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ESTIMATING NIELSEN NUMBERS ON INFRASOLVMANIFOLDS

CHRISTOPHER K. MCCORD

A well-known lower bound for the number of fixed points of a self-map $f: X \to X$ is the Nielsen number $N(f)$. Unfortunately, the Nielsen number is difficult to calculate. The Lefschetz number $L(f)$, on the other hand, is readily computable, but does not give a lower bound for the number of fixed points. In this paper, we investigate conditions on the space X which guarantee either $N(f) = |L(f)|$ or $N(f) \geq |L(f)|$. By considering the Nielsen and Lefschetz coincidence numbers, we show that $N(f) \geq |L(f)|$ for all self-maps on compact infrasolvmanifolds (aspherical manifolds whose fundamental group has a normal solvable subgroup of finite index). Moreover, for infranilmanifolds, there is a Lefschetz number formula which computes $N(f)$.

1. Estimating Nielsen numbers. Consider a continuous self-map $f: X \to X$. Let $\mathrm{Fix}(f)$ denote the fixed point set $\{x \in X | f(x) = x\}$. One of the fundamental problems of fixed point theory is to estimate (preferably from below) the cardinality of this set. The Nielsen number $N(f)$ provides such an estimate: it is an integer homotopy invariant which provides a lower bound on the number of fixed points of g, for all maps g homotopic to f. This estimate is sharp for all compact manifolds save surfaces of negative Euler characteristic. Its one drawback is that it is very difficult to compute $N(f)$ from its definition, so that other means must be sought. At least, since the Nielsen number provides a lower bound for the original topological object $|\mathrm{Fix}(f)|$, it would be useful to find lower bounds for $N(f)$. We will refer to the search for lower bounds to $N(f)$ as the problem of *estimating* $N(f)$; while the search for other algebraic-topological means of finding the exact value of $N(f)$ will be referred to as the problem of *computing* $N(f)$.

The Lefschetz number $L(f)$ is a (reasonably) computable invariant, but in general, there is no relation between $L(f)$ and either $N(f)$ or $|\mathrm{Fix}(f)|$. One approach to computing the Nielsen number is to find conditions on either the space X or the map f which allow $N(f)$ and $L(f)$ to be related. The Jiang condition, for example, is a condition on the map f which, when satisfied, computes $N(f)$ from $L(f)$

and $\operatorname{coker}(1 - f_{1*})$. The other approach, searching for conditions on the space X, begins with the result of Brooks, Brown, Pak and Taylor **[8]**, that $N(f) = |L(f)|$ for all maps on tori. Anosov **[3]** and Fadell and Husseini **[11]** show that the equality holds for all maps on compact nilmanifolds. While counter-examples on the Klein bottle show that equality does not hold for all maps on solvmanifolds, nor on infranilmanifolds, Kwasik and Lee **[16]** show that $N(f) = L(f)$ for homotopically periodic maps on infranilmanifolds, and it is shown in **[18]** that $N(f) \geq |L(f)|$ for all maps on solvmanifolds.

Fixed-point theory has a natural extension to coincidences: if $f, g: X_1 \to X_2$, let $\operatorname{Coin}(f, g) = \{x \in X_1 | f(x) = g(x)\}$. The Nielsen number generalizes to a Nielsen coincidence number $N(f, g)$, which is a homotopy invariant and a lower bound for the number of coincidences. This estimate is sharp when X_1 and X_2 are manifolds with $\dim(X_1) = \dim(X_2) \geq 3$. The Lefschetz coincidence number, on the other hand, is only defined when X_1 and X_2 are orientable manifolds of the same dimension. Jezierski **[14]** and Brooks and Wang **[9]** show that $N(f, g) = |L(f, g)|$ when $X_1 = X_2$ is an infranilmanifold; in **[19]**, it is shown that $N(f, g) \geq |L(f, g)|$ when X_1 and X_2 are compact orientable solvmanifolds of the same dimension, with equality if X_2 is a nilmanifold.

The original goal of this work was to extend the results of **[19]** to nonorientable solvmanifolds. This was to be done by lifting the map to the orientable double cover (which is also a solvmanifold) and applying the existing results there. All that was needed was to understand the relation between the Nielsen and Lefschetz numbers for the original maps and the numbers for the lifts. Once this relation was investigated and understood, it became clear that a broader class of manifolds could be studied in this manner. The present work therefore studies compact *infrasolvmanifolds*, manifolds which admit a finite cover by a compact solvmanifold. While the main results (Theorems 7.4 and 7.9) are stated in terms of coincidence numbers, their specialization to fixed point numbers extends the results of Anosov et al. to the following:

If M is a compact infrasolvmanifold, then $N(f) \geq |L(f)|$ for all $f: M \to M$. If M is a compact infranilmanifold covered by compact nilmanifold $\widetilde{M}$, there is an expression $\widetilde{L}(f, \Gamma_2)$, involving Lefschetz numbers of lifts of f to $\widetilde{M}$, which computes $N(f)$.

It is worth noting that this Lefschetz formula involves Lefschetz coincidence numbers of lifts, even when the original problem involved only fixed point numbers.

The next two sections contain a brief recapitulation of the relevant parts of Nielsen coincidence theory. Sections 4 and 5 explore the relation between coincidence numbers of maps and coincidence numbers of lifts of the maps. Section 6 describes the topology and algebra of infrasolvmanifolds, while §7 combines the results of §5 with the estimates in **[18]** and **[19]** to obtain the estimates for Nielsen numbers on infrasolvmanifolds. The paper concludes with a comparison in §8 between these results and the Jiang condition, and some open questions in §9.

2. Coincidence numbers. We now briefly review the basics of Nielsen coincidence theory, as developed by Brooks **[4]**, **[7]**. We will work in the category of compact connected polyhedra and continuous maps. If X_1, X_2 are polyhedra and $f, g: X_1 \to X_2$ are maps, let $\mathrm{Coin}(f, g) = \{x \in X_1 | f(x) = g(x)\}$ be the coincidence set of f and g. To analyze this set, we begin by partitioning it into coincidence classes $\mathbf{S}(f, g)$ (or just $\mathbf{S}$ when f and g are understood). If $x, y \in \mathrm{Coin}(f, g)$, set $x \sim y$ if there exists a path ω in X_1 from x to y with $f \circ \omega \simeq g \circ \omega (\mathrm{rel}\{0, 1\})$. Clearly, each coincidence class is a union of path components of $\mathrm{Coin}(f, g)$, and so is compact and open in $\mathrm{Coin}(f, g)$. Denote the set of coincidence classes of $\mathrm{Coin}(f, g)$ by $\mathscr{R}(f, g)$.

If $F: f_0 \simeq f_1$ and $G: g_0 \simeq g_1$, then coincidence classes $\mathbf{S}_0 \in \mathscr{R}(f_0, g_0)$ and $\mathbf{S}_1 \in \mathscr{R}(f_1, g_1)$ are *(F, G)-related* if there exist $x_0 \in \mathbf{S}_0$, $x_1 \in \mathbf{S}_1$ and path ω in X_1 such that the paths $\langle F, \omega\rangle$ and $\langle G, \omega\rangle$, defined by $\langle F, \omega\rangle(t) = F_t(\omega(t))$, are homotopic in X_2. A class $\mathbf{S} \in \mathscr{R}(f, g)$ is *topologically essential* if, for every $F: f \simeq f'$, $G: g \simeq g'$, there exists a class $\mathbf{S}' \in \mathscr{R}(f', g')$ which is (F, G)-related to $\mathbf{S}$. That is, the class cannot be "homotoped away". We will denote the set of topologically essential coincidence classes by $\mathscr{E}(f, g)$. The *Nielsen coincidence number* $N(f, g)$ is the number of topologically essential coincidence classes of f and g.

The Nielsen coincidence number is, by construction, a homotopy invariant and a lower bound on the number of coincidences of f' and g' for every $f' \simeq f$ and $g' \simeq g$. We will refer to a pair of manifolds M_1, M_2 as a *Wenken pair* if $N(f, g)$ is a sharp lower bound for every pair of homotopy classes. All manifolds M_1, M_2 with $(M_2 \times M_2, M_2 \times M_2 \backslash \Delta(M_2))$ 2-connected (in particular, if M_2 is an n-manifold with $n \geq 3$) are Wenken pairs **[6]**. However, it cannot be used directly to estimate the number of coincidences of f and g, since complete information about $\mathrm{Coin}(f, g)$ (indeed, about

$\mathrm{Coin}(f', g')$ for all $f' \simeq f$, $g' \simeq g$) is required before $N(f, g)$ can be computed. So, to make it a useful tool for estimating $\mathrm{Coin}(f, g)$, indirect methods of computation are required. One approach begins by replacing the concept of topologically essential classes with that of algebraically essential classes.

This is done by introducing a coincidence index. A variety of indices are possible in different settings, but the following will suffice for our purposes. Suppose M_1 and M_2 are both compact connected orientable n-manifolds, with $(M_1 \times M_2, M_2 \times M_2 \backslash \Delta(M_2))$ 2-connected. For each coincidence class (in fact, for any set $\mathbf{S} \subseteq \mathrm{Coin}(f, g)$ which is open in $\mathrm{Coin}(f, g)$ and compact) a coincidence number $\mathrm{Ind}(f, g, \mathbf{S})$ is defined. If W, V are neighborhoods of $\mathbf{S}$ with $\mathbf{S} \subset W \subset \overline{W} \subset V$ and $\mathrm{Coin}(f, g) \cap V = \mathbf{S}$, then let $(f, g)_\bullet$ be the composition

$$\begin{aligned} &H_n(M_1) \to H_n(M_1, M_1 \backslash W) \\ &\qquad \xleftarrow{\cong} H_n(V, V \backslash W) \xrightarrow{(f_*, g_*)} H_n(M_2 \times M_2, M_2 \times M_2 \backslash \Delta(M_2)). \end{aligned}$$

If $z_1 \in H_n(M_1)$ is the fundamental class of M_1 and $U_2 \in H_n(M_2 \times M_2, M_2 \times M_2 \backslash \Delta(M_2))$ is the Thom class of M_2, then $\mathrm{Ind}(f, g, \mathbf{S}) \equiv \langle U_2, (f, g)_\bullet(z_1) \rangle$. This is independent of V and W, and has the following properties [23]:

