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ALMOST s-TANGENT MANIFOLDS OF HIGHER ORDER

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We introduce the notion of almost s-tangent structures of higher order by abstracting the geometric structure of the space of k-jets $J^k(\mathbb{R}, M)$. These structures are a natural extension of almost tangent structures of higher order.

1. Introduction. Almost tangent structures on even-dimensional manifolds were introduced by Clark and Bruckheimer [**1**] and Eliopoulos [**6**] around 1960 and have been investigated by many authors (see [**15**] and references therein). As it is well known the tangent bundle TM of a manifold M carries a canonical almost tangent structure (hence the name). This almost tangent structure plays an important role in the Lagrangian formulation of particle dynamics [**15**]. Crampin [**2**] showed that integrable almost tangent structures are relevant to study the inverse problem of Lagrangian dynamics and Cantrijn, Cariñena, Crampin and Ibort [**3**] proposed a geometric method of reduction of degenerate Lagrangian systems in this framework.

The notion of almost tangent structure of higher order is due to Eliopoulos [**7**]. An almost tangent structure of order k on a $((k+1)n)$-dimensional manifold is defined by abstracting the geometric structure of the tangent bundle of order k of an n-dimensional manifold. Tangent bundles of higher order are the natural framework to develop the Lagrangian dynamics of higher order (see [**14, 4**]). Recently, de León, Giraldo and Rodrigues [**12, 17**] have obtained results similar to those of Cantrijn et al. in order to reduce degenerate Lagrangian systems of higher order in the framework of higher order integrable almost tangent structures.

In [**20**] Oubiña extended the notion of almost tangent structures to odd-dimensional manifolds and introduced a new type of geometric structures, the so called almost s-tangent structures, their model being the stable tangent bundle $J^1(\mathbb{R}, M) \equiv \mathbb{R} \times TM$. These structures are involved in the study of the non-autonomous Lagrangian systems, and the inverse problem of non-autonomous Lagrangian dynamics can be reformulated in terms of almost s-tangent structures [**13**].

In the present paper we extend the notion of almost s-tangent structure to higher orders. The geometrical model is now the space of k-jets $J^k(\mathbb{R}, M)$, which can be identified to $\mathbb{R} \times T^k M$. In §2, we introduce the notion of almost s-tangent structure (J, ω, ξ) of order k on a manifold V of dimension $(k+1)n+1$. In §3, (J, ω, ξ) is interpreted as a **G**-structure, for a certain Lie subgroup **G** of $\mathrm{Gl}((k+1)n+1, \mathbb{R})$, and it is proved that (J, ω, ξ) is integrable if and only if the Nijenhuis tensor of J vanishes and ω is closed. Moreover, we prove that the existence of an almost s-tangent structure of order k on a manifold V is equivalent to a reduction of the structure group of TV to $O(n) \overset{\times}{=} \cdots \overset{\times}{=} O(n) \times 1$. Section 4 is devoted to showing that some special submanifolds of an almost tangent manifold of higher order inherit an almost s-tangent structure of the same order.

Since $J^{2k-1}(\mathbb{R}, M)$ is the evolution space of a non-autonomous Lagrangian system of order k, we feel that almost s-tangent structures of higher order might be relevant to give a geometric procedure in order to reduce degenerate non-autonomous Lagrangian systems of higher order. They could also be useful in discussing the inverse problem for higher order autonomous systems (see **[10, 12, 16]**).

2. Almost s-tangent structures of higher order. Throughout this paper it is assumed that all differential structures are of C^∞-class.

