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PSEUDO REGULAR ELEMENTS IN A NORMED RING

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Let A be an algebra, and let f be a linear mapping of A into some normed linear space C. For a in A we will write af for the image of a under f. By abf we mean $(ab)f$. Suppose $\|abf\| \leq M\|af\| \cdot \|bf\|$ for some real M, and all a, b in A. Then we will say that f is *pseudo regular* for A.

We study mainly the case when $C = A$ and A is a commutative Banach algebra. We present some conditions which imply pseudo regularity, and some that prevent it. For example, if the non-zero elements of the spectrum of f are bounded away from zero, then f is pseudo regular. A result (5.3) in the other direction is that if $\sum_{-\infty}^{\infty} |tf(t)|dt < \infty$ for a pseudo regular element f of $L^1(\mathbb{Z})$, then the spectrum is bounded away from 0. Concerning the algebra $C^1[a, b]$, any f which has no zero in common with its derivative is pseudo regular.

2. The relation to regularity. Behavior on extension. Let A be a normed commutative algebra and let $f \in A$. One says that f is *subregular in* A if there is another commutative normed algebra B which contains A isomorphically, which has a unit 1, and in which the element corresponding to f in B has an inverse.

(2.1) PROPOSITION. *If f is subregular in A, it is pseudo regular in A.*

Proof. Let f have the inverse g in an algebra B containing A. Let a and b belong to A. Then $\|afbfg\| = \|afbfg\| \leq \|af\| \, \|bf\| \, \|g\|$, so f is pseudo regular in A.

(2.2) PROPOSITION. *Pseudo regularity does not imply subregularity.*

Here an example will suffice. Take A to be the space $C(S, \mathbb{C})$ of continuous functions on some compact Hausdorff space S with a non-trivial open-and-closed subset E. The characteristic function e of E satisfies $e^2 = e$, so it cannot have an inverse in any B. Thus e is not subregular. On the other hand, [**A&G**, Theorem 3.3, or (3.22) below] shows that e is pseudo regular.

The next theorem is needed for our further examples, and comes close to giving the essence of pseudo regularity for algebras resembling function algebras. It is a restatement of **[A&G1**, Th. 3.6].

(2.3) THEOREM **[A&G1**, *Th.* 3.6]. *Let* A *be a function algebra. Let* f *belong to* A *and let* Σ *be the set of values* f *has on the Shilov boundary* ∂_A **[G**, 10]. *If there is now a real* M *such that* $\|a^2\| \leq M\|af\|^2$ *for all* a *in* A *then* 0 *is not a limit point of the rest of* Σ. *On the other hand, if* 0 *is not a limit point of the rest of* Σ, *then* f *is pseudo regular.*

(2.2) THEOREM. *Let* f *be an element of* A, *and let* A *be a subalgebra of a normed algebra* B. *If* f *is pseudo regular in* B, *it is pseudo regular in* A; *but not the other way around.*

The first half is trivial, and for the second, an example will suffice. Let A be the disc algebra $A(D)$ of those functions in $B = C(D, \mathbb{C})$ which are holomorphic in the interior of the disc D. Consider the complex variable z. It has an inverse (namely its complex conjugate) in the superalgebra $C(S^1, \mathbb{C})$ of the disc algebra $A(D)$. By (2.1), it is pseudo regular in A. But consider its spectrum as an element of $B = C(D, \mathbb{C})$. The Shilov boundary is the disc D, as is the set Σ, and 0 is a limit of the punctured disc. Hence, by (2.3), f is not pseudo regular in B.

3. Strongly pseudo regular elements. An element f of a normed algebra shall be called *strongly pseudo regular* if there exists a real number M such that for every pair of elements u, v, of A, and every positive integer n, there hold the inequalities

$$\|uvf^n\| \leq M^n \|uf^n\| \, \|vf^n\|. \tag{3.1}$$

The reason for introducing this concept is not merely its connection with pseudo regularity, but also because it can be neatly characterized.[1]

(3.2) THEOREM. *The statements* (3.21), (3.22), (3.23) *about an element of* f *of a semisimple*[2] *normed algebra* A *with unit are equivalent. The statements* (3.24) *and* (3.25) *are equivalent to each other.*

(3.21) f *is strongly pseudo regular,*

[1] Professor Johnson has shown that strong pseudo regularity is not equivalent to pseudo regularity. See **[J]** or (3.5) below.