1. Coincidence: if $\mathrm{Ind}(f, g, \mathbf{S}) \neq 0$, then $\mathbf{S} \neq \varnothing$.
2. Homotopy: if $F: f_0 \simeq f_1$, $G: g_0 \simeq g_1$ and there exists a V such that $V \cap \mathrm{Coin}(F, G)$ is compact, then $\mathbf{S}_0 = V \cap \mathrm{Coin}(f_0, g_0)$ and $V \cap \mathbf{S}_1 = \mathrm{Coin}(f_1, g_1)$ have $\mathrm{Ind}(f_0, g_0, \mathbf{S}_0) = \mathrm{Ind}(f_1, g_1, \mathbf{S}_1)$.
3. Additivity: if $\mathbf{S} = \bigcup \mathbf{S}_i$, then $\mathrm{Ind}(f, g, \mathbf{S}) = \sum \mathrm{Ind}(f, g, \mathbf{S}_i)$.
4. Products: Given $f, g: M_1 \to M_2$ and $f', g': M_1' \to M_2'$ and $\mathbf{S} \subseteq \mathrm{Coin}(f, g)$, $\mathbf{S}' \subseteq \mathrm{Coin}(f', g')$, then $\mathrm{Ind}(f, g, \mathbf{S})$ and $\mathrm{Ind}(f', g', \mathbf{S}')$ are defined if and only if $\mathrm{Ind}(f \times f', g \times g', \mathbf{S} \times \mathbf{S}')$ is defined. If all are defined, $\mathrm{Ind}(f \times f', g \times g', \mathbf{S} \times \mathbf{S}') = \mathrm{Ind}(f, g, \mathbf{S}) \cdot \mathrm{Ind}(f', g', \mathbf{S}')$.

A coincidence class $\mathbf{S}$ is *algebraically essential* if $\mathrm{Ind}(f, g, \mathbf{S}) \neq 0$. Of course, an algebraically essential class is topologically essential: if $f' \simeq f$ and $g' \simeq g$, then the corresponding class $\mathbf{S}' \neq \varnothing$. Conversely, if $\dim(M_i) \geq 3$, then for every f and g, there exist $f' \simeq f$ and $g' \simeq g$ such that, for each coincidence class $\mathbf{S}$ of f and g, the corresponding class $\mathbf{S}'$ consists of a single point if $\mathbf{S}$ is algebraically essential and is empty if $\mathbf{S}$ is algebraically inessential [22]. Thus the Nielsen number can be defined as the number of algebraically essential coincidence classes.

Contrasting with the Nielsen number is an algebraic count of the essential classes: $\operatorname{Ind}(f, g) = \sum \operatorname{Ind}(f, g, \mathrm{S})$, the sum of the coincidence index over all coincidence classes. Of course, this can be computed directly, using $W = V = M_1$ to generate $(f, g)_\bullet$. This is also a homotopy invariant, but it is not in general a good estimate on the number of essential coincidence classes. This for two reasons. First, there can be cancellation. If $\operatorname{Coin}(f, g)$ has two coincidence classes $\mathrm{S}_1, \mathrm{S}_2$ with $\operatorname{Ind}(f, g, \mathrm{S}_1) = 1$ and $\operatorname{Ind}(f, g, \mathrm{S}_2) = -1$, then every $f' \simeq f$ and $g' \simeq g$ have at least two coincidence points, yet $\operatorname{Ind}(f, g) = 0$ detects none. Second, there can be multiplicity. If $\operatorname{Coin}(f, g)$ consists of a single point c with $\operatorname{Ind}(f, g, c) = 2$, then $\operatorname{Ind}(f, g) = \operatorname{Ind}(f, g, c)$ overestimates the number of coincidences of f and g. However, it is at least true that if $\operatorname{Ind}(f, g) \neq 0$, then $N(f, g) \neq 0$ and $\operatorname{Coin}(f, g) \neq \varnothing$.

Moreover, $\operatorname{Ind}(f, g)$ is relatively easy to compute. In rational coefficients, let $D_i \colon H^p(M_i) \to H_{n-p}(M_i)$ be the duality isomorphism and let $\Theta_p(f, p)$ be the composition

$$H_p(M_1) \xrightarrow{f_*} H_p(M_2) \xrightarrow{D_2^{-1}} H^{n-p}(M_2) \xrightarrow{g^*} H^{n-p}(M_1) \xrightarrow{D_1} H_p(M_1).$$

The *Lefschetz coincidence number* $L(f, g)$ is defined as

$$\sum_{p=0}^{n} (-1)^p \operatorname{tr} \Theta_p(f, g),$$

and the Lefschetz coincidence theorem states that $L(f, g) = \operatorname{Ind}(f, g)$.

3. Coincidence numbers vs. fixed-point numbers. The coincidence theory outlined above is derived from the analogous form of the Nielsen and Lefschetz fixed-point theories. However, it is not strictly speaking a generalization of the fixed-point theory. That is, it is not clear that the fixed-point theory is recovered by setting $X_1 = X_2$ and $g = \mathrm{id}$. For Nielsen numbers, there may be a difference between $N(f)$ and $N(f, \mathrm{id})$: in the definition of $N(f, \mathrm{id})$, both f and id are allowed to be modified by a homotopy; while in the definition of $N(f)$, only the map f is. For Lefschetz numbers, the coincidence number $L(f, g)$ is defined only for orientable manifolds, while the fixed point number $L(f)$ is defined for all polyhedra. All of these (potential) global differences have corresponding local differences: A set may be inessential as a coincidence class of (f, id) yet essential as a fixed-point class of f. Similarly, $L(f, g)$ is only defined for orientable manifolds because the coincidence index is only defined in that setting. The goal of this section is to remove some of these apparent differences between fixed-point theory and coincidence theory.

LEMMA 3.1. *If X is a compact orientable manifold, $\mathbf{S}$ an isolated fixed-point set for $f: X \to X$, then* $\operatorname{Ind}(f, \mathrm{id}, \mathbf{S}) = \operatorname{index}(f, \mathbf{S})$, *where* $\operatorname{index}(f, \mathbf{S})$ *is the fixed-point index of* $\mathbf{S}$.

COROLLARY 3.2. *If X is a compact orientable manifold, then $L(f, \mathrm{id}) = L(f)$ for all $f: X \to X$.*

THEOREM 3.3 [**5**, Cor. 3]. *If X is a compact manifold, then $N(f, \mathrm{id}) = N(f)$ for all $f: X \to X$.*

Though not employing the language of Nielsen coincidence numbers, [**10**] and [**13**] contain similar results.

The following lemma helps to generalize these results, and the definition of the coincidence index, from $g = \mathrm{id}$ to any homeomorphism g.

LEMMA 3.4. *Suppose X_0, X_1, X_2, X_3 are compact polyhedra, $h_0: X_0 \to X_1$ and $h_1: X_2 \to X_3$ are homeomorphisms. Then for all $f, g: X_1 \to X_2$, there is a bijection $\overline{h}_0: \mathscr{R}(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0) \to \mathscr{R}(f, g)$ with $\overline{h}_0(\mathscr{E}(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0)) = \mathscr{E}(f, g)$, and which preserves indices up to a sign (when defined).*

Proof. The homeomorphism h_0 restricts to a homeomorphism

$$h_0: \operatorname{Coin}(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0) \to \operatorname{Coin}(f, g).$$

If $\mathbf{S}_0 \in \mathscr{R}(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0)$ and $x_0, x_1 \in \mathbf{S}_0$, then there is a path ω in X_0 from x_0 to x_1 with $h_1 \circ f \circ h_0 \circ \omega \simeq h_1 \circ g \circ h_0 \circ \omega$. Thus $h_0(\mathbf{S}_0)$ is contained in a single coincidence class $\mathbf{S}_1$ of f and g. Similarly, $\mathbf{S}_1$ maps under h_0^{-1} into a single coincidence class of $h_1 \circ f \circ h_0$ and $h_1 \circ g \circ h_0$, that is, into $\mathbf{S}_0$.

If $\mathbf{S}_1$ is removed by homotopies $F: f \simeq f'$ and $G: g \simeq g'$, then $\mathbf{S}_0$ is removed by homotopies $h_1 \circ F \circ h_0: h_1 \circ f \circ h_0 \simeq h_1 \circ f' \circ h_0$ and $h_1 \circ G \circ h_0: h_1 \circ g \circ h_0 \simeq h_1 \circ g' \circ h_0$, and conversely.

Finally, suppose indices are defined for both (f, g) and $(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0)$. Then either $g = \mathrm{id}$ and $h_1 = h_0^{-1}$, or all of the spaces involved are orientable manifolds. In the first case, both of the indices involved are fixed-point indices, and the result is a standard one. If all of the spaces are orientable, then $(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0)_\bullet$ is given by the composition

$$\begin{aligned} H_n(X_0) \to H_n(X_1) &\xrightarrow{(f,g)_\bullet} H_n(X_2 \times X_2, X_2 \times X_2 \backslash \Delta(X_2)) \\ &\xrightarrow{(h_1 \times h_1)_\bullet} H_n(X_3 \times X_3, X_3 \times X_3 \backslash \Delta(X_3)) \end{aligned}$$

and so differs from $(f, g)_\bullet$ by (at most) $\deg(h_0) \cdot \deg(h_1)$. □

This motivates an extension of the coincidence index: Suppose X_1, X_2 are compact polyhedra, and $f, g: X_1 \to X_2$ are maps with g a homeomorphism. If $\mathbf{S}$ is an isolated coincidence set, define $\operatorname{Ind}(f, g, \mathbf{S}) \equiv \operatorname{index}(g^{-1} \circ f, \mathbf{S})$. Clearly, this inherits all of the standard properties of the index, and agrees with the usual definition of $\operatorname{Ind}(f, g, \mathbf{S})$ when X_1 and X_2 are orientable manifolds. To save tedious repetition, we will assume from now on that whenever the coincidence index is involved, either one of the maps is a homeomorphism, or both of the spaces are compact orientable manifolds of the same dimension. We will consider the utility of this definition in §§5 and 7.

