)_N \times B_V(\chi^{-1})_N$$

is of the form

$$\langle z , z' \rangle = \sum_{w , w' \in W/W_M} a_{w , w'} \langle z_w , z'_{w'} \rangle_0 ,$$

where the $a_{w,w'}$ are constants, $z = (z_w | w \in W/W_M) \in \bigoplus_w i_{M,A}(w\chi)$, $z' = (z'_{w'} | w' \in W/W_M) \in \bigoplus_{w'} i_{M,A}(w'\chi^{-1})$, and $\langle z_w, z'_{w'} \rangle_0$ denotes the pairing on $V_B^M(\chi) = i_{M,A}(\chi)$. Using (2.1), we want to show that $a_{w,w'} = 0$ if $w \neq w'$.

Suppose not: if $a_{w,w'} \neq 0$ for some $w \neq w'$, choose z to be such that every component is zero except z_w and choose z' to be such that every component is zero except $z'_{w'}$. We have

$$r_{M,G} \circ i_{G,A}(\chi)(a)|_{i_{M,A}(w\chi)} : z \mapsto \delta^{1/2}(a)(w\chi)(a)z,$$

and, of course, $(w ; \chi^{-1})(a) = (w'\chi)(a)^{-1}$. In particular,

$$\langle r_{M,G} \circ i_{G,A}(\chi)(a)z, z' \rangle \neq \langle z, r_{M,G} \circ i_{G,A}(\chi^{-1})(a)z' \rangle,$$

for $a \in A^-(\varepsilon)$. This is a contradiction. □

The following result generalizes Macdonald's formula (for split groups). An analogous result is in [W, Lemma I.3.1]. The proof given here, which is more of a verification than a derivation, is different from that in [W] in that we use Macdonald's formula (twice, in fact) to evaluate the coefficients instead of a direct calculation.

LEMMA 2.5. *Let χ be as in* (2.2), *$f \in V_B(\chi)$, and $f' \in V_B(\chi^{-1})$. For $a \in A^-(\varepsilon)$, we have*

$$\langle i_{G,A}(\chi)(a)f, f' \rangle = \sum_{w \in W/W_M} c(w, M, \chi)\delta^{1/2}(a)(w\chi)(a)\langle R_m \circ T_w f, R_m \circ T_w f' \rangle_M,$$

where $\varepsilon > 0$ depends only on the level of f, f'. Here $c(w, M, \chi)$ is given by

$$Q_G^{-1} Q_M \frac{c(w\chi)}{c_w^M(\chi) c_w^M(\chi') c_M(w\chi)}.$$

In particular, if $M = C_G(A)$ then

$$c(w, A, \chi) = Q^{-1} \frac{c(w\chi)}{c_w(\chi) c_w(\chi^{-1})},$$

as in [M] (*for $f = \Phi_{K,\chi}$ and $f' = \Phi_{K,\chi'}$*).

REMARK. Suppose that f is bi-invariant under $K_f \subset K$ and f' is bi-invariant under $K_{f'} \subset K$. We will use the fact that we may choose $\varepsilon > 0$ once and for all with the property that the analogous identity holds true (with this fixed value of ε) even if G is replaced by a

Levi M and f is replaced by $f(w, k) := i_{M,A}(w\chi)R_M T_w f$, where $w \in W/W_M$ and $k \in K$ are arbitrary.

Proof. The identity itself, modulo the evaluation of the constants, follows from Lemmas 2.2 and 2.4. To evaluate the constants when $M = C_G(A)$, take $f = \Phi_{K,\chi}$ and $f' = \Phi_{K,\chi^{-1}}$ (in the notation of [**Car**]). Then $\langle i_{G,A}(\chi)(a)f, f'\rangle = \Gamma_\chi(a)$, by [**Car**, p. 151]. Moreover,

$$T_w f = T_w \Phi_{K,\chi} = c_w(\chi)\Phi_{K,w\chi},$$

and

$$T_w f' = T_w \Phi_{K,\chi^{-1}} = c_w(\chi^{-1})\Phi_{K,w\chi^{-1}},$$

by (1.8), so by the remark following Lemma 2.4, the identity becomes

$$\Gamma_\chi(a) = \sum_{w\in W} c(w, A, \chi)\delta(a)(w\chi)(a)(w\chi^{-1})(a)c_w(\chi)c_w(\chi^{-1}).$$

Comparing this with Macdonald's formula (1.9) gives the result claimed when $M = C_G(A)$.

In general we must proceed as follows. Taking f, f' as above, we find that

$$\begin{aligned}\Gamma_\chi(a) &= \sum_{w\in W/W_M} c(w, M, \chi)\langle i_{M,A}(\chi)(a)T_w\Phi^M_{K^M,\chi}, T_w\Phi^M_{K^M,\chi^{-1}}\rangle_M \\ &= \sum_{w\in W/W_M} c(w, M, \chi)\Gamma^M_{w\chi}(a)c^M_w(\chi)c^M_w(\chi') \\ &= \delta(a)^{1/2}\sum_{w'\in W}(w'\chi)(a)c(w'\chi).\end{aligned}$$

The last equality is just Macdonald's formula for G. On the other hand, Macdonald's formula for M states that

$$\Gamma^M_{w\chi}(a) = \delta_M(a)^{1/2}\sum_{v\in W_M}(vw\chi)(a)c_M(vw\chi).$$