[2] **[L**, 62]. We can do without the semi-simplicity by using the argument of Th. 3.4 below.

(3.22) *There is a superalgebra* B *of* A *which has an element* m *such that* $f = f^2 m$;

(3.23) 0 *is not a limit point of the rest of* Σ.

(3.24) *There is an element* m *of* A *such that* $f = f^2 m$,

(3.25) 0 *is not a limit point of the rest of the spectrum of* f.

Proof. Assume (3.21). Replace u in (3.1) by u^n and v by v^n. Take the nth root of both sides, and using [G, 5.2], obtain $\| u_A v_A f_A \| \leq M \| u_A f_A \| \| v_A f_A \|$. Here the heavy bars indicate the spectral norm and the suffix A denotes the Gel′fand transform. This says that F_A is pseudo regular as an element of the algebra of Gel′fand transforms. We refer to (2.3) and declare that (3.23) holds.

Next assume (3.23). Let B be the algebra of all continuous complex valued functions on the Shilov boundary S. Define m to be $1/f$ where f is not 0, and 0 otherwise. Then $f^2 m = f$, which shows (3.22); and (3.22) obviously implies (3.21).

Next, assume (3.25). Let σ be the spectrum of f. Let U_1 be a neighborhood of the origin in the complex plane. Let U_2 be a neighborhood of σ minus the origin. These U_i can and shall be chosen to be disjoint, precisely because of (3.31). Let η be a function which is 1 on the first set and 0 on the second. This function is holomorphic on a neighborhood of σ. We can use [G, 5.1 Theorem, 10] to obtain an element e of B which is 1 at the points of S where f is 0, and 0 where f is not 0. Clearly $fe = 0$. Moreover, the element $f + e$ is never 0 on the space of maximal ideals. So there is an m in A such that $m(f + e) = 1$. Hence $f = mf^2$, which is (3.24).

A comparison of (3.2) and (2.3) shows that for an algebra $C(S, \mathbb{C})$, pseudo regularity implies strong pseudo regularity.

Another application, (3.3), of (3.2) shows the same for a convolution algebra. Let G be a compact abelian group, and let A be the algebra $L^1(G)$ of integrable[3] complex valued functions on G, under convolution [L, 35D]. The space of maximal ideals is the character group Γ. Given an f in A, it has a Gel′fand transform f_A whose value at the point m in Γ is the Fourier coefficient [L, loc. cit.]

$$f_A(m) = \int_G f(\theta) m(\theta)\, d\theta .$$

[3] with respect to normalized Haar measure.

(3.3) THEOREM. *An* f *in* $L^1(G)$ *for which*

(3.31) *only a finite number of Fourier coefficients are non-zero*

is strongly pseudo regular. Conversely, (3.31) *holds if* f *is pseudo regular.*

Proof. Assume (3.31). Let m belong to Γ. If the mth Fourier coefficient of f is c_m and is not 0, let the mth Fourier coefficient of g be the reciprocal of c_m. Otherwise let it be also zero. This clearly defines a linear combination g of characters and thus an element of $L^1(G)$. It is easy to see that the Fourier transform of $f*f*g-f$ is 0 and hence that $f*f*g-f$ is 0. By (3.22), f is strongly pseudo regular.

Now suppose f is pseudo regular. Then there is a real M such that $\|u*u*f\| \leq M\|u*f\|^2$. Let u be one of the characters m [L, 38C]. Then $u*u=u$ and $\|u*f\|=|c_m|$. Call this positive number C. So $C(1-CM)\leq 0$. Thus if C is not 0 then it is not less than $1/M$, so of course there can be only finitely many C not 0, because Fourier transforms of L^1 functions vanish at infinity [L, 154-5].

We will go beyond (3.2) in two ways. In the first way, we consider algebras which are not semisimple. In the second, we enlarge f to be a finite set of elements.

For the remainder of this section, let A be a commutative Banach algebra with unit. For an element a of A there is the Gel'fand transform a_A, a function defined on the space of maximal ideals of A. It may happen that a_A vanishes identically. Then a is a *radical* element.

(3.4) THEOREM. *Let* A *be as above and let* f *be an element of* A. *Suppose that* (*as in* (3.23))

(3.41) 0 *is not a limit point of the spectrum* σ *of* f.

Then f *differs from a strongly pseudo regular element* g *in* A *by at most a radical element* r.