COROLLARY 3.5. *Suppose X_0, X_1, X_2, X_3 are compact polyhedra, $h_0: X_0 \to X_1$ and $h_1: X_2 \to X_3$ are homeomorphisms. Then*

$$N(h_1 \circ f \circ h_0, h_1 \circ g \circ h_0) = N(f, g) \quad \textit{for all } f, g: X_1 \to X_2.$$

In particular, if $g: X_1 \to X_2$ is a homeomorphism between compact polyhedra, then

$$N(f \circ g^{-1}, \mathrm{id}) = N(f, g) = N(g^{-1} \circ f, \mathrm{id}) \quad \textit{for all } f: X_1 \to X_2.$$

COROLLARY 3.6. *If $g: M \to M$ is a homeomorphism between compact manifolds, then $N(f \circ g^{-1}) = N(f, g) = N(g^{-1} \circ f)$ for all $f: M \to M$.*

COROLLARY 3.7. *Suppose X_1 and X_2 are compact orientable n-manifolds and g is a homotopy equivalence with homotopy inverse g'. Choose orientations for X_1 and X_2 so that $\deg(g) = \deg(g') = 1$. Then $L(f, g) = L(g' \circ f) = L(f \circ g')$.*

4. Covering spaces and lifts. Given $f, g: X_1 \to X_2$, we want to relate the Nielsen and Lefschetz numbers of f and g to Nielsen and Lefschetz numbers of lifts $\tilde{f}, \tilde{g}: \widetilde{X}_1 \to \widetilde{X}_2$, where $\widetilde{X}_1$ and $\widetilde{X}_2$ are finite covers of X_1 and X_2.

Fix base points $x_1 \in X_1$, $x_2 \in X_2$, and for convenience assume that $f(x_1) = x_2 = g(x_1)$. Let π_i denote $\pi(X_i, x_i)$ and define $\mathscr{C}(\pi_i) = \{\Gamma \lhd \pi_i | [\pi_i : \Gamma] < \infty\}$. There is a one-to-one correspondence between elements of $\mathscr{C}(\pi_i)$ and finite regular covers of X_i. Because the Lefschetz coincidence number requires orientable manifolds, we will be particularly interested in finding orientable covering spaces when X_1 and X_2 are manifolds. Recall that any manifold has an orientable cover. This cover corresponds to the subgroup $O_i = O(\pi_i) \lhd \pi_i$ consisting of $\alpha \in \pi_i$ such that the tangent space map $D\tau_\alpha \colon T_{x_i}X_i \to T_{x_i}X_i$ induced by translation along α acts orientably (i.e. has positive determinant). $O_i = \pi_i$ if X_i is orientable, and is a subgroup of index 2 if X_i is nonorientable. Then the orientable finite regular covers correspond to $\mathscr{CO}(\pi_i) = \{\Gamma \in \mathscr{C}(\pi_i) | \Gamma \subseteq O_i\}$.

Now consider $f, g \colon X_1 \to X_2$. Fix $\Gamma_2 \in \mathscr{C}(\pi_2)$ and corresponding finite regular cover $p_2 \colon \widetilde{X}_2 \to X_2$. Given a cover $p_1 \colon \widetilde{X}_1 \to X_1$ and corresponding $\Gamma_1 \in \mathscr{C}(\pi_1)$, f and g lift to some $\tilde{f}, \tilde{g} \colon \widetilde{X}_1 \to \widetilde{X}_2$ if and only if $f_\#, g_\# \colon \pi_1 \to \pi_2$ have $f_\#, g_\#(\Gamma_1) \subseteq \Gamma_2$. So define $\mathscr{C}(f, g, \Gamma_2) = \{\Gamma_1 \in \mathscr{C}(\pi_1) | f_\#, g_\#(\Gamma_1) \subseteq \Gamma_2\}$. Note that $f_\#^{-1}(\Gamma_2) \cap g_\#^{-1}(\Gamma_2) \in \mathscr{C}(f, g, \Gamma_2)$, so $\mathscr{C}(f, g, \Gamma_2)$ is nonempty. We will refer to the lifting diagram

$$\begin{array}{ccc} \widetilde{X}_1 & \overset{\tilde{f}, \tilde{g}}{\rightrightarrows} & \widetilde{X}_2 \\ \downarrow p_1 & & \downarrow p_2 \\ X_1 & \overset{f, g}{\rightrightarrows} & X_2 \end{array}$$

as the $\Gamma_1 - \Gamma_2$ lifting diagram of f and g.

For any lifting diagram, the lifts $\tilde{f}, \tilde{g}$ have a Nielsen coincidence number defined. However, they may not have a coincidence index defined, even if f and g did. That is, if the spaces are not manifolds but g is a homeomorphism, they have a coincidence index defined. Then $\tilde{f}$ and $\tilde{g}$ will only have an index defined if $\Gamma_1 = g_\#^{-1}(\Gamma_2)$ has $f_\#(\Gamma_1) \subseteq \Gamma_2$. But of course, this may not be the case for all f and g. We therefore define for every $\Gamma_2 \in \mathscr{C}(\pi_2)$ the set

$$\mathscr{IC}(f, g, \Gamma_2) = \{\Gamma_1 \in \mathscr{C}(\pi_1) | f_\#(\Gamma_1), g_\#(\Gamma_1) \subseteq \Gamma_2 \text{ and an index is defined for the } \Gamma_1 - \Gamma_2 \text{ lifts}\}.$$

If X_1 and X_2 are manifolds and $\Gamma_2 \in \mathscr{CO}(\pi_2)$, then $f_\#^{-1}(\Gamma_2) \cap g_\#^{-1}(\Gamma_2) \cap O_1 \in \mathscr{IC}(f, g, \Gamma_2)$, so $\mathscr{IC}(f, g, \Gamma_2)$ is nonempty. If X_1 and X_2 are not manifolds, or if $\Gamma_2 \notin \mathscr{CO}(\pi_2)$, then $\mathscr{IC}(f, g, \Gamma_2)$ is nonempty if and only if there is a $\Gamma_1 \in \mathscr{C}(\pi_1)$ with $f_\#(\Gamma_1) \subseteq \Gamma_2$ and $g_\#$ an isomorphism from Γ_1 to Γ_2.

In any $\Gamma_1 - \Gamma_2$ lifting diagram, Γ_i has covering group $\Phi_i = \pi_i/\Gamma_i$. $f_\#$ and $g_\#$ induce maps $\overline{f}, \overline{g}\colon \Phi_1 \to \Phi_2$. If $\tilde{f}, \tilde{g}\colon \widetilde{X}_1 \to \widetilde{X}_2$ are fixed "reference lifts" of f and g, then all lifts have the form $\beta \circ \tilde{f} \circ \alpha$, $\beta \circ \tilde{g} \circ \alpha$ with $\alpha \in \Phi_1$, $\beta \in \Phi_2$. But $\tilde{f} \circ \alpha = \overline{f}(\alpha) \circ \tilde{f}$, so we need only consider lifts of the form $\beta \circ \tilde{f}$, $\beta \circ \tilde{g}$.

This has all been done relative to fixed base points $x_1 \in X_1$, $x_2 \in X_2$. However, the sets are really base point-independent. Consider $x_1, x_1' \in \mathrm{Coin}(f, g)$, with $x_2 = f(x_1)$, $x_2' = f(x_1')$. Fix a path γ from x_1 to x_1'. Then there are commutative diagrams

$$\begin{tikzcd} \pi(X_1, x_1) \arrow[r, "f_\#"] \arrow[d, "\gamma_\#"] & \pi(X_2, x_2) \arrow[d, "(f\circ\gamma)_\#"] \\ \pi(X_1, x_1') \arrow[r, "f_\#'"] & \pi(X_2, x_2') \end{tikzcd}$$

and

$$\begin{tikzcd} \pi(X_1, x_1) \arrow[r, "g_\#"] \arrow[d, "\gamma_\#"] & \pi(X_2, x_2) \arrow[d, "(g\circ\gamma)_\#"] \\ \pi(X_1, x_1') \arrow[r, "g_\#'"] & \pi(X_2, x_2'). \end{tikzcd}$$

The maps $(f \circ \gamma)_\#$ and $(g \circ \gamma)_\#$ differ by an inner automorphism, so if $\Gamma_2 \in \mathscr{C}(\pi(X_2, x_2))$, $(f\circ\gamma)_\#(\Gamma_2)$ and $(g\circ\gamma)_\#(\Gamma_2)$ are both normal in $\pi(X_2, x_2')$, and so are equal. That is, there is a well defined bijection between $\mathscr{C}(\pi(X_2, x_2))$ and $\mathscr{C}(\pi(X_2, x_2'))$, and a similar bijection between $\mathscr{C}(\pi(X_1, x_1))$ and $\mathscr{C}(\pi(X_1, x_1'))$. From the diagrams above, these bijections preserve the set $\mathscr{C}(f, g, \Gamma_2)$.

If X_i is a manifold, the bijection between $\mathscr{C}(\pi(X_i, x_i))$ and $\mathscr{C}(\pi(X_i, x_i'))$ preserves the orientation subgroup O_i and the set $\mathscr{C}\mathscr{O}(\pi_i)$. So if the $\Gamma_1 - \Gamma_2$ lifting diagram has an index defined because $\Gamma_i \in \mathscr{C}\mathscr{O}(\pi_i)$, this is independent of the base-point. On the other hand, if the index is defined because $g_\#$ is an isomorphism from Γ_1 to Γ_2, then this too is a base-point independent property. That is, the bijection between $\mathscr{C}(\pi(X_1, x_1))$ and $\mathscr{C}(\pi(X_1, x_1'))$ preserves the set $\mathscr{I}\mathscr{C}(f, g, \Gamma_2)$.

The ability to construct lifts, or lifts which admit an index, is then independent of the base-point chosen. We now turn to the coincidence theory for lifts. We will see that here, the choice of base-point is significant.

5. Coincidence numbers for covering spaces. We now consider the relation between the coincidence sets, Nielsen numbers and Lefschetz numbers of f and g and those of the lifts. Much of this is simply a modification of the universal cover/lifting approach to Nielsen numbers, as found in **[15]**.

PROPOSITION 5.1. $\mathbf{S}$ *is a coincidence class in* $\operatorname{Coin}(\tilde{f}, \tilde{g})$ *if and only if it is a coincidence class in* $\operatorname{Coin}(\beta \circ \tilde{f}, \beta \circ \tilde{g})$ *for all* $\beta \in \Phi_2$.