Plugging this into the above equation gives

$$\begin{aligned}\Gamma_\chi(a) = \delta_M(a)^{1/2}Q_M^{-1}&\sum_{w\in W/W_M} c(w, M, \chi) \\ &\times \sum_{v\in W_M}(vw\chi)(a)c_M(vw\chi)c^M_w(\chi^{-1}).\end{aligned}$$

Comparing these two equations gives

$$c(w, M, \chi) = \frac{c(vw\chi)}{c_M(vw\chi)c^M_w(\chi)c^M_w(\chi^{-1})},$$

for any $v \in W/W_M$. Taking $v = 1$ gives the lemma. $\square$

LEMMA 2.6. *Let χ, χ' be two regular unramified characters of $A(F)$ and let $u \in V_B(\chi^{-1})$ and $u' \in V_B(\chi'^{-1})$. For all $a \in A^-(\varepsilon)$, with $\varepsilon > 0$ as in* (2.5),

$$\begin{aligned}\int_{KaK} &\langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u\rangle_G \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u'\rangle_G \, dg \\ &= \operatorname{meas}(KaK) \sum_{w, w' \in W/W_M} c_M(a, w, \chi) c_M(a, w', \chi') \\ &\qquad\qquad \times \langle u(w, *), u'(w', *)\rangle_G,\end{aligned}$$

where $u(w, g) := \langle i_{G,A}(\chi)(a) R_M T_w \Phi_{K,\chi}, R_M T_w \pi(g) u\rangle_M$.

Proof. Using Lemma 2.5, we have

$$\begin{aligned}\int_{KaK} &\langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u\rangle_G \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u'\rangle_G \, dg \\ &= \operatorname{meas}(KaK) \int_k \langle i_{G,A}(\chi)(a)\Phi_{K,\chi}, \pi(k)u\rangle_G \\ &\qquad\qquad \times \langle i_{G,A}(\chi')(a)\Phi_{K,\chi'}, \pi(k)u'\rangle_G \, dk \\ &= \operatorname{meas}(KaK) \sum_{w, w' \in W/W_M} c(w, M, \chi) \\ &\quad \times c(w', M, \chi') \int_K \langle i_{G,A}(\chi)(a) R_M T_w \Phi_{K,\chi}, R_M T_w \pi(k) u\rangle_M \\ &\quad \times \langle i_{G,A}(\chi')(a) R_M T_{w'} \Phi_{K,\chi'}, R_M T_{w'} \pi(k) u'\rangle_M \, dk,\end{aligned}$$

where $\pi := i_{G,A}(\chi)|_K = i_{G,A}(\chi')|_K$ is independent of (unramified) χ. Plugging Casselman's (1.8) into this, we find that the above equation equals

$$\begin{aligned}\delta(a) &\operatorname{meas}(KaK) \sum_{w, w' \in W/W_M} c(w, M, \chi) \\ &\times c(w', M, \chi') c_w(\chi)(w\chi)(a) c_{w'}(\chi)(w'\chi')(a) \\ &\times \int_K \langle R_M \Phi_{K, w\chi}, R_M T_w \pi(k) u'\rangle_M \\ &\qquad\qquad \times \langle R_M \Phi_{K, w', \chi'}, R_M T_{w'} \pi(k) u''\rangle_M \, dk.\end{aligned}$$

Putting these equations together gives the desired result. □

A truncated inner product of matrix coefficients. Let $T \in \mathscr{A}_{A,\mathbf{R}}$ and assume that $d(T) := \inf_{\alpha\in\Delta}\langle T, \alpha\rangle$ is positive, so T belongs to the positive Weyl chamber. The set

$$\begin{aligned}(2.7)\qquad \mathscr{A}_A(T) := \{a \in X_*(A) \bmod \nu_A(Z(G(F))) \mid &\langle a, \alpha\rangle \geq 0, \\ &\langle \alpha^\vee, a - T\rangle < 0, \ \forall \alpha \in \Delta\},\end{aligned}$$

is obviously finite. By (1.4), we may choose T as above so that

$$\nu_A^{-1}(\mathscr{A}_A(T)^c) \subset A^-(\varepsilon),$$

where $\mathscr{A}_A(T)^c = \nu_A(A^-) - \mathscr{A}_A(T)$. Define $G(T) := K\nu_A^{-1}(\mathscr{A}_A(T)^c)K \subset K \cdot A^-(\varepsilon) \cdot K$.

Using the bijection (1.5), we define

$$\begin{aligned}(2.8)\qquad & J^T(\chi, \chi', u', u'') \\ &:= \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u'\rangle_G \\ &\qquad\qquad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u''\rangle_G \, dg.\end{aligned}$$