Proof. Construct the open set U_i and the function η as above in the proof of (3.2).

(3.42) Let γ be the function which is 0 on U_1 and z on U_2.

Then $\gamma + \eta$ is never 0 on the union U of U_1 and U_2. Obviously there is a function μ holomorphic on U such that $\mu(\gamma + \eta) = 1$. Hence $\gamma\mu(\gamma + \eta) = \gamma$ and indeed

(3.43) $$\gamma\mu\gamma = \mu$$

because $\gamma\eta = 0$, as is easily verified.

We now apply the analytic-functional calculus as established in [**A1**, see 5.1, p. 427; G, ch. 3] The four functions z, γ, η, and μ give rise to four elements f, g, e, and m of A and they satisfy the relation $\mu g = g$ because the relation (3.43) is preserved under the functional calculus. Comparing this with (3.22), we see that g is strongly pseudo regular.

We observe that $z - \gamma$ is 0 on the spectrum σ. Therefore $f_A - g_A$ is 0 on the space of maximal ideals whence $f - g$ is a radical element.

Thus (3.4) is established.

Professor B. E. Johnson has kindly communicated to me the next theorem, and its consequence (3.6). See [**J**].

(3.5) THEOREM[4]. *Let A be the Banach algebra $C^1[a, b]$. Let $f(t) = t$. Then f is a pseudo regular element of A.*

Proof. If 0 does not lie in $[a, b]$ then of course f is pseudo regular, but in any case the following argument will work.

Let J be the ideal of elements which vanish at 0. Let i belong to J. Define $q(i)$ as the function whose value is $i(t)/t$ for $t \neq 0$, and $i'(0)$ otherwise. In this proof, let the supremum of the absolute value of any complex valued bounded function h be denoted by $S(h)$. The norm $\|h\|$ of an element of A is $S(h) + S(h')$.

By the theorem of the mean

(3.51) $$S(q(i)) \leq S(i') \leq \|i\|.$$

If j is another element of J we have $S(q(i)j) \leq \|i\| \cdot \|j\|$.

We turn to $(q(i)j)'$. Its value at $t \neq 0$ is $tq(i)'(t)[j(t)/t] + q(i)(t)j'(t)$, and the obvious limit thereof for $t = 0$. Thus $S((q(i)j)') \leq S(tq(i)'(t))S[j(t)/t] + S(q(i)(t))S(j'(t))$. Now $S[j(t)/t] \leq \|j\|$ by (3.51). So $S((q(i)j)') \leq S(tq(i)'(t))\|j\| + \|i\| \cdot \|j\|$. As to $tq(i)'(t)$, it is $i'(t) + i(q)/t$, so again by (3.51), $S(tq(i)'(t)) \leq S(i') + S(i') \leq 2\|i\|$. Thus $S((q(i)j)') \leq 3\|i\| \cdot \|j\|$. Therefore

(3.52) $$\|q(i)j\| \leq 4\|i\| \cdot \|j\|.$$

[4] generalized in (3.9) below.

Take $i(t)$ to be $b(t)t$ where b is any element of A. So $i = bf$. It is easy to verify that $q(i) = b$ itself, so $\|bj\| \le 4\|bf\| \cdot \|j\|$. Now take $j = af$, and obtain the assertion that f is pseudo regular with $M \le 4$.

(3.6) COROLLARY (*B. E. Johnson*). *Pseudo regularity does not imply strong pseudo regularity.*

Indeed, when 0 lies in the interval $[a, b]$, the f above is not strongly pseudo regular by (3.23).

To this counterexample we may add another, namely (3.8) below. First another theorem.

(3.7) THEOREM. *Assume the hypotheses of* (3.5), *and take* $[a, b] = [0, 1]$. *Then* f^2 *is not pseudo regular.*

Proof. We use the S-notation of (3.5), but we use the norm $N(h) = |h(0)| + S(h')$ in A. This is equivalent to $\| \cdot \|$. We will study $f(t) = 1 - t$, rather than t. This helps in the notation. Assume $N(f^2g^2) \le N(f^2g)^2$ with $g(t) = t^n$. We will estimate $N(f^2g)$, which is $S((f^2g)')$. Now

$$(f^2g)' = [t^n - t^{n-1}][(n+2)t - n]. \tag{3.71}$$

The extremal points of this expression are the two zeros of $(n+1)(n+2)t^2 - 2n(n+1)t + (n-1)n$. We expand these roots in powers of $z = 1/n$. They are $t_{1,2} = 1 + \zeta z + \cdots$ where $\zeta = -2 \pm \sqrt{2}$. Inserting either of these into (3.71) gives an expression of the order of z, so

$$N(f^2g) \text{ is of the order of } z = 1/n. \tag{3.72}$$

This implies that $N(f^2g^2)$ is of the order of $1/2n$. So we get $K/n \le M(L/n)^2$ for all sufficiently large n. This forces M to be infinite. In other words, f^2 couldn't have been pseudo regular.