PROPOSITION 5.2. *If* $\tilde{x} \in \operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g})$ *for some* $\beta \in \Phi_2$, *then* $x = p_1(\tilde{x}) \in \operatorname{Coin}(f, g)$. *Conversely, if* $x \in \operatorname{Coin}(f, g)$ *and* $\tilde{x} \in p^{-1}(x)$, *then there is a unique* $\beta \in \Phi_2$ *such that* $\tilde{x} \in \operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g})$. *That is,*

$$p_1^{-1}(\operatorname{Coin}(f, g)) = \bigsqcup_{\beta \in \Phi_2} \operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g}).$$

Proof. If $\tilde{x} \in \operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g})$, then $f(x) = f \circ p_1(\tilde{x}) = p_2 \circ \tilde{f}(\tilde{x}) = p_2 \circ \tilde{g}(\tilde{x}) = g \circ p_1(\tilde{x}) = g(x)$. If $p_2 \circ \tilde{f}(\tilde{x}) = f \circ p_1(\tilde{x}) = f(x) = g(x) = g \circ p_1(\tilde{x}) = p_2 \circ \tilde{g}(\tilde{x})$, then $\tilde{g}(\tilde{x})$ and $\tilde{f}(\tilde{x})$ lie in the same fiber, and there is a unique $\beta \in \Phi_2$ so that $\tilde{g}(\tilde{x}) = \beta \circ \tilde{f}(\tilde{x})$. □

PROPOSITION 5.3. *If* $\widetilde{\mathbf{S}}$ *is a coincidence class in* $\operatorname{Coin}(\tilde{f}, \tilde{g})$ *and* $\alpha \in \Phi_1$, *then* $\mathbf{S} = p_1(\widetilde{\mathbf{S}})$ *is a coincidence class in* $\operatorname{Coin}(f, g)$ *and* $\alpha(\widetilde{\mathbf{S}})$ *is a coincidence class in* $\operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g})$, *where* $\beta = \overline{g}(\alpha) \cdot \overline{f}^{-1}(\alpha)$.

Proof. If $\tilde{x}_1, \tilde{x}_2 \in \widetilde{\mathbf{S}}$, then there is a path $\tilde{\omega}$ in $\widetilde{X}_1$ from $\tilde{x}_1$ to $\tilde{x}_2$ with $\tilde{f} \circ \tilde{\omega} \simeq \tilde{g} \circ \tilde{\omega}$. If $x_i = p_1(\tilde{x}_i)$, then $\omega = p_1 \tilde{\omega}$ is a path from x_1 to x_2 with $H\colon f \circ \omega \simeq g \circ \omega$. Conversely, suppose $x_1, x_2 \in \operatorname{Coin}(f, g)$ are connected by a path ω with $f \circ \omega \simeq g \circ \omega$. If $\tilde{x}_1 \in \operatorname{Coin}(\tilde{f}, \tilde{g})$ lies over x_1, let $\tilde{\omega}$ be the unique lift of ω based at $\tilde{x}_1$, and let $\widetilde{H}$ be the unique lift of H based at $\tilde{f}(\tilde{x}_1)$. Let $\tilde{x}_2 = \tilde{\omega}(1)$. Then $\tilde{f}(\tilde{x}_2) = \tilde{g}_2(\tilde{x}_2)$ and $\widetilde{H}\colon \tilde{f} \circ \tilde{\omega} \simeq \tilde{g} \circ \tilde{\omega}$.

Now, if $\tilde{x} \in \widetilde{\mathbf{S}}$ and $\alpha \in \Phi_1$, then $\tilde{f}(\alpha(\tilde{x})) = \overline{f}(\alpha) \circ \tilde{f}(\tilde{x})$ and $\tilde{g}(\alpha(\tilde{x})) = \overline{g}(\alpha) \circ \tilde{g}(\tilde{x})$. Thus $\beta \circ \tilde{f}(\alpha(\tilde{x})) = \tilde{g}(\alpha(\tilde{x}))$ if and only if $\beta = \overline{g}(\alpha) \cdot \overline{f}^{-1}(\alpha)$. And if $\widetilde{H}\colon \tilde{f} \circ \tilde{\omega} \simeq \tilde{g} \circ \tilde{\omega}$, then $\overline{g}(\alpha) \circ \widetilde{H}\colon \beta \circ \tilde{f} \circ \alpha_{-1}\tilde{\omega} \simeq \tilde{g} \circ \alpha_{-1}\tilde{\omega}$, so $\alpha(\widetilde{\mathbf{S}})$ is a coincidence class in $\operatorname{Coin}(\beta \circ \tilde{f}, \tilde{g})$. □

If $\mathbf{S}$ is a coincidence class of f and g, then define $C_\#(f, g, \mathbf{S}) = \{\alpha \in \pi_1 | f_\#(\alpha) = g_\#(\alpha)\}$, where $f_\#$ and $g_\#$ are based at some $x \in \mathbf{S}$. $C_\#(f, g, \mathbf{S})$ is a subgroup of π_1, but is not necessarily normal. Note that $C_\#(f, g, \mathbf{S})$ depends on $\mathbf{S}$, but not on the base point $x \in \mathbf{S}$ chosen. Similarly, if X_1 and X_2 are manifolds of the same dimension, let $CO_\#(f, g, \mathbf{S}) = C_\#(f, g, \mathbf{S}) \cap O(\pi_1)$.

COROLLARY 5.4. *If $\widetilde{\mathbf{S}}$ is a coincidence class in* $\mathrm{Coin}(\beta \circ \tilde{f}, \tilde{g})$, *then $\widetilde{\mathbf{S}}$ is covering space over $\mathbf{S}$ with deck transformations*

$$C_\#(f, g, \mathbf{S})/\Gamma_1 \cap C_\#(f, g, \mathbf{S}).$$

That is, if $\mathbf{S}$ is a coincidence class in $\mathrm{Coin}(f, g)$, then Φ_1 acts transitively on the coincidence classes in $p^{-1}(\mathbf{S})$ with isotropy group of $C_\#(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1$. Thus $\mathbf{S}$ is covered by $[\pi_1 : C_\#(f, g, \mathbf{S}) \cdot \Gamma_1]$ coincidence classes. In particular, the number of classes covering $\mathbf{S}$ depends on $\mathbf{S}$.

Of course, to compute the Nielsen number of lifts $\tilde{f}$ and $\tilde{g}$, we must consider more than just the number of coincidence classes. We must also consider which classes are essential and which are inessential. The ideal situation would be that a class $\mathbf{S} \in \mathscr{R}(f, g)$ is essential if and only if all of the classes $\widetilde{\mathbf{S}}$ covering it are. This is almost, but not quite, true.

PROPOSITION 5.5. *If $\widetilde{\mathbf{S}}$ is an essential coincidence class in* $\mathrm{Coin}(\tilde{f}, \tilde{g})$, $\alpha \in \Phi_1$, *then $\mathbf{S} = p_1(\widetilde{\mathbf{S}})$ is an essential coincidence class in* $\mathrm{Coin}(f, g)$ *and $\alpha(\widetilde{\mathbf{S}})$ is an essential coincidence class in* $\mathrm{Coin}(\beta \circ \tilde{f}, \tilde{g})$, *where* $\beta = \overline{g}(\alpha) \circ (\overline{f}(\alpha))^{-1}$.

Proof. Suppose $\mathbf{S}$ is inessential. Then there exist $f' \simeq f$ and $g' \simeq g$ such that $\mathbf{S}$ continues to an empty coincidence class $\mathbf{C}'$. Now lift $f' \simeq f$ and $g' \simeq g$ to $\tilde{f}' \simeq \tilde{f}$ and $\tilde{g}' \simeq g'$. The class $\widetilde{\mathbf{S}}$ continues to a class $\widetilde{\mathbf{C}}'$ which covers $\mathbf{C}'$, and so is empty. Thus $\widetilde{\mathbf{S}}$ is inessential. Similarly, if $\alpha(\widetilde{\mathbf{S}})$ is inessential, then the homotopy which removes it can be carried by the deck transformations to a homotopy which removes $\widetilde{\mathbf{S}}$. □

Thus every essential class in $\mathrm{Coin}(f, g)$ is covered by either no essential classes, or $[\pi_1 : \Gamma_1 \cdot C_\#(f, g, \mathbf{S})]$ essential classes, as β ranges over Φ_2. Intuitively, if $\mathbf{S}$ is essential, a class $\widetilde{\mathbf{S}}$ in $\mathrm{Coin}(\tilde{f}, \tilde{g})$ covering $\mathbf{S}$ is inessential if it can be removed via homotopies $\widetilde{F} : \tilde{f} \simeq \tilde{f}'$ and $\widetilde{G} : \tilde{g} \simeq \tilde{g}'$ which are not equivariant under the covering group actions, and so do not project to homotopies on X_1. It is difficult to find topological conditions that eliminate this possibility. The following result, though restricted to the manifold setting, will prove to be sufficient for our purposes.

THEOREM 5.6. *Suppose* X_1, X_2 *are compact manifolds of the same dimension, and neither is a surface with negative Euler characteristic. Suppose* $\mathbf{S}$ *is an essential coincidence class in* $\mathscr{E}(f, g)$.

1. *If* $C_\#(f, g, \mathbf{S}) \subseteq \Gamma_1$, *then all coincidence classes covering* $\mathbf{S}$ *in the* $\Gamma_1 - \Gamma_2$ *lifting diagram of* f *and* g *are essential.*
2. *If* $\Gamma_1 \in \mathscr{IC}(f, g, \Gamma_2)$, *then all coincidence classes covering* $\mathbf{S}$ *in the* $\Gamma_1 - \Gamma_2$ *lifting diagram of* f *and* g *are essential if and only if*

$$\begin{aligned} O_1 \cdot \Gamma_1/\Gamma_1 &\cap C_\#(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 \\ &= f_\#^{-1}(O(\pi_2)) \cdot \Gamma_1/\Gamma_1 \cap C_\#(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 . \end{aligned}$$

Before beginning the proof, some comments on the condition in (2) are in order. The idea is simply that, in moving from one class covering $\mathbf{S}$ to another, the index can only change by changing sign. Changing sign, in turn, occurs when the two covering transformations (or their corresponding elements in the fundamental group) involved in moving one class to another have different "parity"—that is, one changes the orientation, and the other does not. The condition in (2), while awkward, is precisely the condition needed to rule this out.

Proof of Theorem 5.6. The class of manifolds specified is closed under finite covers, and all pairs of manifolds in the class are Wenken pairs. More precisely, the maps f and g (up to homotopy) may be assumed to have each coincidence class contain a single point, which can be removed by a perturbation supported on an arbitrarily small neighborhood if the class is inessential.