Let M be a manifold of dimension n and $T^k M$ be the tangent bundle of order k of M. Then $T^k M$ carries a canonical integrable almost tangent structure F of order k (see **[14]**), which is actually the $(k-1)$-lift of the identity tensor field of M to $T^k M$ in the sense of Morimoto **[19]**. Let (z^i) be local coordinates for M and $(z_0^i, z_1^i, \ldots, z_k^i)$ the induced coordinates of $T^k M$. Then the components of F in every such coordinate system are

$$\begin{pmatrix} 0 & 0 \\ I_{kn} & 0 \end{pmatrix}.$$

This notion can be extended to the space of jets of order k, i.e. $J^k(\mathbb{R}, M)$. In fact, the $((k+1)n+1)$-dimensional manifold $J^k(\mathbb{R}, M)$ can be identified with $\mathbb{R} \times T^k M$ in a very natural way. Then we can define a canonical tensor field on $J^k(\mathbb{R}, M)$ given by $J = F + (\partial/\partial t) \otimes dt$, where t is the global coordinate function for $\mathbb{R}$. Hence J has rank $kn+1$ and satisfies $J^{k+1} = (\partial/\partial t) \otimes dt$.

This suggests the following

DEFINITION. Let V be a differentiable manifold of dimension $(k+1)n+1$. A triple (J, ω, ξ), where J is a tensor field of type

$(1, 1)$, ω is a 1-form and ξ is a vector field on V such that

$$\omega(\xi) = 1, \tag{1}$$

$$J^{k+1} = \omega \otimes \xi, \tag{2}$$

$$\operatorname{rank} J = kn + 1, \tag{3}$$

will be called an *almost s-tangent structure of order* k and the manifold V an *almost s-tangent manifold of order* k.

The following proposition follows from (1) and (2).

PROPOSITION 2.1. *We have* $J\xi = \lambda\xi$ *and* $\omega J = \lambda\omega$, *where* $\lambda^{k+1} = 1$.

Proof. Clearly $J\xi = J(J^{k+1}\xi) = J^{k+1}(J\xi) = \omega(J\xi)\xi$ and so $J\xi = \lambda\xi$, being $\lambda^{k+1} = 1$. Also $\omega(JX)\xi = J^{k+1}(JX) = J(J^{k+1}X) = \omega(X)J\xi = \lambda\omega(X)\xi$, that is $\omega J = \lambda\omega$. □

Let Q be the $((k+1)n)$-dimensional distribution defined by the condition $\omega = 0$, i.e. $Q = \ker\omega$. By Proposition 2.1 we have $JQ \subset Q$ and hence $J^{k+1}_{|Q} = 0$. Since $\operatorname{rank}(J_{|Q}) = kn$, J acts on Q as an almost tangent structure operator of order k. Also we have

$$\ker J \subset \ker J^2 \subset \cdots \subset \ker J^k \subset \ker J^{k+1} = Q$$

and

$$\dim \ker J^r = rn, \qquad 1 \leq r \leq k+1.$$

REMARK. It is to be noted that, by virtue of well known properties of nilpotent operators on vector spaces, conditions (1) and (2) in the definition imply that $\operatorname{rank} J \leq kn+1$. Thus the condition (3) requires $\operatorname{rank} J$ to be maximal.

EXAMPLE 2.1. The space $J^k(\mathbb{R}, M)$ of jets of order k on a differentiable manifold M is an almost s-tangent manifold of order k.

EXAMPLE 2.2. (*A principal* 1-*bundle over an almost tangent manifold of order* k.) Let $V(M, G)$ be a principal bundle over a manifold M, G being a connected Lie group of dimension 1 with Lie algebra $\mathfrak{g}$. Let F be an almost tangent structure of order k on M. If $a \in \mathfrak{g} \equiv \mathbb{R}$, we denote by a^* the fundamental vector field on V corresponding to a. In particular, if e is the unit vector of $\mathfrak{g} \equiv \mathbb{R}$, we shall denote $\xi = e^*$. Let ω be a connection form on V and, for every vector field

X on M, denote by X^H the horizontal lift of X with respect to ω. We define a tensor field J of type $(1, 1)$ on V by putting

$$JX^H = (FX)^H, \qquad Ja^* = a\xi$$

for all vector fields X on M and for all $a \in \mathfrak{g} \equiv \mathbb{R}$. Then (J, ω, ξ) is an almost s-tangent structure of order k on V and the distribution Q on V, defined by $\omega = 0$, is the horizontal distribution of the connection.