(3.8) COROLLARY. *The product of pseudo regular elements need not be pseudo regular.*

COROLLARY. *Let* f *belong to* A *and suppose* $f'(x)$ *is never* 0. *Then* f *is pseudo regular.*

To prove this, just change the variable to $t = f(x)$, and use (3.5).

So now we know that if either f is never 0 in $C^1[a, b]$, or f' is never 0, then f is pseudo regular. In fact, we can generalize this and (3.5) in one theorem.[5]

(3.9) THEOREM. *Let f belong to $C^1[a, b]$ and suppose f and f' have no common zero. Then f is pseudo regular.*

Proof. We will adapt Johnson's line of reasoning as presented in (3.5). Let J be the ideal elements of $C^1[a, b]$ which vanish on the set $Z = \{t_1, \ldots, t_n\}$ of zeros of f. For an element i of J we define $q(i)$ as $i(t)/f(t)$ or as $i'(t)/f'(t)$ according to whether t is not, or is, a zero of f.

For each k there is an open interval V_k containing t_k on which f' is bounded away from 0. Let V be the union of these V_k. Then $|f'| > r$ in V for some positive r. Moreover, $|f| > s$ for some positive s, outside of V. By multiplying f by some constant, we can make sure that 1 will serve as r and s.

Now suppose t is outside of V. Then $|q(i)(t)| \leq |i(t)|/|f(t)| \leq S(i)$.

Next suppose t is in V. If t is a zero of f we have $|q(i)(t)| = |i'(t)|/|f'(t)| \leq S(i')$. If t is not a zero of f then we can find a z which is a zero and such that the interval $[z, t]$ lies in V, then

$$q(i)(t) = \frac{i(t) - i(z)}{f(t) - f(z)} = \frac{i'(v)}{f'(v)}$$

for some v in $[z, t]$. Hence

(3.91) $$|q(i)(t)| \leq S(i') \quad \text{for all } t \text{ in } V.$$

We can therefore assert that

(3.92) $$S(q(i)) \leq S(i) + S(i') = \|i\|, \qquad \text{and} \qquad S(q(i)j) \leq \|i\| \cdot \|j\|$$

just as in (3.51). We now examine the $(q(i)j)'$. $q(i)j$ is ij/f off Z. Using Leibniz' rule yields $(q(i)j)' = i'q(j) + j'q(i) - q(i)q(j)f'$ on the (dense) complement of Z. Hence $S((q(i)j)') \leq S(i')S(q(j)) + S(j')S(q(i)) - S(q(i))S(q(j))S(f') \leq (2 + \|f'\|)\|i\| \cdot \|j\|$, by a multiple use of (3.92). We must also consider the difference quotients where one or both points are on Z. The derivative there is easily found to be $i'j'/f'$, since j is 0 on Z. Hence

(3.94) $$\|q(i)j\| = S(q(i)j) + S((q(i)j)') \leq (3 + \|f'\|)\|i\| \cdot \|j\|.$$

[5]Functions of several variables are discussed in §7 below.

Now take $i = af$ and $j = bf$, and conclude that f is pseudo regular.

Statement (3.7) shows that when f has a repeated zero, pseudo regularity may indeed fail.

4. Pseudo regular systems.

DEFINITION. Let F be a subset $\{f_1, \dots, f_N\}$ of A. Let A be a subalgebra of a second Banach algebra in which there exist elements $b_1, \dots, b_N$ such that $f_1 b_1 + \cdots + f_N b_N = 1$. Then F is called *subregular.*

DEFINITION. Let F be a subset $\{f_1, \dots, f_N\}$ of A. For each a in A define $T_F(a)$ to be $\|f_1 a\| + \cdots + \|f_N a\|$. Then F is a *pseudo regular system* if there is a real constant M such that for any a, b in A one has $T_F(ab) \le M T_F(a) T_F(b)$.