First, suppose $C_\#(f, g, \mathbf{S}) \subseteq \Gamma_1$. Then every coincidence class $\widetilde{\mathbf{S}}$ covering $\mathbf{S}$ consists of a single point. Then any perturbation supported in a neighborhood of it passes down to a perturbation of f and g in a neighborhood of $\mathbf{S}$. If it is inessential and that perturbation removes $\widetilde{\mathbf{S}}$, then the corresponding perturbation removes $\mathbf{S}$ as a coincidence class.

Now suppose $\Gamma_1 \in \mathscr{IC}(f, g, \Gamma_2)$. We will consider the case $\Gamma_i \in \mathscr{CO}(\pi_i)$: the case of $\tilde{g}$ a homeomorphism is similar. Suppose $\Gamma_2 \subseteq O(\pi_2)$, $\Gamma_1 \in \mathscr{IC}(f, g, \Gamma_2)$ and

$$\begin{aligned} O_1 \cdot \Gamma_1/\Gamma_1 &\cap C_\#(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 \\ &= f_\#^{-1}(O(\pi_2)) \cdot \Gamma_1/\Gamma_1 \cap C_\#(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 . \end{aligned}$$

If $\tilde{x} \in \widetilde{\mathbf{S}}$, then $\tilde{x}$ is an isolated coincidence, and $\operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x})$ is nonzero. Any other element of $\widetilde{\mathbf{S}}$ has the form $\alpha(\tilde{x})$ with $\alpha \in C_{\#}(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1$. To relate the indices of $\tilde{x}$ and $\alpha(\tilde{x})$ choose neighborhoods $W \subseteq V$ which isolate $\tilde{x}$ as a coincidence. Then $\alpha(W) \subseteq \alpha(V)$ isolate $\alpha(\tilde{x})$, and the indices are related by the diagrams

$$\begin{array}{ccccc} H_n(\widetilde{X}_1) & \to & H_n(\widetilde{X}_1, \widetilde{X}_1\backslash W) & \overset{\cong}{\leftarrow} & H_n(V, V\backslash W) \\ \alpha_* \downarrow & & \alpha_* \downarrow & & \alpha_* \downarrow \\ H_n(\widetilde{X}_1) & \to & H_n(\widetilde{X}_1, \widetilde{X}_1\backslash \alpha(W)) & \overset{\cong}{\leftarrow} & H_n(\alpha(V), \alpha(V)\backslash \alpha(W)) \end{array}$$

and

$$\begin{array}{ccc} H_n(V, V\backslash W) & \xrightarrow{(f_*, g_*)} & H_n(\widetilde{X}_2 \times \widetilde{X}_2, \widetilde{X}_2 \times \widetilde{X}_2\backslash\Delta(\widetilde{X}_2)) \\ \alpha_* \Big\downarrow & & \overline{f}(\alpha)_* \Big\downarrow \\ H_n(\alpha(V), \alpha(V)\backslash\alpha(W)) & \xrightarrow{(f_*, g_*)} & H_n(\widetilde{X}_2 \times \widetilde{X}_2, \widetilde{X}_2 \times \widetilde{X}_2\backslash\Delta(\widetilde{X}_2)) \end{array}$$

so $\operatorname{Ind}(\tilde{f}, \tilde{g}, \alpha(\tilde{x})) = \deg(\alpha)\deg(\overline{f}(\alpha))\operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x})$.

But the condition

$$\begin{aligned} O_1 \cdot \Gamma_1/\Gamma_1 &\cap C_{\#}(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 \\ &= f_{\#}^{-1}(O(\pi_2)) \cdot \Gamma_1/\Gamma_1 \cap C_{\#}(f, g, \mathbf{S}) \cdot \Gamma_1/\Gamma_1 \end{aligned}$$

is precisely the condition needed to guarantee that $\deg(\alpha) = \deg(\overline{f}(\alpha))$ for $\alpha \in p_{1\#}(C_{\#}(f, g, \mathbf{S}))$, so $\operatorname{Ind}(\tilde{f}, \tilde{g}, \alpha(\tilde{x})) = \operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x})$. Then

$$\operatorname{Ind}(\tilde{f}, \tilde{g}, \widetilde{\mathbf{S}}) = [C_{\#}(f, g, \mathbf{S}) \cdot \Gamma_1 : \Gamma_1]\operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x}) \neq 0.$$

Thus $\widetilde{\mathbf{S}}$ is essential. On the other hand, if the condition fails, exactly half of the elements in $p_{1\#}(C_{\#}(f, g, \mathbf{S}))$ will have $\deg(\alpha) = \deg(\overline{f}(\alpha))$, while the other half will have $\deg(\alpha) = -\deg(\overline{f}(\alpha))$. The corresponding points in $\widetilde{\mathbf{S}}$ will have $\operatorname{Ind}(\tilde{f}, \tilde{g}, \alpha(\tilde{x})) = \operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x})$ and $\operatorname{Ind}(\tilde{f}, \tilde{g}, \alpha(\tilde{x})) = -\operatorname{Ind}(\tilde{f}, \tilde{g}, \tilde{x})$ respectively, so $\operatorname{Ind}(\tilde{f}, \tilde{g}, \widetilde{\mathbf{S}}) = 0$ and $\mathbf{S}$ is inessential. □

Note that if $C_{\#}(f, g, \mathbf{S}) \subseteq \Gamma_1$, all coincidence classes covering $\mathbf{S}$ are homeomorphic to $\mathbf{S}$.

COROLLARY 5.7. *If $\pi_1 \in \mathscr{JC}(f, g, \pi_2)$, then for every $\Gamma_1 - \Gamma_2$ lifting diagram, essential classes lift to essential classes. Moreover, if $\mathbf{S} \in \mathscr{R}(f, g)$ and $\mathscr{S}$ is the set of coincidence classes covering $\mathbf{S}$ in the $\Gamma_1 - \Gamma_2$ lifting diagram, then $|\Phi_1|\operatorname{Ind}(f, g, \mathbf{S}) = \sum_{\widetilde{\mathbf{S}} \in \mathscr{S}} \operatorname{Ind}(\beta \circ \tilde{f}, \tilde{g})$.*

Proof. If $\pi_1 \in \mathscr{JC}(f, g, \pi_2)$, then either $\pi_i = O(\pi_i)$ for $i = 1, 2$, or g is a homeomorphism. In the former case, condition (ii) of

Theorem 5.6 is trivially satisfied for all f and g. If g is a homeomorphism, we may assume without loss that $g = \mathrm{id}$. Then $O(\pi_2) = O(\pi_1)$ and $f_\#^{-1}(O(\pi_2)) = g_\#^{-1}(O(\pi_1)) = O(\pi_1)$. In either case, if $x \in \mathrm{Coin}(f, g)$ is an isolated coincidence and $\tilde{x} \in p_1^{-1}(x)$ is in $\mathrm{Coin}(\tilde{f}, \tilde{g})$, then $\mathrm{Ind}(\tilde{f}, \tilde{g}, \tilde{x}) = \mathrm{Ind}(f, g, x)$. Since $\mathbf{S}$ may be assumed to consist of isolated coincidences,

$$\begin{aligned}|\Phi_1|\mathrm{Ind}(f, g, \mathbf{S}) &= \sum_{\tilde{x}\in p_1^{-1}(\mathbf{S})} \mathrm{Ind}(\beta \circ \tilde{f}, \tilde{g}, \tilde{x}) \\ &= \sum_{\tilde{\mathbf{S}}\in\mathcal{S}} \mathrm{Ind}(\beta \circ \tilde{f}, \tilde{g}). \qquad \square\end{aligned}$$

It will be useful to note what these results imply in the special case $\Gamma_2 = \pi_2$, $\Gamma_1 \neq \pi_1$ and $\Gamma_2 \neq \pi_2$, $\Gamma_1 = \pi_1$. In the first case, $\Gamma_2 = \pi_2$, $\Gamma_1 \neq \pi_1$, the maps f and g each have only one lift: $\tilde{f} = f \circ p_1$ and $\tilde{g} = g \circ p_1$. If $\mathbf{S} \in \mathcal{R}(f, g)$, then $\mathbf{S}$ is covered by $[\pi_1 : C_\#(f, g, \mathbf{S}) \cdot \Gamma_1]$ coincidence classes. Each is a covering space over $\mathbf{S}$ of order $[C_\#(f, g, \mathbf{S}) \cdot \Gamma_1 : \Gamma_1]$, and either all are essential or all are inessential. To compare the Lefschetz numbers of (f, g) and $(\tilde{f}, \tilde{g})$, we need $\Gamma_1, \pi_1 \in \mathcal{JC}(f, g, \pi_2)$. This is only possible if both X_1 and X_2 are orientable manifolds of the same dimension. Then $\Gamma_1 \subset \pi_1 = O(\pi_1)$ and $\pi_2 = O(\pi_2)$. In particular, the covering map $p_1 : \tilde{X}_1 \to X_1$ has $\deg(p_1) = |\Phi_1|$ and $L(\tilde{f}, \tilde{g}) = \deg(p_1) \cdot L(f, g) = |\Phi_1| \cdot L(f, g)$.

On the other hand, if $\Gamma_2 \neq \pi_2$, $\Gamma_1 = \pi_1$, then there are $|\Phi_2|$ lifts $\beta \circ \tilde{f}$ and $\beta \circ \tilde{g}$ for f and g, with $f = p_2 \circ \beta \tilde{f}$ and $g = p_2 \circ \beta \tilde{g}$. In this case, the decomposition $p_1^{-1}(\mathrm{Coin}(f, g)) = \bigcup_{\rho\in\Phi_2} \mathrm{Coin}(\beta \circ \tilde{f}, \tilde{g})$, becomes a partition of $\mathrm{Coin}(f, g)$, which preserves coincidence classes. That is, there is a bijection $\mathcal{R}(f, g) \leftrightarrow \bigcup \mathcal{R}(\beta \circ \tilde{f}, \tilde{g})$. In the Wenken manifold setting of Theorem 5.6, this bijection preserves essential classes, so $N(f, g) = \sum_{\beta\in\Phi_2} N(\beta \circ \tilde{f}, \tilde{g})$. To compare the Lefschetz numbers of (f, g) and $(\beta \circ \tilde{f}, \tilde{g})$, we now need $\pi_1 \in \mathcal{JC}(f, g, \pi_2)$ and $\pi_1 \in \mathcal{JC}(f, g, \Gamma_2)$. Once again, this is only possible if X_1 and X_2 are orientable manifolds of the same dimension. In that case, any $\mathbf{S} \in \mathcal{R}(f, g)$ has $\mathrm{Ind}(f, g, \mathbf{S}) = \deg(p_2) \cdot \mathrm{Ind}(\beta \circ \tilde{f}, \tilde{g})$, so $L(f, g) = \sum_{\beta\in\Phi_2} L(\beta \circ \tilde{f}, \tilde{g})$.