3. The structure group. Next we shall describe an almost s-tangent structure of order k as a G-structure.

Let (J, ω, ξ) be an almost s-tangent structure of order k on a manifold V of dimension $(k+1)n+1$. Let P be the one-dimensional distribution determined by ξ. We have $T_xV = Q_x \oplus P_x$, for each $x \in V$. If S_x is a complementary subspace of $\ker J_x^k$ in Q_x then we have $T_xV = S_x \oplus \ker J_x^k \oplus P_x$. Thus, if $\{X_1, \dots, X_n\}$ is a basis for S_x then $\{X_i, JX_i, \dots, J^kX_i, \xi_x\}$ is a basis for T_xV called an *adapted frame.* if S'_x is another complementary subspace of $\ker J_x^k$ in Q_x and $\{X'_i, JX'_i, \dots, J^kX'_i, \xi_x\}$ another adapted frame, then we have

$$\begin{aligned} X'_i &= A_i^jX_j + (A_1)_i^jJX_j + \cdots + (A_k)_i^jJ^kX_j, \\ JX'_i &= A_i^jJX_j + (A_1)_i^jJ^2X_j + \cdots + (A_{k-1})_i^jJ^kX_j, \\ &\cdots\cdots \\ J^kX'_i &= A_i^jJ^kX_j, \\ \xi_x &= \xi_x. \end{aligned}$$

Therefore two adapted frames are related by a matrix $\mathbf{A} \in \mathrm{Gl}((k+1)n+1, \mathbb{R})$ of the form

$$\mathbf{A} = \begin{pmatrix} A & 0 & 0 & \cdots & 0 & 0 & 0 \\ A_1 & A & 0 & \cdots & 0 & 0 & 0 \\ A_2 & A_1 & A & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ A_{k-1} & A_{k-2} & A_{k-3} & \cdots & A & 0 & 0 \\ A_k & A_{k-1} & A_{k-2} & \cdots & A_1 & A & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{G}$ be the set of such matrices. Then $\mathbf{G}$ is a closed subgroup of $\mathrm{Gl}((k+1)n+1, \mathbb{R})$ and therefore a Lie subgroup of $\mathrm{Gl}((k+1)n+1, \mathbb{R})$. Let $B_{\mathbf{G}}$ be the set of adapted frames at all points of V. It can be easily proved that $B_{\mathbf{G}}$ defines a $\mathbf{G}$-structure on V.

Notice that, with respect to an adapted frame, J is represented by the matrix

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_{kn} & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

where $\lambda^{k+1} = 1$, $J\xi = \lambda\xi$. In fact, the group $\mathbf{G}$ can be described as the invariance group of the matrix J_0, i.e. $\mathbf{A} \in \mathbf{G}$ if and only if $\mathbf{A}J_0\mathbf{A}^{-1} = J_0$.

Suppose now given a $\mathbf{G}$-structure B on V. Then we may define a tensor field J of type $(1, 1)$ on V as follows:

$$J_x(X) = p \circ J_0 \circ p^{-1}(X),$$

where $X \in T_xV$, $x \in V$ and $p \in B$ is a linear frame at x, $p \colon \mathbb{R}^{(k+1)n+1} \to T_xV$. From the definition of $\mathbf{G}$, $J_x(X)$ is independent of the choice of p. In other words, J_x is defined as the linear endomorphism of T_xV which has at x the matrix representation J_0 with respect to one of the linear frames determined at x by B, and hence with respect to any other. Also, we define a vector field ξ and a 1-form ω on V as follows. If $\{e_1, \dots, e_{(k+1)n+1}\}$ is the canonical basis for $\mathbb{R}^{(k+1)n+1}$, then $\xi_x = p(e_{(k+1)n+1}) = p(0, \dots, 0, 1)$, and $\omega_x(p(e_\alpha)) = 0$, for $1 \le \alpha \le (k+1)n$ and $\omega_x(p(e_{(k+1)n+1})) = 1$. One can easily check that (J, ω, ξ) defines an almost s-tangent structure of order k on V.