Before calculating this, we need the following

Lemma 2.9. *Let χ be a regular unramified character of $A(F)$ and let $T = \sum_\alpha T_\alpha \alpha^\vee \in X_*(A) \subset \mathscr{A}_{A,\mathbf{R}}$ be as above. For each subset $\omega \subset \Delta$ with corresponding parabolic $P = P_\omega$, there is an entire function of χ, denoted $F_{\omega,T}(\chi) = F_{P,T}(\chi)$, uniformly bounded on the support of the Plancherel measure* (1.7), *such that*

$$\begin{aligned}\sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} \chi(a) &= \sum_{\omega \subset \Delta} F_{\omega,T}(\chi) \prod_{\alpha \in \omega} (1 - \chi(a_\alpha))^{-1} \\ &= \prod_{\alpha \in \Delta} (1 - \chi(a_\alpha))^{-1} \sum_{\omega \subset \Delta} F_{\omega,T}(\chi) \prod_{\substack{\alpha \notin \omega \\ \alpha \in \Delta}} (1 - \chi(a_\alpha)) \\ &= \tilde{\theta}(\chi)^{-1} \sum_{P_0 \subset P} F_{P,T}(\chi) \prod_{\alpha \in \Delta - \Delta_P} (1 - \chi(a_\alpha)),\end{aligned}$$

where $\tilde{\theta}$ is defined in (2.13) *below.* (*Of course, the zeros of $F_{\omega,T}(\chi)$ cancel with the poles of $\prod_{\alpha\in\omega}(1-\chi(a_\alpha))^{-1}$, since $\nu_A^{-1}(\mathscr{A}_A(T))$ is finite.*) *In fact, $F_{\omega,T}(\chi)$ may be written as*

$$F_{\omega,T}(\chi) = \prod_\alpha \chi(a_\alpha)^{T_\alpha} \cdot F_\omega(\chi),$$

where $F_\omega(\chi)$ is independent of T.

Remark. The statement of the lemma remains true if we replace $\nu_A^{-1}(\mathscr{A}_A(T))$ by $A^-(1)$, provided the sum is defined (either χ is in the product of half-planes where the sum converges absolutely, or, if χ belongs to the complement of this region define the sum by analytic continuation). In this case, the poles of this meromorphic function of χ are precisely those of $\prod_{\alpha\in\omega}(q - \chi(a_\alpha))^{-1}$.

Proof. First consider the part of $\nu_A^{-1}(\mathscr{A}_A(T))$ away from the walls:

$$\mathscr{A}_A(T)_{\mathrm{reg}} := \{a \in X_*(A) \mod \nu_A(Z(G(F)))| \\ \langle a, \alpha\rangle > 0, \langle \alpha^\vee, a - T\rangle < 0, \ \forall \alpha \in \Delta\}.$$

We can assume $\chi(a)$ is of the form

$$\chi(a) = \prod_{\alpha\in\Delta} q^{s_\alpha n_\alpha} = \prod_{\alpha\in\Delta} q^{s_\alpha\langle a, \alpha\rangle},$$

where $\mathrm{Re}\, s_\alpha < 0$. In this case, one can see directly that

$$\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T)_{\mathrm{reg}})} \chi(a) = F_{\Delta, T}(\chi) \prod_{\alpha\in\Delta}(1-\chi(a_\alpha))^{-1},$$

where $F_{\Delta,T}(\chi)$ is a polynomial in the $\chi(a_\alpha)$. Now let

$$\mathscr{A}_A(T)_\omega := \{a \in \mathscr{A}_A(T)|\langle a, \alpha\rangle < 0, \\ \forall\alpha \in \omega, \langle a, \alpha\rangle = 0, \ \forall \alpha \in \Delta - \omega\},$$

so $\mathscr{A}_A(T)_\Delta = \mathscr{A}_A(T)_{\mathrm{reg}}$ and

$$\mathscr{A}_A(T) = \coprod_{\omega\in\Delta} \mathscr{A}_A(T)_\omega. \tag{2.10}$$

In each case, one can see directly that

$$\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T)_\omega)} \chi(a) = F_{\omega, T}(\chi) \prod_{\alpha\in\omega}(1-\chi(a_\alpha))^{-1}, \tag{2.11}$$

and the result follows. □

From [M, Proposition 3.2.15] we find that, for $a \in A^-(1)$,

$$\mathrm{meas}(KaK) = \delta(a)^{-1} Q_a, \tag{2.12}$$

where $Q_a = Q_\omega \in \mathbf{Z}$ is constant on each $\nu_A^{-1}(\mathscr{A}_A(T)_\omega)$. The case of the above lemma which we will need is the following. Assume that χ, χ' are unramified regular unitary characters of $A(F)$. Then

$$\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T))} \sum_{w, w'\in W} (w\chi)(a)(w'\chi')(a) Q_a,$$

equals

$$\sum_{w, w'\in W} \tilde{\theta}(w\chi w'\chi')^{-1} \sum_{P_0\subset P} Q_P F_{P,T}(w\chi w'\chi') \\ \times \prod_{\alpha\in\Delta-\Delta_P} (1-(w\chi w'\chi')(a_\alpha)), \tag{2.13}$$

where

$$\tilde{\theta}(\chi) := \prod_{\alpha\in\Delta}(1-\chi(a_\alpha)).$$

PROPOSITION 2.14. *Assume that χ, χ' are unramified regular characters of $A(F)$, that $u \in V_B(\chi^{-1})$, $u' \in V_B(\chi'^{-1})$, and that $T \in \mathscr{A}_{A,\mathbf{R}}$ is chosen as above (depending on G and the "level" of u, u'). We have*

$$J^T(\chi, \chi', u, u') = \sum_{w, w' \in W} C^T(\chi, \chi', w, w')\langle T_w u, T_{w'} u'\rangle_G.$$

If $T \in X_*(A)$ is chosen sufficiently regular as in Lemma 2.9, then $C^T(\chi, \chi', w, w')$ may be effectively calculated using Lemma 2.9 and (2.13). In any case, $C^T(\chi, \chi', w, w')$ is meromorphic in χ and χ' and has no poles on the support of the Plancherel measure (1.7). (Note that the sum here is over W and not W/W_M.)