To collate all of this information, we now introduce a Nielsen-type coincidence number for the $\Gamma_1 - \Gamma_2$ lifting diagram. The following data is required: maps $f, g : X_1 \to X_2$, $\Gamma_2 \in \mathcal{C}(\pi_2)$, $\Gamma_1 \in$

$\mathscr{C}(f, g, \Gamma_2)$ and lifts $\tilde{f}$, $\tilde{g}$ of f and g. We then define

$$\widetilde{N}(f, g, \Gamma_1) = \frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} N(\beta \circ \tilde{f}, \tilde{g}).$$

If $\Gamma_1 \in \mathscr{J\!C}(f, g, \Gamma_2)$, we can also define a Lefschetz-type coincidence number

$$\widetilde{L}(f, g, \Gamma_2) = \frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}, \tilde{g})|.$$

Apparently, both numbers depend on both Γ_1 and Γ_2. The following result justifies the notation.

THEOREM 5.8. 1. *$\widetilde{N}(f, g, \Gamma_1)$ is independent of Γ_2. That is, if $\Gamma_1 \in \mathscr{C}(f, g, \Gamma_2) \cap \mathscr{C}(f, g, \Gamma_2')$, and (f, g) lift to $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ in the $\Gamma_1 - \Gamma_2$ and $\Gamma_1 - \Gamma_2'$ lifting diagrams respectively, then*

$$\frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} N(\beta \circ \tilde{f}, \tilde{g}) = \frac{1}{|\Phi_1|} \sum_{\beta' \in \Phi_2'} N(\beta' \circ \tilde{f}', \tilde{g}').$$

2. *When defined, $\widetilde{L}(f, g, \Gamma_2)$ is independent of Γ_1. That is, if $\Gamma_1, \Gamma_1' \in \mathscr{J\!C}(f, g, \Gamma_2)$, and (f, g) lift to $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ in the $\Gamma_1 - \Gamma_2$ and $\Gamma_1' - \Gamma_2'$ lifting diagrams respectively, then*

$$\frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}, \tilde{g})| = \frac{1}{|\Phi_1'|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}', \tilde{g}')|.$$

Proof. To show that $\widetilde{N}(f, g, \Gamma_1)$ is independent of Γ_2, suppose $\Gamma_2' \subset \Gamma_2$ and $\Gamma_1 \in \mathscr{C}(f, g, \Gamma_2')$. We may assume that $\tilde{f}$ and $\tilde{g}$ are covered by $\tilde{f}'$ and $\tilde{g}'$. If $s\colon \Phi_2 \to \Phi_2'$ is a section of the natural projection $\Phi_2' \to \Phi_2$, then for every $\beta \in \Phi_2$, the map $s(\beta) \circ \tilde{f}'$ can be used as a reference lift for $\beta \circ \tilde{f}$. Let $\Psi_2 = \Phi_2'/\Phi_2$. Then $N(\beta \circ \tilde{f}, \tilde{g}) = \sum_{\psi \in \Psi_2} N(\psi \circ s(\beta) \circ \tilde{f}', \tilde{g}')$, and

$$\begin{aligned} \frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} N(\beta \circ \tilde{f}, \tilde{g}) &= \frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} \sum_{\psi \in \Psi_2} N(\psi \circ s(\beta) \circ \tilde{f}', \tilde{g}') \\ &= \frac{1}{|\Phi_1|} \sum_{\beta' \in \Phi_2'} N(\beta' \circ \tilde{f}', \tilde{g}'). \end{aligned}$$

Of course, if $\Gamma_2' \not\subset \Gamma_2$, both contain $\Gamma_2' \cap \Gamma_2$ and $\Gamma_1 \in \mathscr{C}(f, g, \Gamma_2' \cap \Gamma_2)$, so Γ_2' can be replaced by $\Gamma_2' \cap \Gamma_2$.

To show that $L(f, g, \Gamma_2)$ is independent of Γ_1, suppose $\Gamma_1' \subset \Gamma_1$ and $\Gamma_1', \Gamma_1 \in \mathscr{IE}(f, g, \Gamma_2)$. This requires $\Gamma_i \subseteq O(\pi_i)$. Suppose $p_1 : \widetilde{X}_1 \to X_1$ and $p_1' : \widetilde{X}_1' \to X_1$ are the covering spaces corresponding to Γ_1 and Γ_1' respectively, $p : \widetilde{X}_1' \to \widetilde{X}_1$ is the covering map of $\widetilde{X}_1'$ over $\widetilde{X}_1$, and $\tilde{f}, \tilde{g}$ are lifts of f and g to $\widetilde{X}_1$. Then $\tilde{f} \circ p$ and $\tilde{g} \circ p$ are the lifts of f and g to $\widetilde{X}_1'$, and $L(\tilde{f} \circ p, \tilde{g} \circ p) = \deg(p) \cdot L(\tilde{f}, \tilde{g}) = |\Gamma_1 : \Gamma_1'| \cdot L(\tilde{f}, \tilde{g})$. Then

$$\begin{aligned} \frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}, \tilde{g})| &= \frac{1}{|\Phi_1'|} \frac{1}{|\Gamma_1 : \Gamma_1'|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}, \tilde{g})| \\ &= \frac{1}{|\Phi_1'|} \sum_{\beta \in \Phi_2} |L(\beta \circ \tilde{f}', \tilde{g}')|. \qquad \square \end{aligned}$$

Note that, in contrast to $\widetilde{L}(f, g, \Gamma_2)$, the quantity

$$\frac{1}{|\Phi_1|} \sum_{\beta \in \Phi_2} L(\beta \circ \tilde{f}, \tilde{g})$$

(when defined) is independent of both Γ_1 and Γ_2. This is one reason for the introduction of the absolute value in the definition of $\widetilde{L}(f, g, \Gamma_2)$: without it, no new information is obtained. Also, we note that, in general, $\widetilde{N}$ does depend on Γ_1 and $\widetilde{L}$ does depend on Γ_2. We now examine that dependence.

THEOREM 5.9. 1. *If* $\Gamma_1, \Gamma_1' \in \mathscr{E}(f, g, \Gamma_2)$ *for some* $\Gamma_2 \in \mathscr{E}(\pi_2)$, *and* $\Gamma_1' \subseteq \Gamma_1$, *then* $\widetilde{N}(f, g, \Gamma_1) \geq \widetilde{N}(f, g, \Gamma_1')$.

2. *Suppose* X_1, X_2 *are manifolds of the same dimension, and neither is a surface with negative Euler characteristic. If* $C_\#(f, g, \mathbf{S}) \cap \Gamma_1 = C_\#(f, g, \mathbf{S}) \cap \Gamma_1'$ *for every* $\mathbf{S} \in \mathscr{E}(f, g)$ *then* $\widetilde{N}(f, g, \Gamma_1) = \widetilde{N}(f, g, \Gamma_1')$.

Proof. Recall that if $(\tilde{f}, \tilde{g})$ covers (f, g) in the $\Gamma_1 - \Gamma_2$ lifting diagram, and $p : \widetilde{X}_1' \to \widetilde{X}_1'$ is the covering space corresponding to $\Gamma_1' \to \Gamma_1$, then $(\tilde{f} \circ p, \tilde{g} \circ p)$ covers (f, g) in the $\Gamma_1' - \Gamma_2$ lifting diagram. Then $\widetilde{\mathbf{S}} \in \mathscr{E}(\tilde{f}, \tilde{g})$ is covered by $[\Gamma_1 : C_\#(\tilde{f}, \tilde{g}, \widetilde{\mathbf{S}}) \cdot \Gamma_1'] = [\Gamma_1 : (C_\#(f, g, \mathbf{S}) \cap \Gamma_1) \cdot \Gamma_1']$ coincidence classes, which are either all essential or all inessential. That is, each $\widetilde{\mathbf{S}} \in \mathscr{E}(\tilde{f}, \tilde{g})$ is covered by at most $[\Gamma_1 : \Gamma_1']$ essential classes. If $C_\#(f, g, \mathbf{S}) \cap \Gamma_1 = C_\#(f, g, \mathbf{S}) \cap \Gamma_1'$ then there are exactly $[\Gamma_1 : \Gamma_1']$ classes, and Theorem 5.6 implies that they are all essential. $\square$

COROLLARY 5.10. *Suppose X_1, X_2 are manifolds of the same dimension, and neither is a surface with negative Euler characteristic. If $C_\#(f, g, \mathbf{S}) \subset \Gamma_1$ for every $\mathbf{S} \in \mathscr{E}(f, g)$ and every $\Gamma_1 \in \mathscr{C}(\pi_1)$, then $\widetilde{N}$ is independent of Γ_1: $\widetilde{N}(f, g, \Gamma_1) = N(f, g)$ for every Γ_1.*

THEOREM 5.11. 1. *If $\Gamma_2', \Gamma_2 \in \mathscr{C}(\pi_2)$ with both $\mathscr{H}\mathscr{C}(f, g, \Gamma_2)$ and $\mathscr{H}\mathscr{C}(f, g, \Gamma_2')$ nonempty and $\Gamma_2' \subset \Gamma_2$, then $\widetilde{L}(f, g, \Gamma_2) \leq \widetilde{L}(f, g, \Gamma_2')$.*

2. *Let $p: \Phi_2' \to \Phi_2$ be the natural projection, and let $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ be the lifts of (f, g) corresponding to Γ_2 and Γ_2' respectively. Then $\widetilde{L}(f, g, \Gamma_2) = \widetilde{L}(f, g, \Gamma_2')$ if and only if, for every $\beta \in \Phi_2$, all lifts $(\beta' \circ \tilde{f}', \tilde{g})$, $\beta' \in p^{-1}(\beta)$, have $L(\beta' \circ \tilde{f}', \tilde{g})$ with the same sign.*

Proof. Both statements follow immediately from the equality

$$L(\beta \circ \tilde{f}, \tilde{g}) = \sum_{\beta' \in p^{-1}(\beta)} L(\beta' \circ \tilde{f}', \tilde{g}'). \qquad \square$$

6. Infrasolvmanifolds. The goal of this work is to use the machinery developed in the previous section to extend the results of **[18]** and **[19]** to infrasolvmanifolds. In this section, we briefly review the topology of infrasolvmanifolds, before turning to coincidence theory on infrasolvmanifolds in the next section. Three subclasses of infrasolvmanifolds will merit special attention: nilmanifolds, infranilmanifolds and solvmanifolds. The topology of nilmanifolds is described in **[17]**; solvmanifolds in **[20]** and **[24]**; infranilmanifolds in **[1]**; and infrasolvmanifolds in **[12]**.