Summing up, we have proved the following

PROPOSITION 3.1. *A $((k+1)n+1)$-dimensional manifold admits an almost s-tangent structure of order k if and only if the structure group of its tangent bundle is reducible to* $\mathbf{G}$.

COROLLARY 3.1. *Every almost s-tangent manifold of odd order is orientable.*

The integrability of (J, ω, ξ) as a $\mathbf{G}$-structure means that around each point of V there exists a coordinate system in which J is represented by the constant matrix J_0, being $\lambda^{k+1} = 1$, $J\xi = \lambda\xi$. In other words, around each point of V there is a coordinate system $(z_0^i, z_1^i, \dots, z_k^i, t)$ such that

$$(4) \quad J\left(\frac{\partial}{\partial z_r^i}\right) = \frac{\partial}{\partial z_{r+1}^i}, \quad 0 \le r \le k-1, \qquad J\left(\frac{\partial}{\partial z_k^i}\right) = 0,$$
$$J\left(\frac{\partial}{\partial t}\right) = \lambda\left(\frac{\partial}{\partial t}\right), \quad \xi = \frac{\partial}{\partial t}, \quad \omega = dt.$$

Let us remark that the canonical almost s-tangent structure of order k on $J^k(\mathbb{R}, M)$ is integrable.

PROPOSITION 3.2. *An almost s-tangent structure* (J, ω, ξ) *of order* k *is integrable if and only if* $N_J = 0$ *and* $d\omega = 0$, *where* N_J *denotes the Nijenhuis tensor of* J.

Proof. Clearly, if (J, ω, ξ) is integrable then $N_J = 0$ and $d\omega = 0$. Conversely, suppose that $d\omega = 0$ and $N_J = 0$. Hence, Q and of course P are integrable distributions. Then, around each point of V, there exists a cubic coordinate neighbourhood U, with local coordinates $(\bar{z}^i_r, \bar{t})$, $-\varepsilon < \bar{t} < \varepsilon$, such that

$$Q = \left\langle \frac{\partial}{\partial \bar{z}^i_r} \right\rangle, \qquad P = \left\langle \frac{\partial}{\partial \bar{t}} \right\rangle.$$

Then $\xi = f(\partial/\partial\bar{t})$ and $\omega = g\, d\bar{t}$, with $g = 1/f$. Since $d\omega = 0$, we deduce that $\partial g/\partial \bar{z}^i_r = 0$ and so $g = g(\bar{t})$. Now, we introduce a new coordinate system (z'^i_r, t') where $z'^i_r = \bar{z}^i_r$, $t' = h(\bar{t})$, with $h(\bar{t})$ a primitive function of $g(\bar{t})$. With respect to (z'^i_r, t'), we have

$$Q = \left\langle \frac{\partial}{\partial z'^i_r} \right\rangle, \quad \xi = \frac{\partial}{\partial t'}, \quad \omega = dt'.$$

Since $N_J = 0$ and Q is integrable, we know that $J_{|Q}$ is an integrable almost tangent structure of order k on the integral manifolds of Q ([**11**]). Consider the submanifold W of U defined by $t' = 0$. Then, there exists a coordinate open U' in W, with local coordinates $(\tilde{z}^i_r)$ such that

$$J\left(\frac{\partial}{\partial \tilde{z}^i_r}\right) = \frac{\partial}{\partial \tilde{z}^i_{r+1}}, \quad 0 \leq r \leq k-1, \qquad J\left(\frac{\partial}{\partial \tilde{z}^i_k}\right) = 0.$$

Consider the open coordinate $U' \times (-\varepsilon, \varepsilon)$ in V, with local coordinates $(z^i_r = \tilde{z}^i_r, t = t')$. We set

$$J\left(\frac{\partial}{\partial z^i_r}\right) = \sum_{\substack{0 \leq p \leq k \\ 1 \leq l \leq n}} (A^i_r)^p_l \frac{\partial}{\partial z^l_p} + B^i_r \frac{\partial}{\partial t}.$$