REMARK. We only indicate below the formal derivative of the formula, referring the proof of the statement about the poles and meromorphicity to Lemma 2.9 and (2.13).

Proof. The proof is by induction on the semi-simple rank of G.

If the semi-simple rank of G is 0 then G is a torus and the result follows immediately from definition (2.8), (2.12), Lemma 2.6 and the case of Lemma 2.9 mentioned in (2.13). Indeed, in this case

$$\nu_A^{-1}(\mathscr{A}_A(T)) = \nu_A^{-1}(\mathscr{A}_A(T)^c)$$

so (2.12) gives

$$\text{(2.15a)} \qquad \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} \delta(a)(w\chi)(a)(w'\chi')(a)\,\mathrm{meas}(KaK) = \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} (w\chi)(a)(w'\chi')(a) Q_A.$$

By (2.13), this is

$$\text{(2.15b)} \qquad \sum_{\omega \subset \Delta} Q_\omega F_{\omega, T}(w\chi w'\chi') \prod_{\alpha \in \omega} (1 - (w\chi w'\chi')(a_\alpha))^{-1}.$$

Putting these together gives

$$\begin{aligned} J^T(\chi, \chi', u, u') &= \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u\rangle_G \\ &\qquad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u'\rangle_G \, dg \\ &= \sum_{w, w' \in W} \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} c_M(a, w, \chi) c_M(a, w', \chi')\langle T_w u, T_{w'} u'\rangle_G, \end{aligned}$$

by Lemma 2.6.

Now suppose that the semi-simple rank of G is greater than zero. By the induction hypothesis the result holds true for all proper Levi subgroups of G.

Let $D \subset \nu_A(Z(G)) \otimes \mathbf{R}$ be a subset invariant under translation by $\nu_A(Z(G))$, let $T, U \in \mathscr{A}_{A,\mathbf{R}}$ be such that $d(U-T) > 0$ with T as above. Let $\xi(M, D, T, U)$ denote the characteristic function of the set of $X \in \mathscr{A}_{A,\mathbf{R}}$ such that

$$\begin{array}{lll} & X_G \in D, & \\ \langle \alpha, X^M \rangle \geq 0, & \langle \hat{\alpha}, X^M \rangle \leq \langle \hat{\alpha}, T^M \rangle, & \forall \alpha \in \Delta^M, \\ \langle \alpha, X_M \rangle > \langle \alpha, T_M \rangle, & \langle \hat{\alpha}, X_M \rangle \leq \langle \hat{\alpha}, U_M \rangle, & \forall \alpha \in \Delta^G - \Delta^M. \end{array}$$

Let $I_D(X)$ equal 1 if $X_G \in D$ and equal 0 otherwise. Observe that the statement and proof of [**W**, Lemma II.3.1], in the context of $\mathrm{GL}(n)$, is valid without change for the more general class of groups G used here. Multiplying both sides of the equation in [**W**, Lemma II.3.1] by I_D we obtain the following equation (see also [**W**, p. 15]):

$$\xi(G, D, U, U) = \sum_{M \subset G} \xi(M, D, T, U).$$

We will use the same notation for the pull-back of $\xi(M, D, T, U)$ to $A(F)$ via ν_A in (1.4).

Let

$$\begin{aligned} (2.16)\ J_D^U(\chi, \chi', u, u') := & \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(U))} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u \rangle_G \\ & \qquad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u' \rangle_G \, dg \\ = & \sum_{a \in A^-} \xi(G, D, U, U)(a) \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u \rangle_G \\ & \qquad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u' \rangle_G \, dg \\ = & \sum_{M \subset G} \sum_{a \in A^-} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u \rangle_G \\ & \qquad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u' \rangle_G \, dg \cdot \xi(M, D, T, U)(a) \\ := & \sum_{M \subset G} J_D^{T,U}(M, \chi, \chi', u, u'), \end{aligned}$$

where each $J_D^{T,U}(M, \chi, \chi', u, u')$, defined by the identity above, depends on T and U but the sum over M of them depends only on U.

To verify the proposition we calculate the $J_D^{T,U}(M, \chi, \chi', u, u')$ inductively. We consider the cases $M = G$ and $M \neq G$ separately.

Case $M = G$. In this case there is no dependence on U:

$$\begin{aligned}(2.17)\ J_D^{T,U}(G, \chi, \chi', u, u') &= \sum_{a \in A^-} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u\rangle_G \\ &\quad \times \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u'\rangle_G \, dg \cdot \xi(G, D, T, T)(a) \\ &= J_D^T(\chi, \chi', u, u').\end{aligned}$$

Case $M \neq G$. In this case the semi-simple rank of M is strictly less than that of G, so the induction hypothesis is applicable to M. Suppose $X \in a_B$ is such that $\xi(M, D, T, U)(X) = 1$. For each $\alpha \in \Sigma^G - \Sigma^M$ with $\alpha > 0$ there exists a $\beta \in \Delta^G - \Delta^M$ such that $\alpha - \beta$ is a positive root, so $\langle \alpha, X\rangle \geq \langle \beta, X\rangle$. We thus have $\langle \alpha, X\rangle \geq \langle \beta, X\rangle \geq \langle \beta, T\rangle$. With T chosen sufficiently regular, Lemma 2.6 gives