Pseudo regular system is the same sort of generalization of regular system [A] as pseudo regular element is of regular element.

Just for the record, we state without proof the obvious analogue of (2.1).

(4.1) PROPOSITION. *If F is subregular in A, it is pseudo regular in A.*

More substantial is the analogue of (3.2).

(4.2) THEOREM. *Let A be as above and let F be a finite set $\{f_1, \dots, f_N\}$ of elements of A. Suppose that*

(4.21) *the origin $\mathbf{0}$ is not a limit point of the joint spectrum σ of F.*

Then F differs from a pseudo regular system G in A by an additive N-tuple $(r_1, r_2, \dots, r_N)$ where the r_i are radical elements.

Proof. Find disjoint open sets U_1 and U_2 in complex N space where U_1 contains the origin $\mathbf{0}$ and U_2 contains the rest of σ. Define a function η to be 1 on U_1 and 0 on U_2. Using the analytic functional calculus gives us an idempotent e such that e_A is 1 when all the f_{iA} are 0, and 1 otherwise. Now $f_i = ef_1 + (1-e)f_i$. Let $g_i = f_i(1-e)$. Then $f_i - g_i = ef_i$. Now e_A is 0 when any of the f_{iA} are not 0 and f_{iA} is of course 0 when all the f_{jA} are 0. So $f_i - g_i$ is a radical element.

I declare that the N tuple $(e_A + (1 - e_A)f_{1A}, \dots, e_A + (1 - e_A)f_{NA})$ have no common 0. For when e_A is 1, then they are all 1, and when e_A is 0 they have the values of $(f_{1A}, \dots, f_{NA})$, which are not all 0 when e_A is 0.

So the g_i form a regular system, and there are elements m_i such that $1 = m_1(e + g_1) + \cdots + m_N(e + g_N)$. Fix a value of j and obtain $g_j = \sum_i m_i g_j g_i$ because $g_i e = 0$ for all i. Select any pair a, b from A and you have $g_j ab = \sum_i m_i g_j a g_i b$. Hence $\sum_j k\|g_j ab\| \le \sum_{i,j} \|g_j a\| \cdot \|g_i b\| \ M$, where M is the greatest of the norms of the m_i. Thus the g_i form a pseudo regular system.

5. Conditions preventing pseudo regularity. Let A be a subalgebra of a function algebra $C(S, \mathbb{C})$ where S is some compact Hausdorff space, and suppose that A separates [**G**, pp. 15, 4] the points of S and contains the unit.

REMARK. Let f be an element of such an algebra A. Suppose f is not pseudo regular in A. Then f must vanish somewhere on the Shilov boundary ∂_A, or it is not pseudo regular.

(5.1) THEOREM. *Let f be a non-zero element of such an algebra, and suppose the Shilov boundary ∂_A is connected. Then f is pseudo regular if and only if it does not vanish on ∂_A.*

For if f does vanish on ∂_A then the part of ∂_A where f is not 0 must be an open set Z. If Z is empty then f is 0 (and thus pseudo regular in a trivial way) or ∂_A is not connected.

We now turn to a normed algebra with a norm other than the sup norm. Consider $A = L^1(\mathbb{Z})$ as an algebra under convolution

(5.2) THEOREM. *Let $f = \{c(n) : n \in \mathbb{Z}\}$ belong to $L^1(\mathbb{Z})$ and suppose the Gel'fand transform series*

$$\sum_n c(n)e^{in\theta} \tag{5.21}$$

never vanishes on the unit circle. Then f is pseudo regular.

We make this well-known statement only to draw attention to the converse. If (5.21) is sometimes 0, must it be *not* pseudo regular? We can *almost* prove it.[6] Let $\|\cdot\|$ denote the usual norm in $A = L^1(\mathbb{Z})$. The operations are linear combination and convolution. We write simply fg for the convolution of f and g.

[6] *almost* because we also assume (5.31).

(5.3) **Theorem** [*Compare* **A&G2**, 3.1]. *Let* $f \in L^1(\mathbb{Z})$ *and suppose*

$$\sum_{-\infty}^{\infty} |tf(t)|\,dt < \infty, \tag{5.31}$$

(5.32) $\|u^2 f\| \le M\|uf\|^2$ *for some real* M *and at least for all* u *of finite support for which* $u(t) = 0$ *when* $t < 0$.