Since $N_J = 0$ and $d\omega = 0$ we deduce that $L_\xi J = 0$. Then

$$\frac{\partial (A^i_r)^p_l}{\partial t} = 0, \qquad \frac{\partial B^i_r}{\partial t} = 0,$$

and, consequently,

$$(A_r^i)_l^p(z_s^j, t) = (A_r^i)_l^p(z_s^j, 0) = \delta_j^i \delta_s^{r+1}, \qquad 0 \le i \le k-1,$$
$$(A_k^i)_l^p(z_s^j, t) = (A_k^i)_l^p(z_s^j, 0) = 0,$$
$$(B_r^i)(z_s^j, t) = (B_r^i)(z_s^j, 0) = 0, \qquad 0 \le r \le k.$$

Hence J, ω and ξ are locally expressed by (4). □

Now, let h be an arbitrary Riemannian metric on V and let S be the subbundle of Q orthogonal to $\ker J^k$ with respect to h. Then $J^p_{|S}: S \to Q$ is injective and $J^pS \cap J^qS = 0$, for $0 \le p, q \le k$, $p \ne q$, where J^0 is the identity tensor field. Also we have a direct sum decomposition

$$J^rS_x \oplus J^{r+1}S_x \oplus \cdots \oplus J^kS_x = (\ker J^{k-r+1})_x, \qquad 0 \le r \le k,$$

at each point x of V and, therefore,

$$(5) \qquad TV = \underbrace{S \oplus \overbrace{JS \oplus \cdots \oplus J^kS}^{\ker J^k}}_{Q} \oplus P.$$

Let $\rho^r: TV \to J^rS$ be the canonical projection. We define a Riemannian metric on V by

$$g_x(X, Y) = \sum_{r=0}^{k} h(J^{k-r}\rho^r(X), J^{k-r}\rho^r(Y)) + \omega(X)\omega(Y),$$

for all $X, Y \in T_x(V)$, $x \in V$. It is clear that $g(X, \xi) = \omega(X)$, so that Q and P are orthogonal with respect to g; moreover, $g_x(X, Y) = 0$ if $X \in J^pS$, $Y \in J^qS$, $0 \le p < q \le k$, and $g_x(X, Y) = g_x(J^rXJ^rY)$ for $X, Y \in S_x$, $1 \le r \le k$. Hence the decomposition (5) is orthogonal with respect to g and we have

PROPOSITION 3.3. *Let $V(J, \omega, \xi)$ be an almost s-tangent manifold of order k, P the one-dimensional distribution determined by ξ and $Q = \ker\omega$. Then there exists a subbundle S of Q and a Riemannian metric g on V such that $S, JS, \dots, J^kS$ and P are mutually orthogonal and*

(i) *$g(X, \xi) = \omega(X)$ for all vector fields X on V,*
(ii) *$g_x(X, Y) = g_x(J^rX, J^rY)$ for all $X, Y \in S_x$, $x \in V$, $1 \le r \le k$.*

We shall say that a metric g with the properties stated in this proposition is a *compatible metric.*

Now we can obtain a reduction of the structural group $\mathbf{G}$ in Proposition 3.1.

PROPOSITION 3.4. *A* $((k+1)n+1)$*-dimensional differentiable manifold* V *admits an almost s-tangent structure of order* k *if and only if the structure group of its tangent bundle is reducible to* $O(n) \stackrel{\times}{=} \stackrel{k-1}{\cdots} \stackrel{\times}{=} O(n) \times 1$, *where* $O(n) \stackrel{\times}{=} O(n)$ *denotes the group of diagonal products of* $O(n)$.

Proof. Suppose (J, ω, ξ) is an almost s-tangent structure of order k on V and let g be a compatible metric. We can choose a local orthonormal basis $\{X_\alpha\}_{\alpha=1,\dots,n}$ for S in a coordinate neighbourhood U; then $\{J^r X_\alpha\}_{\alpha=1,\dots,n}$ is an orthonormal basis for $J^r S$ on U, $1 \leq r \leq k$. We call the local orthonormal basis

$$\{X_\alpha, JX_\alpha, \dots, J^k X_\alpha, \xi\}_{\alpha=1,\dots,n}$$

a J*-basis.* With respect to this frame, the matrix representing J is

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_{kn} & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