$$\begin{aligned}(2.18)\ \int_{KaK} &\langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u\rangle_G \langle i_{G,A}(\chi')(g)\Phi_{K,\chi'}, u'\rangle_G \, dg \\ &= \sum_{w, w' \in W/W_M} \mathrm{vol}(KaK) \\ &\quad \times \int_K \langle i_{G,A}(w\chi)(a)R_M\Phi_{K,w\chi}, R_M T_w \pi(k)u'\rangle_M \\ &\quad \times \langle i_{G,A}(\omega'\chi')(a)R_M\Phi_{K,w',\chi'}, R_M T_{\omega'}\pi(k)u''\rangle_M \, dk \\ &= \sum_{w, w' \in W/W_M} \mathrm{vol}(KaK) \\ &\quad \times \int_K \langle i_{G,A}(w\chi)(a)\Phi^M_{K^M,w\chi}, u(w, k)\rangle_M \\ &\qquad\qquad \times \langle i_{G,A}(w'\chi')(a)\Phi^M_{K^M,w'\chi'}, u'(w', k)\rangle_M \, dk,\end{aligned}$$

where $u(w, k) := R_M T_w \pi(k)u$. Concerning the integral in this last expression, reversing the reasoning in the proof of Lemma 2.6 above gives

$$(2.19)\quad \int_K \langle i_{G,A}(w\chi)(a)\Phi^M_{K^M,w\chi}, u(w,k)\rangle_M \times \langle i_{G,A}(w'\chi')(a)\Phi^M_{K^M,w'\chi'}, u'(w',k)\rangle_M\, dk$$
$$= \int_K \int_{K^M} \langle i_{G,A}(w\chi)(a)\Phi^M_{K^M,w\chi}, u(w,hk)\rangle_M \times \langle i_{G,A}(w'\chi')(a)\Phi^M_{K^M,w'\chi'}, u'(w',hk)\rangle_M\, dh\, dk$$
$$= \delta_M(a)^{-1}\operatorname{vol}(K^M a K^M)^{-1} \times \int_K \int_{K^M a K^M} \langle i_{M,A}(w\chi)(m)\Phi^M_{K^M,w\chi}, u(w,k)\rangle_M \times \langle i_{M,A}(w'\chi')(m)\Phi^M_{K^M,w'\chi'}, u'(w',k)\rangle_M\, dm\, dk.$$

(This is the analog of the calculation on [**W**, bottom of p. 16].) We will now show that this last expression is the integral over K of the summand of $J_{D'}^{T,U,M}(w\chi, w'\chi', u(w,k), u(w',k))$, where D' will be defined below. This inner product

$$J_{D'}^{T,U,M}(w\chi, w'\chi', u(w,k), u(w',k))$$

is an M-analog of our original inner product, so the induction hypothesis applies.

In more detail, by (2.12) there is a constant c_M independent of a such that

$$c_M\delta_M(a)\delta(a)^{-1} = \operatorname{vol}(KaK)\operatorname{vol}(K^M a K^M)^{-1}.$$

Now plug (2.19) into (2.18) to get

$$(2.20)\quad J_D^{T,U}(M,\chi,\chi',u,u') = c_M \sum_{w,w'\in W/W_M} \int_K \left[\sum_{a\in A^-} \xi(M,D,T,U)(a) \times \int_{K^M a K^M} \langle i_{M,A}(w\chi)(m)\Phi^M_{K^M w\chi}, u(w,k)\rangle_M \times \langle i_{M,A}(w'\chi')(m)\Phi^M_{K^M,w'\chi'}, u'(w',k)\rangle_M\, dm \right] dk.$$

Denote by $D_M(T,U)$ the set of $X \in \mathscr{A}_{A,\mathbf{R}}$ such that

$$X_G \in D, \quad \langle\alpha, X-T\rangle > 0 \quad\text{and}\quad \langle\hat\alpha, X-U\rangle \le 0, \quad \forall\alpha\in\Delta^G-\Delta^M.$$

We have ([**W**, p. 17])

$$\xi(M, D, T, U) = \xi^M(M, D_M(T, U), T, T), \tag{2.21}$$

which gives our D' mentioned above. Putting together (2.20), (2.21), and the definition of the inner product integral, we obtain

$$J^{T,U}(M, \chi, \chi', u, u') = c_M \sum_{w, w' \in W/W_M} \int_K J^T_{D_M(T,U)}(w\chi, w'\chi', u(w, k), u'(w', k))\, dk.$$

Note that the T chosen above depends only on G and the "level" of u and u'. We want to apply the induction hypothesis with u and u' replaced by $u(w, k)$ and $u'(w', k)$, but with the same T. To check that this is valid it suffices to check that the level of $u(w, k)$ and $u'(w', k)$ in M is not worse than the level of u and u' in G. Since W is finite, K is compact, and u, u' are supported in some fixed compact set, we may fix T so large that the induction hypothesis applies to u, u' and all the $u(w, k)$, $u'(w', k)$. Applying the induction hypothesis to $J^T_{D_M(T,U)}(w\chi, w'\chi', u(w, k), u'(w', k))$, we obtain

$$\begin{aligned} J_D^{T,U}(M, \chi, \chi', u, u') &= c_M \sum_{w, w' \in W/W_M} \int_K \sum_{v, v' \in W_M} C_M^T(w\chi, w'\chi', v, v') \\ &\qquad \times \langle T_v^M u(w, k) T_{v'}^M u'(w', k)\rangle_M \, dk. \end{aligned} \tag{2.22a}$$