Then f *has an inverse in* $L^1(\mathbb{Z})$ *or* $f = 0$.

We insert two lemmas. We omit the proof of the first.

(5.4) **Lemma.** *Let* $s(p) = \sum_{-\infty}^{p} f(t)\,dt$. *Let* N *be a positive integer. Define* u *by setting* $u(t)$ *be* 1 *when* t *lies in the interval* $[0, N]$, *and* 0 *otherwise. Then*

$$uf(t) = s(t) - s(t - N - 1). \tag{5.41}$$

(5.5) **Lemma.** *Let* $J = \sum_{-\infty}^{0} |tf(t)|\,dt$. *Let*

$$D = f^{\wedge}(0) = \sum_{-\infty}^{\infty} f(t)\,dt$$

where $f^{\wedge}$ *is the Fourier transform of* f. *Let* $K = \sum_{0}^{\infty} |tf(t)|\,dt$. *Then*

$$\sum_{-\infty}^{0} |s(t)|\,dt \le J \tag{5.51}$$

and

$$\sum_{0}^{\infty} |s(t) - D|\,dt \le K. \tag{5.52}$$

These are readily obtained by reversing the order of summation. To derive (5.52) one starts by observing that

$$s(p) + \sum_{p+1}^{\infty} f(t) = D. \tag{5.53}$$

Define T by

$$T = \sum_{-\infty}^{\infty} |tf(t)|. \tag{5.54}$$

Then $J + K = T$.

(5.55) Define $h(t)$ to be 0 for $t < 0$ and 1 for all other values.

PROPOSITION. *According to* (5.41), $uf = s - s^{N-1}$ *where* s^{N+1} *is* s *shifted* $N+1$ *units to the right. As to its norm,*

(5.6) $$\|uf\| \leq (N+1)|D| + 2T.$$

Proof. $s - s^{N+1} = s - Dh + Dh - Dh^{N+1} + Dh^{N+1} - s^{N+1}$. Therefore $\|s - s^{N+1}\| \leq \|s - Dh\| + \|Dh - Dh^{N+1}\| + \|Dh^{N+1} - s^{N+1}\|$.

Now $\|s - Dh\| = \|Dh^{N+1} - s^{N+1}\|$ which is not greater than $J + K$ by (5.51) and (5.52). The term $\|Dh - Dh^{N+1}\| = (N+1)|D|$. From this, (5.6) follows.

We resume the proof of (5.3) by deducing from this that the right side of the inequality in (5.32) is $M[(N+1)|D| + 2T]^2$, and turn to the left side.

A real number α represents a point of the space of maximal ideals of A. The value of the Gel'fand transform of f there is $f^\wedge(\alpha)$. This has to be numerically at most equal to the norm of $u^2 f$, and hence, assuming (5.32),

$$|u^\wedge(\alpha)^2|\,|f^\wedge(\alpha)| \leq M[(N+1)|D| + 2T]^2.$$

Let us evaluate this for $\alpha = 0$, noting $u^\wedge(0) = N+1$. So $(N+1)^2|D| \leq M[(N+1)|D| + 2T]^2$. Since N is arbitrary, we have $|D| \leq M|D|^2$. Thus either $f^\wedge(0) = 0$, or $1/M \leq |f^\wedge(0)|$.

The property of pseudo regularity has the invariance property that for each character α, $e^{-i\alpha t} f(t)$ is pseudo regular if f is. Thus we either have

(5.7) $$f^\wedge(\alpha) = 0,$$

or

(5.8) $$1/M \leq |f^\wedge(\alpha)|.$$

If (5.7) holds for some α, it must hold for all, because $f^\wedge$ is continuous, and f must be 0. If (5.8) holds then $f^\wedge$ does not vanish anywhere on the space of maximal ideals, and hence f has an inverse.

It almost goes without saying that the converse is true, too.

6. The situation based on the action of $L^1(\mathbb{Z})$ on $L^2(\mathbb{Z})$. A $u \in L^1(\mathbb{Z})$ works on an element f in $L^2(\mathbb{Z})$ by sending it into $u * f$ in $L^2(\mathbb{Z})$. So we are defining uf as $u * f$ in this situation. Then f is pseudo regular if there is a real M such that $\|(u * v) * f\| \leq M\|u * f\|\,\|v * f\|$ where the norm is that of $L^2(\mathbb{Z})$.