We begin with the constructive definition of infrasolvmanifolds. Let S be a solvable connected simply connected Lie group, and consider the Lie group $G = S \rtimes \operatorname{Aut}(S)$. G acts on $\mathscr{S}$ by $(s, \alpha) \cdot s' = s\alpha(s')$. If $\pi \subset G$ is a torsion-free subgroup with finite projection Φ onto $\operatorname{Aut}(S)$, then $M = \pi \backslash S$ is an *infrasolvmanifold.* If $\Gamma = \pi \cap S$, then $\widetilde{M} = \Gamma \backslash S$ is a covering space of M with covering group $\Phi = \pi / \Gamma$. M is connected, and is compact if and only if π is uniform in $S \rtimes K$, or equivalently if and only if Γ is uniform in S. We will restrict ourselves to the compact case. This same construction generates the three subclasses of manifolds mentioned: M is an *infranilmanifold* if S is nilpotent, and a *nilmanifold* if S is nilpotent and $\Phi = 1$. M is a *solvmanifold* if Φ is solvable, and a *special solvmanifold* if $\Phi = 1$. Clearly, every infrasolvmanifold has a finite regular cover by a solvmanifold (indeed, by a special solvmanifold) and every infranilmanifold has a finite regular cover by a nilmanifold. In all cases, the universal cover S is contractible, so the manifold is aspherical with $\pi_1(M) \cong \pi$.

Compact infrasolvmanifolds are determined up to homeomorphism by their fundamental group, which must be torsion-free and finitely generated. Recall that for any property P of groups, a group G is *virtually* P if there is a normal subgroup of finite index H which has property P. Similarly, a group is *poly*-P if there is a normal series $\{G_i\}$ for G such that each subquotient G_i/G_{i+1} has property P. In particular, G is polycyclic if there is a normal series with $G_i/G_{i+1} \cong \mathbf{Z}$. Another group-theoretic definition: A strongly torsion-free $\mathscr{S}$-group is a group π with a finitely generated torsion-free nilpotent $\Gamma \triangleleft \pi$ such that π/Γ is free abelian.

THEOREM 6.1. *There is a one-to-one correspondence between homeomorphism classes of nilmanifolds, infranilmanifolds, solvmanifolds and infrasolvmanifolds and isomorphism classes of the following categories of torsion-free finitely generated groups: nilpotent groups, virtually nilpotent groups, strongly torsion-free solvable groups and virtually polycyclic groups.*

If π is the fundamental group of M, then π is virtually polycyclic, with $\dim(M) = rk(\pi) = n$, where n is the number of infinite cyclic summands in the "virtual" polycyclic decomposition of π.

In general, infranilmanifolds, solvmanifolds and infrasolvmanifolds are not orientable as manifolds. The Klein bottle, for example, is both an infranilmanifold and a solvmanifold. However, all nilmanifolds are orientable, and of course, if M is a non-orientable infrasolvmanifold, there is an orientable infrasolvmanifold covering it, which is a solvmanifold or infranilmanifold if M is. It is well known that all solvmanifolds have zero Euler characteristic; hence all infrasolvmanifolds have $\chi(M) = 0$. Finally, all pairs of infrasolvmanifolds of the same dimension are Wenken pairs. For dimensions 3 or more, this follows from the result mentioned in §2. In dimensions 1 and 2, the requirement $\chi(M) = 0$ limits the possibilities to the circle, torus and Klein bottle. All of these are in fact solvmanifolds, for which the result is established in (among other places) **[19]**.

7. Coincidence numbers on infrasolvmanifolds. The tools we will use to study coincidence numbers for infrasolvmanifolds are the machinery developed in §5 and the following results:

THEOREM 7.1 [**19**, Thm. 1]. *If* M_1, M_2 *are compact connected orientable solvmanifolds of the same dimension, and* $\mathbf{S} \in \mathscr{R}(f, g)$, *then* $\operatorname{Ind}(f, g, \mathbf{S}) \in \{-1, 0, 1\}$.

THEOREM 7.2 [**19**, Thm. 2]. *If* M_1, M_2 *are compact connected orientable solvmanifolds of the same dimension, then* $N(f, g) \geq |L(f, g)|$ *for every* $f, g\colon M_1 \to M_2$. *Moreover, if* M_2 *is a nilmanifold, then* $N(f, g) = |L(f, g)|$ *for every* (f, g).

It is now a simple matter to combine these and obtain our main results.

THEOREM 7.3. *If* M_1 *and* M_2 *are compact connected infrasolvmanifolds of the same dimension, and* $\pi_1 \in \mathscr{IC}(f, g, \pi_2)$, *then* $\operatorname{Ind}(f, g, \mathbf{S}) \in \{-1, 0, 1\}$ *for every* $\mathbf{S} \in \mathscr{R}(f, g)$.

Proof. Choose a solvable $\Gamma_2' \in \mathscr{CO}(\pi_2)$ and a solvable $\Gamma_1' \in \mathscr{IC}(f, g, \Gamma_2')$. In the $\Gamma_1' - \Gamma_2'$ lifting diagram, $\widetilde{M}_1$ and $\widetilde{M}_2$ are orientable solvmanifolds, and by Theorem 7.1, any coincidence class $\widetilde{\mathbf{S}}$ covering $\mathbf{S}$ has $\operatorname{Ind}(\tilde{f}, \tilde{g}, \widetilde{\mathbf{S}}) \in \{-1, 0, 1\}$. But

$$|\Phi_1| \operatorname{Ind}(f, g, \mathbf{S}) = \sum_{\widetilde{\mathbf{S}} \in \mathscr{S}} \operatorname{Ind}(\beta \circ \tilde{f}, \tilde{g}),$$

so

$$\begin{aligned} |\Phi_1| |\operatorname{Ind}(f, g, \mathbf{S})| &\leq \sum_{\widetilde{\mathbf{S}} \in \mathscr{S}} |\operatorname{Ind}(\beta \circ \tilde{f}, \tilde{g})| \\ &\leq |\mathscr{S}| = [\pi_1 : C_\#(f, g, \mathbf{S})\Gamma_1]. \end{aligned}$$

But $\mathbf{S}$ is covered by at most $|\Phi|$ coincidence classes, so

$$|\Phi_1| |\operatorname{Ind}(f, g, \mathbf{S})| \leq |\Phi| \quad \text{and} \quad |\operatorname{Ind}(f, g, \mathbf{S})| \leq 1. \quad \square$$

THEOREM 7.4. *If* M_1 *and* M_2 *are compact connected infrasolvmanifolds of the same dimension, then* $\widetilde{N}(f, g, \Gamma_1) \geq \widetilde{L}(f, g, \Gamma_2)$ *for every* $f, g\colon M_1 \to M_2$ *and every* $\Gamma_2 \in \mathscr{C}(\pi_2)$, $\Gamma_1 \in \mathscr{IC}(f, g, \Gamma_2)$. *In particular,* $N(f, g) \geq \max\{\widetilde{L}(f, g, \Gamma_2) | \mathscr{IC}(f, g, \Gamma_2) \neq \varnothing\}$.

Proof. Choose a solvable $\Gamma_2' \in \mathscr{CO}(\pi_2)$ and a solvable $\Gamma_1' \in \mathscr{IC}(f, g, \Gamma_2')$. In the $\Gamma_1' - \Gamma_2'$ lifting diagram, $\widetilde{M}_1$ and $\widetilde{M}_2$ are orientable solvmanifolds, and by Theorem 7.2, $N(\beta \circ \tilde{f}, \tilde{g}) \geq |L(\beta \circ \tilde{f}, \tilde{g})|$. Then for any such choice of Γ_2' and Γ_1', $\widetilde{N}(f, g, \Gamma_1') \geq \widetilde{L}(f, g, \Gamma_2')$. But for any Γ_1, Γ_2 with $\Gamma_1 \in \mathscr{IC}(f, g, \Gamma_2)$, there exists a solvable $\Gamma_2' \in \mathscr{CO}(\pi_2)$ and a solvable $\Gamma_1' \in \mathscr{FC}(f, g, \Gamma_2')$ with $\Gamma_i' \subseteq \Gamma_i$. Then $\widetilde{N}(f, g, \Gamma_1) \geq \widetilde{N}(f, g, \Gamma_1') \geq \widetilde{L}(f, g, \Gamma_2') \geq L(f, g, \Gamma_2)$. □

COROLLARY 7.5. *If M is a compact connected infrasolvmanifold, then $N(f) \geq |L(f)|$ for every $f: M \to M$.*

Can the numbers $\widetilde{L}(f, g, \Gamma_2)$ ever compute $N(f, g)$? That is, is there a systematic choice of $\Gamma_2 \in \mathscr{C}(\pi_2)$ with $\widetilde{L}(f, g, \Gamma_2) = N(f, g)$ for all (f, g)? The natural route to such an equality is to find $\Gamma_1 \in \mathscr{FC}(f, g, \Gamma_2)$ with $N(f, g) = \widetilde{N}(f, g, \Gamma_1) = \widetilde{L}(f, g, \Gamma_2)$. If Γ_2 is nilpotent, the second equality will hold [**19**, Thm. 2]. To obtain the first, we need $C_\#(f, g, \mathbf{S}) \subseteq \Gamma_1$ for all essential coincidence classes $\mathbf{S}$. The following results give sufficient conditions for this.