In fact, since $u(w, k) := R_M T_w \pi(k) u$ it follows that

$$\int_K \sum_{v, v' \in W_M} \langle T_v^M u(w, k), T_{v'}^M u'(w', k)\rangle_M \, dk = \sum_{v, v' \in W_M} \langle T_\sigma u, T_{\sigma'} u'\rangle_G$$

where $\sigma = w_l^G w_l^M v w$, $\sigma' = w_l^G w_l^M v' w'$, and w_l^M denotes the longest element of W_M (see [**W**, p. 17, eqs. (2), (3)]). Therefore,

$$\begin{aligned} J_D^{T,U}(M, \chi, \chi', u, u') &= c_M \sum_{w, w' \in W} C_M^{T,U}(v, v') \langle T_w u, T_{w'} u'\rangle_G \, dk, \end{aligned} \tag{2.22b}$$

where $C_M^{T,U}(v, v')$ takes the form

$$C_M^{T,U}(v, v') = c_M(-1)^{\mathrm{rk}(M)} \sum_{a \in A^-} \chi(M)(a) \gamma_a(\chi, \chi', v, v'). \tag{2.23}$$

Here $\gamma_a(\chi, \chi', v, v')$ is meromorphic in χ and χ', having no poles on the support of the Plancherel measure (1.7), and $\chi(M)$ denotes the characteristic function of the set of $a \in A^-$ such that

$$\begin{gathered} \nu_A(a)_G \in D, \\ \langle \alpha, \nu_A(a)_M - T_M \rangle > 0, \quad \langle \hat{\alpha}, \nu_A(a)_M - U_M \rangle > 0, \quad \forall \alpha \in \Delta^G - \Delta^M, \\ \langle \hat{\alpha}, \nu_A(a)^M - T^M \rangle > 0, \quad \forall \alpha \in \Delta^M. \end{gathered}$$

Collecting equations (2.16), (2.17), (2.22), and (2.23), we get

(2.24)

$$\begin{aligned} J_D^{T,U}&(\chi, \chi', u, u') \\ &= J_D^{T,U}(G, \chi, \chi', u, u') + \sum_{\substack{M \subset G \\ M \neq G}} J_D^{T,U}(M, \chi, \chi', u, u') \\ &= J_D^T(\chi, \chi', u, u') + \sum_{w, w' \in W} \langle T_w u, T_{w'} u' \rangle_G \sum_{\substack{M \subset G \\ M \neq G}} C_M^{T,U}(v, v') \\ &= J_D^T(\chi, \chi', u, u') + c_M \sum_{w, w' \in W} \langle T_w u, T_{w'} u' \rangle_G \\ &\qquad \times \sum_{a \in A^-} \gamma_a(\chi, \chi', v, v') \sum_{\substack{M \subset G \\ M \neq G}} (-1)^{\mathrm{rk}(M)} \chi(M)(a). \end{aligned}$$

Here is where we apply a combinatorial lemma. In the notation of **[Art3]**, we have

$$\begin{aligned} (2.25)\quad \chi(M)(a) \\ &= 1_D(\nu_G(a)_G) \Gamma_M^G(\nu_G(a) - T, U - T) \hat{\tau}^M(\nu_G(a) - T), \end{aligned}$$

where 1_D denotes the characteristic function of D and $\Gamma_M^G(X, Y) = \tau_M^G(X - Y)\hat{\tau}^M(Y - X)$. By **[Morn**, Lemma 13.1.3, lecture 13] (or **[Art2**, §2]), we have

$$\begin{aligned} (2.26)\quad &\sum_{\substack{M \subset G \\ M \neq G}} (-1)^{\mathrm{rk}(M)} \chi(M)(a) \\ &\qquad = 1_D(\nu_G(a)_G)(-1)^{\mathrm{rk}(G)} [\hat{\tau}^M(\nu_A(a) - U) - \hat{\tau}^M(\nu_A(a) - T)]. \end{aligned}$$

The function $1_D(\nu_G(a)_G)\hat{\tau}^M(\nu_G(a) - T)$ is the characteristic function of $\nu_A^{-1}(\mathscr{A}_A(T)^c)$. From (2.25) and (2.26), we obtain

$$\sum_{a\in A^-}\gamma_a(\chi, \chi', v, v')\sum_{\substack{M\subset G\\ M\neq G}}(-1)^{\mathrm{rk}(M)}\chi(M)(a)$$
$$= (-1)^{\mathrm{rk}(G)}\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(U)^c)}\gamma_a(\chi, \chi', v, v') - (-1)^{\mathrm{rk}(G)}$$
$$\times\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T)^c)}\gamma_a(\chi, \chi', v, v')$$
$$= (-1)^{\mathrm{rk}(G)}\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T))}\gamma_a(\chi, \chi', v, v') - (-1)^{\mathrm{rk}(G)}$$
$$\times\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(U))}\gamma_a(\chi, \chi', v, v').$$

Plugging these into (2.24), we obtain the proposition. Note that the dependence on T in the final expression is fictitious since the left-hand side depends only on U.