(6.1) THEOREM. *Let* $f \in L^1(\mathbb{Z})$ *and suppose*

$$\sum_{-\infty}^{\infty} |tf(t)|\, dt < \infty, \tag{6.11}$$

(6.12) $\|u^2 f\| \le M\|uf\|^2$ *for some real* M *and at least for all those* u *in* $L^2(\mathbb{Z})$ *for which* $u(t) = 0$ *when* $t < 0$.

Then $f = 0$.

Proof. The norm $\|\cdot\|$ shall now refer to $L^2(\mathbb{Z})$. Define D, h, u and s as in (5.3)–(5.55). We want an upper estimate for $\|s - s^{N+1}\|$. $\|s - s^{N+1}\| \le \|s - Dh\| + \|Dh - Dh^{N+1}\| + \|Dh^{N+1} - s^{N+1}\|$. Now $\|Dh^{N+1} - s^{N+1}\| = \|s - Dh\|$, and our next step is to show that

$$\|s - Dh\| \quad \text{is finite}. \tag{6.13}$$

$$\begin{aligned}\|s - Dh\|^2 &= \sum_{p=-\infty}^{-1} |s(p)|^2 + \sum_{p=0}^{\infty} |s(p) - D|^2 \\ &= \sum_{p=-\infty}^{-1} \left|\sum_{t=-\infty}^{p} f(t)\right|^2 + \sum_{p=0}^{\infty} \left|\sum_{t=p+1}^{\infty} f(t)\right|^2 \\ &\le \sum_{p=-\infty}^{-1} \left|\sum_{t=-\infty}^{p} f_t\right|^2 + \sum_{p=0}^{\infty} \left|\sum_{t=p+1}^{\infty} f_t\right|^2\end{aligned}$$

where f_t is an abbreviation for $|f(t)|$. Let these two terms be called S_1 and S_2 respectively, for a moment. Take S_1 and let the index p be called $-q$. Then

$$S_1 = \sum_{q=1}^{\infty} \left|\sum_{t=q}^{\infty} f_{-t}\right|^2$$

which we will call T^-. Concerning S_2 we can surely say $S_2 \le \sum_{q=1}^{\infty} |\sum_{t=q}^{\infty} f_t|^2$ which sum we shall call T^+. We may rewrite T^+ as $\sum_{q=1}^{\infty} \sum_{t=q}^{\infty} \sum_{u=q}^{\infty} f_t f_u$. This is twice the sum over all lattice points for which $u \ge t \ge q \ge 0$. Thus

$$T^+ = 2 \sum_{u \ge t \ge 0} \sum_{q=0}^{t} f_t f_u \sum_{u \ge t \ge 0} (t+1) f_t f_u \le 2(T + U)U$$

where T is given in (5.54), and U is the L^1 norm of f. The reason for the $\leq$ is that in the summing for T^+ only the values of f_t with nonnegative suffix are used, whereas in U all suffixes are involved.

It is easy to see that T^- also is not greater than $2(T+U)U$, and consequently (6.13) is true.

Concerning $\|Dh - Dh^{N+1}\|$ it is easy to see that it is equal to $|D|\sqrt{N+1}$. This gives the dominant term on the right side of the inequality $\|u^2 f\| \leq M\|uf\|^2$ in (6.12). Squaring both sides it implies $\|u^2 f\| \leq (b + |D|\sqrt{N+1})^2$. By Parseval,

$$\|u^2 f\|^2 = (1/2\pi) \int_{-\pi}^{\pi} |u^\wedge|^4 |f^\wedge|^2 \, d\theta .$$

It is not hard to compute that $(1/2\pi)\int_{-\pi}^{\pi} |u^\wedge|^4 \, d\theta$ is a polynomial N_3 of degree 3 in N, and that $|u^\wedge|^4/N_3$ satisfy the conditions (i), (ii), (iii) of **[Z, 3.201]** associated with the concept of an approximate identity. Therefore $(1/2\pi)\int_{-\pi}^{\pi} p(1/N_3)|u^\wedge|^4|f^\wedge|^2 \, d\theta$ tends to $|f^\wedge(0)|^2$ as N goes to ∞. But $(b+|D|\sqrt{N+1})^2/N_3$ tends to 0, so $f^\wedge(0) = 0$.