COROLLARY 7.6. *If M_1 and M_2 are compact connected infrasolvmanifolds of the same dimension, and $\pi_1 \in \mathscr{FC}(f, g, \pi_2)$, then $C_\#(f, g, \mathbf{S}) \subset \Gamma_1$ for every $\Gamma_1 \in \mathscr{C}(\pi_1)$ and every essential coincidence class $\mathbf{S} \in \mathscr{E}(f, g)$.*

Proof. If $\mathbf{S}$ is essential, then from Theorem 7.3, $\operatorname{Ind}(f, g, \mathbf{S}) = \pm 1$, and $|\Phi_1| = [\pi_1 : C_\#(f, g, \mathbf{S})\Gamma_1]$. That is, $C_\#(f, g, \mathbf{S}) \subset \Gamma_1$.

LEMMA 7.7 [**19**, Thm. 3.4]. *If M_1 and M_2 are compact connected solvmanifolds of the same dimension, then for every $f, g: M_1 \to M_2$, $C_\#(f, g, \mathbf{S}) = 1$ for every $\mathbf{S} \in \mathscr{E}(f, g)$.*

LEMMA 7.8. *Suppose M_1, M_2 are infrasolvmanifolds of the same dimension and M_2 is a nilmanifold* (*resp. solvmanifold, infranilmanifold*). *Then either M_1 is a nilmanifold* (*resp. solvmanifold, infranilmanifold*), *or $N(f, g) = 0$ for all $f, g: M_1 \to M_2$.*

Proof. We will prove the case of M_2 an infranilmanifold, M_1 not an infranilmanifold. Since M_2 is an infranilmanifold, there is a nilpotent $\Gamma_2 \in \mathscr{C}(\pi_2)$. The derived series $\Gamma_2^k = [\Gamma_2, \Gamma_2^{k-1}]$ descends to 1: there is an n with $\Gamma_2^n = 1$. Take $\Gamma_1 \in \mathscr{C}(f, g, \Gamma_2)$. Since M_1 is not an infranilmanifold, Γ_1 is not nilpotent, and $\Gamma_1^n \neq 1$. But $f_\#, g_\#(\Gamma_1) \subseteq \Gamma_2$, so $f_\#, g_\#(\Gamma_1^n) \subseteq \Gamma_2^n = 1$. Then $K = \ker(f_\#) \cap \ker(g_\#)$ has $\operatorname{rk}(K) > 1$ and $Q = \pi_1/K$ is a finitely generated torsion-free virtually poly-$\mathbf{Z}$ group. Of course, if $\rho: \pi_1 \to Q$ is the natural projection, there are homomorphisms $\phi, \gamma: Q \to \pi_2$ with $f_\# = \phi \circ \rho$ and

$g_{\#} = \gamma \circ \rho$. From [2], there is then a compact connected infrasolvmanifold $\overline{M}$ with $\pi_1(\overline{M}) = Q$ and $\dim(\overline{M}) = \mathrm{rk}(Q) < \mathrm{rk}(\pi_1) = \dim(M_1) = \dim(M_2)$. Moreover, since M_1, M_2 and $\overline{M}$ are all aspherical, the fundamental group diagram

$$\begin{array}{ccc} \pi_1 & \overset{f_{\#},\, g_{\#}}{\rightrightarrows} & \pi_2 \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle\phi,\gamma} & \\ & Q & \end{array}$$

can be realized as the diagram of maps

$$\begin{array}{ccc} M_1 & \overset{f',\, g'}{\rightrightarrows} & M_2 \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \overline{f},\overline{g}} & \\ & \overline{M} & \end{array}$$

with $f' \simeq f$ and $g' \simeq g$. Moreover, $\overline{f}$ and $\overline{g}$ may be assumed to be smooth.

Now consider the map $\overline{M} \overset{\overline{f}\times\overline{g}}{\to} M_2 \times M_2$. By transversality, $\overline{f}\times\overline{g}$ is homotopic to a map $\overline{f}'\times\overline{g}'$ which is transverse to the diagonal $\Delta(M_2)$. Since $\dim(\overline{M}) + \dim(\Delta(M_2)) < \dim(M_2 \times M_2)$, this really means that $\overline{f}' \times \overline{g}'$ misses the diagonal, or that $\mathrm{Coin}(\overline{f}', \overline{g}') = \varnothing$. Then the maps f'', $g''\colon M_1 \to M_2$ given by $f'' = \overline{f}' \circ p$, $g'' = \overline{g}' \circ p$ are homotopic to f and g respectively, and have $\mathrm{Coin}(f'', g'') = \varnothing$. □

THEOREM 7.9. *Suppose M_1 an n-dimensional infrasolvmanifold and either M_2 is an n-dimensional infranilmanifold and solvmanifold, or $\pi_1 \in \mathscr{HC}(f, g, \pi_2)$. Then $N(f, g) = \widetilde{L}(f, g, \Gamma_2)$ for every nilpotent $\Gamma_2 \in \mathscr{C}(\pi_2)$.*

Proof. First, suppose M_2 is a solvmanifold. If M_1 is not an infranilmanifold, or is not a solvmanifold, Lemma 7.8 implies that $N(f, g) = 0$ for all (f, g). Then for all Γ_2, $\widetilde{L}(f, g, \Gamma_2) = 0$ as well. We can therefore assume that M_1 is likewise both an infranilmanifold and a solvmanifold. Since it is a solvmanifold, Corollary 5.10 implies $N(f, g) = \widetilde{N}(f, g, \Gamma_1)$ for all Γ_1. On the other hand, since it is an infranilmanifold, $\widetilde{N}(f, g, \Gamma_1) \geq \widetilde{L}(f, g, \Gamma_2)$ for every nilpotent $\Gamma_2 \in \mathscr{C}(\pi_2)$ and $\Gamma_1 \in \mathscr{HC}(f, g, \Gamma_2)$. The argument for the case $\pi_1 \in \mathscr{HC}(f, g, \pi_2)$ is the same, only substituting Corollary 7.6 for Lemma 7.7. □

COROLLARY 7.10. *If M is an infranilmanifold, then $N(f) = \widetilde{L}(f, \mathrm{id}, \Gamma_2)$ for every nilpotent $\Gamma_2 \in \mathscr{C}(\pi_2)$ and every $f\colon M \to M$.*

The Klein bottle K is perhaps the best-known example of a space which is both a solvmanifold and an infranilmanifold, but not a nilmanifold. The fundamental group $\pi_1(K) \cong \mathbf{Z} \rtimes \mathbf{Z}$, where $n \in \mathbf{Z}$ acts on $m \in \mathbf{Z}$ by $n \cdot m = (-1)^n m$. The orientation subgroup $O(\pi) = 2\mathbf{Z} \rtimes \mathbf{Z} \cong \mathbf{Z}^2$ is invariant under all homomorphisms, so $O(\pi) \in \mathscr{IC}(f, g, O(\pi))$ for all (f, g). If α is the generator of $\pi/O(\pi)$ and $(\tilde{f}, \tilde{g})$ are lifts of (f, g) in the $O(\pi) - O(\pi)$ lifting diagram, then $N(f, g) = |L(\tilde{f}, \tilde{g})| + |L(\alpha \circ \tilde{f}, \tilde{g})|$. In particular, $N(f) = |L(\tilde{f})| + |L(\alpha \circ \tilde{f})|$.

8. The Jiang condition. As was mentioned in §1, the Jiang subgroup and Jiang condition **[15]** provide additional tools for estimating and calculating Nielsen numbers. In this section, we consider how the results described in this paper relate to the Jiang condition. Given a map $f: X \to X$ and $x \in \mathrm{Fix}(f)$, define

$$J(f) = \{\omega \in \pi = \pi_1(X, x) | \omega = [H(x, -)] \text{ for some } H: f \simeq f\}.$$

Some of the important properties of $J(f)$ are:

1. $J(f)$ is independent of the base point chosen.
2. $J(f) \subseteq \mathbf{C}_\pi(f_\#(\pi))$, the centralizer of $f_\#(\pi)$ in π, with equality if X is aspherical.
3. $J(X) = J(\mathrm{id}) \subseteq J(f)$ for all f.
4. If $f_\#(\pi) \subseteq J(f)$, then all fixed point classes have the same index.

In particular, in our setting of an aspherical manifold, if $f_\#(\pi)$ is abelian, then $L(f) = N(f) \cdot \mathrm{Ind}(f, \mathbf{S})$ for any fixed point class $\mathbf{S}$, with $N(f) = 0$ if $\mathrm{Ind}(f, \mathbf{S}) = 0$. Of course, $f_\#(\pi)$ will be abelian for all f if and only if π is abelian. Thus the only infrasolvmanifolds for which the Jiang condition is satisfied for all maps are tori. Thus the Jiang condition does not contribute to our goal of finding conditions on an aspherical manifold which allow $N(f)$ to be estimated or computed for all self-maps. However, for some maps on infrasolvmanifolds, it does allow us to sharpen the inequality $|L(f)| \leq N(f)$ to equality. Namely, we have:

THEOREM 8.1. *If $f: S \to S$ is a self map of an infrasolvmanifold with $f_\#(\pi)$ abelian, then $N(f) = |L(f)|$.*

9. Conclusion. Two directions for further study naturally suggest themselves: refining these results in the infrasolvmanifold category, and extending the results to larger categories of manifolds. To refine these results, we need to know two things: for which spaces is

$\widetilde{N}(f, g, \Gamma_1)$ independent of Γ_1 for all f and g? For which spaces is $N(f, g) = |L(f, g)|$ for all f and g? Of course, the first question hinges on the behavior of $C_\#(f, g, \mathbf{S})$ for essential classes $\mathbf{S}$. It seems likely that $C_\#(f, g, \mathbf{S}) = 1$ for essential classes on all infranilmanifolds; the situation is less clear for infrasolvmanifolds. As to the question of equality of the Nielsen number and Lefschetz number, the natural conjecture is that equality occurs for all f and g if and only if the image manifold is a nilmanifold.

To extend the results to larger classes of manifolds, consider infrasolvmanifolds as spherical manifolds with torsion-free virtually polycyclic fundamental groups. There are several natural extensions of this class of groups: torsion-free elementary groups; torsion-free amenable groups, torsion-free "no-free groups" (i.e. groups which do not contain F_2, cf. **[21]**). For each, there is a corresponding category of compact aspherical manifolds. To what extent does the Lefschetz number estimate or compute the Nielsen number in these larger categories? In particular, is it true in any of these larger categories that $N(f) \geq |L(f)|$ for all self-maps f?

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Institute for Dynamics
University of Cincinnati
Cincinnati, OH 45221-0025