In fact, these sums can be rewritten using Lemma 2.9—see also (2.13). □

3. Integrating the kernel. *The Fourier transform of a truncated orbital integral.* Let

$$G(T) := \bigcup_{a\in\nu_A^{-1}(\mathscr{A}_A(T))} KaK, \tag{3.1}$$

and recall

(3.2) $\Gamma_\chi(g^{-1}hg)$
$$= \langle i_{G,A}(\chi)(h)i_{G,A}(\chi)(g)\Phi_{K,\chi},\ i_{G,A}(\chi^{-1})(g)\Phi_{K,\chi^{-1}}\rangle_G.$$

We wish to calculate, for $\phi\in C_c^\infty(G)$, the Fourier transform

$$\begin{aligned} I^T(\chi, \phi) &:= \int_{G(F)}\Gamma_\chi(h)\int_{G(T)}\phi(g^{-1}hg)\,dg\,dh \\ &= \int_{G(T)\times G(F)}\Gamma_\chi(g^{-1}hg)\phi(h)\,dh\,dg. \end{aligned}\tag{3.3}$$

The idea is to expand (3.2) into a double series using an orthogonal basis and, for each term in the expansion, use the computations of the previous section to evaluate (3.3).

PROPOSITION 3.4. *Let $T \in X_*(A)$ be as in Proposition 2.14 and let χ be unramified, regular character of $A(F)$. We have*

$$I(\chi, \phi) = \lim_{\chi' \to \chi} \sum_{w, w' \in W} Q^T(\chi, \chi', w, w', \phi)\tilde{\theta}(w\chi/w'\chi')^{-1},$$

independent of T(!), where $\tilde{\theta}$ is as in (2.13), and

$$\begin{aligned} &Q^T(\chi, \chi', w, w', \phi) \\ &\quad := \operatorname{tr}[T_w^* T_{w'} i_{G,A}(\chi)(\phi)] C^T(\chi, \chi'^{-1}, w, w')\tilde{\theta}(w\chi/w'\chi'), \end{aligned}$$

in the notation of (2.14). The map $\phi \mapsto I(\chi, \phi)$ is an invariant G-admissible distribution on G_{ell} in the sense of [HC]. Moreover, if $w \neq w'$ then

$$\lim_{\chi' \to \chi} Q^T(\chi, \chi', w, w', \phi)\tilde{\theta}(w\chi/w'\chi')^{-1} = 0.$$

Proof of 3.4. The operator adjoint to

$$T_w \colon V_B(\chi^{-1}) \to V_{B^w}(w\chi^{-1})$$

is

$$T_w \colon V_B(w\chi) \to V_B(\chi),$$

so

$$\langle T_w u, T_{w'} u' \rangle_G = \langle u, T_w^* T_{w'} u' \rangle_G. \tag{3.5}$$

Let $\{u_i | i \in I\}$ denote an $(A(F) \cap K)$-bi-invariant orthonormal basis for $V_B(\chi)$, $\{u_i^* | i \in I\}$ its dual basis for $V_B(\chi^{-1})$ (so $\langle u_i, u_j^* \rangle_G = \delta_{ij}$). Expand

$$i_{G,a}(\chi)(g)\Phi_{K,\chi} = \sum_i \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u_i^* \rangle u_i,$$

and

$$i_{G,A}(\chi^{-1})(g)\Phi_{K,\chi^{-1}} = \sum_j \langle i_{G,A}(\chi^{-1})(g)\Phi_{K,\chi^{-1}}, u_j \rangle u_j^*.$$

This and (3.2) give

$$\begin{aligned} \Gamma_\chi(g^{-1}hg) = &\sum_{i,j \in I} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u_i^* \rangle_G \\ &\times \langle i_{G,A}(\chi)(h)u_i, u_j^* \rangle_G \langle i_{G,A}(\chi^{-1})(g)\Phi_{K,\chi^{-1}}, u_j \rangle_G. \end{aligned}$$

Plugging this into the definition of $I^T(\chi, \phi)$ gives

$$\begin{aligned}
I(\chi, \chi', \phi) = &\sum_{a\in\nu_A^{-1}(\mathscr{A}_A(T))} \sum_{i,j\in I} \int_{KaK} \langle i_{G,A}(\chi)(g)\Phi_{K,\chi}, u_i^*\rangle_G \\
&\qquad\qquad\qquad\qquad \times \langle i_{G,A}(\chi^{-1})(g)\Phi_{K,\chi^{-1}}, u_j\rangle_G \, dg \\
&\times \int_{G(F)} \phi(h)\langle i_{G,A}(\chi)(h)u_i, u_j^*\rangle_G \, dh \\
= &\sum_{i,j\in I} J^T(\chi, \chi^{-1}, u_i^*, u_j^*) \\
&\times \int_{G(F)} \phi(h)\langle i_{G,A}(\chi)(h)u_i, u_j^*\rangle_G \, dh \\
= &\sum_{i,j\in I} J^T(\chi, \chi^{-1}, u_i^*, u_j^*)\langle i_{G,A}(\chi)(\phi)u_i, i_j^*\rangle_G .
\end{aligned}$$

By Proposition 2.14 and (3.5), this is

$$\begin{aligned}
&\sum_{i,j\in I} \sum_{w,w'\in W} C^T(\chi, \chi^{-1}, w, w') \\
&\qquad \times \langle u_i, T_w^* T_{w'} u_j^*\rangle_G \langle i_{G,A}(\chi)(\phi)u_i, u_j^*\rangle_G .
\end{aligned}$$

Since $\{u_i\}$, $\{u_i^*\}$ are orthonormal bases,

$$\sum_{i,j\in I} \langle u_i, T_w^* T_{w'} u_j^*\rangle_G \langle i_{G,A}(\chi)(\phi)u_i, u_j^*\rangle_G = \operatorname{tr}[T_w^* T_{w'} i_{G,A}(\chi)(\phi)] .$$

Collecting these results gives the first statement of the proposition, except for the claim that the result is independent of $T \in X_*(A)$. Putting together Proposition 3.4 and the evaluation of the coefficients $C^T(\chi, \chi', w, w')$ in (2.13), we obtain the last part of the proposition. The claim that the result is independent of T follows from Proposition 2.14.