Appealing again to the invariance which led to (5.7), we conclude that $f = 0$.

One might wonder what about (5.8). Apparently regular elements are not pseudo regular in this situation. If the identity element of $L^1(\mathbb{Z})$ were pseudo regular, then $L^2(\mathbb{Z})$ would be a Banach algebra.

7. C^1 algebras of functions of several variables. The general idea is that if the differential df is not 0 anywhere on the set Z of zeros of f, then f should be pseudo regular. In order to prove any theorems, we have to augment this hypothesis with some technical details which cannot be overlooked. We will assume that M is a Riemannian manifold, or a closed interval in some $\mathbb{R}^m$, and consider the algebra of C' functions on M. For any numerical valued function f on M we define $S(f)$ to be the sup of the values $|f(t)|$ for t in M. If f is C^1, we denote by f' the *gradient* of f. Let $|f'|$ be the length of f' and abbreviate $S(|f'|)$ by $S(f')$. We consider the algebra of those C^1 functions f for which $\|f\| = S(f) + S(f')$ is finite.

We will now define an f', u *curve* in M, where $u > 0$. It is a C^1 curve with tangents T such that $T \cdot f' > u|T| \cdot |f'|$.

Let V^r be the set of points where $|f| < r$.

We now enumerate the precise conditions imposed on f.

(7.1) There are r, $s > 0$ such that $|f'| > s$ on V^r.

(7.2) There is a $u > 0$ such that given any point t of V^r, there is an f', u curve lying in V^r and connecting t to Z.[7]

Let f satisfy these. It is enough to treat the case $r = s = 1$. We will abbreviate V^1 to V. We will follow the line of reasoning of (3.9). Let J be the ideal of elements of $C^1[M]$ which vanish on the set Z of zeros of F. Let i belong to J. We define $q(i)$ as

$$i(t)/f(t) \tag{7.3}$$

or as

$$i'(t) \cdot f'(t)/f'(t) \cdot f'(t) \tag{7.31}$$

according to whether t is not, or is, a zero of f. (7.31) obviously defines a function continuous on Z. It is not hard to show that (7.3) approaches (7.31) as t approaches a point of Z. So q is continuous on M.

Now suppose t is outside of V. Then $|q(i)(t)| \leq |i(t)|/|f(t)| \leq S(i)$. If t is in Z, we can see from (7.31) that $|q(i)(t)| \leq |i'(t)| \leq S(i') \leq \|i\|$ for such t.

Now suppose t is in V. Then we can find an f', u curve c leading from z in Z to t, c lying in V. Now $i(t) = \int i' \cdot T\, ds$, where the integral is over c, T is the unit tangent to c, and s is the arc length. By the theorem of the mean, $i(t) = i' \cdot Ts$, where now $i' \cdot T$ is evaluated somewhere along c, so inside V. The same thing holds for f. Now $|i' \cdot T| \leq S(i') \leq \|i\|$, and $uS(f')| \leq |f' \cdot T|$, so

$$|q(i)(t)| \leq \|i\|. \tag{7.32}$$

We can therefore assert that

$$S(q(i)) \leq S(i) + S(i') = \|i\|, \quad \text{and} \quad S(q(i)j) \leq \|i\| \cdot \|j\|. \tag{7.34}$$

We now examine the $(q(i)j)'$. $q(i)j$ is ij/f off Z. Using Leibniz' rule yields $(q(i)j)' = i'q(j) + j'q(i) - q(i)q(j)f'$ on the (dense) complement of Z. Hence $S((q(i)j)') \leq S(i')(q(j)) + S(j')S(q(i)) - S(q(i))S(q(j))S(f') \leq (2 + \|f'\|)\|i\| \cdot \|j\|$, by a multiple use of (7.34). We must also consider the difference quotients where one or both points are on Z, on which j is 0. The derivative there is

[7] Let M be the closed first quadrant in $\mathbb{R}^2$ and let $f = x^2 - y$. Then for each u, (7.2) does not hold for $t = (0, v)$ when v is sufficiently small. It fails because Z is tangent to the boundary of M.

$(i' \cdot f'/f' \cdot f')j'$. Hence

(7.35) $$\|q(i)j\| = S(q(i)j) + S((q(i)j)') \le (3 + \|f'\|)\|i\| \cdot \|j\|.$$

Now take $i = af$ and $j = bf$, and observe that f is pseudo regular.

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