It remains to prove the admissibility. From [**HC**, §14] it follows that the distribution $\phi \mapsto \operatorname{tr}[T_w^* T_{w'} i_{G,a}(\chi)(\phi)]$ is a meromorphic family of admissible distributions. Therefore, $\phi \mapsto Q^T(\chi, \chi', w, w', \phi)$ satisfies the G-admissibility property of [**HC**, §14]. Since $I(\chi, \phi)$ is the limit of a linear combination of the $Q^T(\chi, \chi', w, w', \phi)$, it also satisfies the G-admissibility property. This proves the proposition completely. □

Integrating the kernel. Let $\phi \in C_c^\infty(G)$ have $\operatorname{supp}\phi \subset G_{\mathrm{ell}}$. The support of the distribution

$$(3.7)\qquad f \mapsto I(f, \phi) := \int_{G(F)^1 \times G(F)^1} f(g^{-1}hg)\phi(h)\,dg\,dh = \int_{G(F)^1} f(h)\Phi_\phi(h)\,dh = \int_{G(F)^1} \Phi_f(h)\phi(h)\,dh,$$

for $f \in C_c^\infty(G)$, is compactly generated by a well-known lemma of Harish-Chandra. Here, for $h \in G_{\text{ell}}$,

$$\Phi_\phi(h) := \int_{G(F)^1} \phi(g^{-1}hg)\,dh,$$

with respect to ordinary Haar measure on $G(F)^1$. Let $f \in C_c^\infty(G)$ and let ϕ be any locally constant function with compact support in G_{ell} so that, writing

$$F(G)^1 = \bigcup_{a \in A^-(1)} KaK,$$

there are only finitely many cosets KaK which support f and ϕ.

The following lemma is a corollary of a well-known lemma of Harish-Chandra.

LEMMA 3.8. *Let ϕ be any locally constant function with compact support in G_{ell} and let $f \in \mathscr{H}(G, K)$. There is a compact set $C_{f,\phi} \subset G(F)$ for which $f(g^{-1}hg)\phi(h) \neq 0$ implies $g \in C_{f,\phi}$.*

For f and ϕ as above, by Lemma 3.8 we have

$$(3.9)\qquad \operatorname{supp}\left(\int_{G(F)^1} f(g^{-1}hg)\phi(h)\,dh\right) \subset K\nu_A^{-1}(\mathscr{A}_A(T))K,$$

where T is sufficiently large and satisfies the conditions of Proposition 2.14. Fix such a $T = T(f, \phi)$ and let

$$I^T(f, \phi) := \int_{G(T) \times G(F)^1} f(g^{-1}hg)\phi(h)\,dg\,dh,$$

so $I^T(f, \phi) = I(f, \phi)$. For $f \in \mathscr{H}(G, K)$, the Plancherel formula (1.16) gives

$$f(g^{-1}hg) = \int_{\mathscr{A}_{A,\mathbf{C}}^*/L} f^\vee(\chi)\Gamma_\chi(g^{-1}hg)\,d\mu(\chi).$$

From this we obtain the following “spectral expansion”:

THEOREM 3.10. *Let f and ϕ be as in Lemma* 3.8, *and* $T = T(f, \phi)$ *as in* (3.9). *We have*

$$\begin{aligned} I(f, \phi) &= \sum_{a \in \nu_A^{-1}(\mathscr{A}_A(T))} \int_{KaK \times G(F)^1} \\ &\qquad \times \int_{\mathscr{A}^*_{A,\mathbf{C}}/L} f^\vee(\chi) \Gamma_\chi(g^{-1}hg)\, d\mu(\chi) \phi(h)\, dg\, dh \\ &= \int_{\mathscr{A}^*_{A,\mathbf{C}}/L} f^\vee(\chi) I(\chi, \phi)\, d\mu(\chi), \end{aligned}$$

where $I(\chi, \phi)$ *is given by Proposition* 3.4.

The Weyl integration formula states that

$$\int_{G(F)^1} \psi(h)\, dh = \sum_T \frac{1}{|W_T|} \int_{T(F)} \Delta(t)^2 \operatorname{meas}(T(F)) \times \int_{T(F) \backslash G(F)^1} \psi(g^{-1}tg) \frac{dg}{dt}\, dt, \tag{3.11}$$

where T runs over a complete set of representatives of non-conjugate Cartans of $G(F)^1$ and W_T denotes the Weyl group of T. Taking f, ϕ as in Lemma 3.8, we have that $\int_C \phi(c)\, dc = 0$, for any regular non-elliptic conjugacy class $C \subset G(F)^1$. Plugging $\psi = f\Phi_\phi$ into (3.11), we obtain the "geometric" expansion:

$$I(f, \phi) = \sum_T \frac{1}{|W_T|} \int_{T(F)} \Delta(t)^2 \Phi_f(t) \Phi_\phi(t)\, dt, \tag{3.12}$$

where Φ_f is the orbital integral of f as above.

The equality between (3.10) and (3.12) may be regarded as a special case of Arthur's local trace formula.

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