where $\lambda^{k+1} = 1$, $J\xi = \lambda\xi$. With respect to another J-basis it is easily verified that the transformation matrix will be of the form

$$\begin{pmatrix} A & & 0 & 0 \\ & \overset{k}{\ddots} & & \\ 0 & & A & 0 \\ 0 & & 0 & 1 \end{pmatrix}$$

where the $n \times n$ matrix A is orthogonal. The converse is trivial. □

EXAMPLE 3.1. (*The Riemannian Heisenberg manifold* $\Gamma \backslash H_{k+1}$.) The real Heisenberg group H_{k+1} is the Lie subgroup of $\mathrm{Gl}(k+2, \mathbb{R})$ consisting of all matrices of the form

$$a = \begin{pmatrix} 1 & X & z \\ 0 & I_{k+1} & {}^tY \\ 0 & 0 & 1 \end{pmatrix}$$

where $X = (x_0, \ldots, x_k)$, $Y = (y_0, \ldots, y_k) \in \mathbb{R}^{k+1}$ and $z \in \mathbb{R}$. H_{k+1} is a connected simply connected Lie group of dimension $2(k+1)+1$ (see [**9**]).

A global system of coordinates (x^i, y^i, z), $1 \leq i \leq k+1$, on H_{k+1} is defined by

$$x^i(a) = x_i, \quad y^i(a) = y_i, \quad z(a) = z, \qquad 0 \leq i \leq k.$$

A basis for the left invariant 1-forms on H_{k+1} is given by

$$dx^i, \quad dy^i, \quad \gamma = dz - \sum_{i=0}^{k} x^i\, dy^i, \qquad 0 \leq i \leq k,$$

and its dual basis of left invariant vector fields is

$$\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z}, \frac{\partial}{\partial z}; 0 \leq i \leq k \right\}.$$

If we put

$$\widehat{J} = \sum_{i=0}^{k-1} \left\{ dx^i \otimes \frac{\partial}{\partial x^{i+1}} + dy^i \otimes \left(\frac{\partial}{\partial y^{i+1}} + x^{i+1} \frac{\partial}{\partial z} \right) \right\} + \gamma \otimes \frac{\partial}{\partial z}$$

then $(\widehat{J}, \gamma, \partial/\partial z)$ is an almost s-tangent structure of order k on H_{k+1} and the left invariant metric defined by

$$\langle\ ,\ \rangle = \sum_{i=0}^{k} \{ dx^i \otimes dx^i + dy^i \otimes dy^i \} + \gamma \otimes \gamma$$

is a compatible metric.

Let Γ be the subgroup of matrices of H_{k+1} with integer entries and consider the space of right cosets $\Gamma \backslash H_{k+1}$. If g is the unique Riemannian metric on $\Gamma \backslash H_{k+1}$ for which the canonical projection π from $(H_{k+1}, \langle\ ,\ \rangle)$ to $\Gamma \backslash H_{k+1}$ is a Riemannian covering, the pair $(\Gamma \backslash H_{k+1}, g)$ is called a Riemannian Heisenberg manifold (see [**8**]). Since the tensor fields that we have considered on H_{k+1} are left invariant they descend to $\Gamma \backslash H_{k+1}$. So we obtain an almost s-tangent structure (J, ω, ξ) of order k on $\Gamma \backslash H_{k+1}$, with

$$J \circ \pi_* = \pi_* \circ \widehat{J}, \quad \pi^* \omega = \gamma, \quad \xi = \pi_* \left(\frac{\partial}{\partial z} \right).$$

Moreover g is a compatible metric. Consider the subbundle S of $Q = \ker \omega$ that is orthogonal to $\ker J^k$ with respect to the metric g. Then each distribution $J^r S$ is generated by the vector fields

$X_r = \pi_*(\partial/\partial x^r)$ and $Y_r = \pi_*(\partial/\partial y^r + x^r \partial/\partial z)$. Since $[X_r, Y_r] = \xi$ for $0 \le r \le k$ and the other bracket products are all zero, then $N_J(X_r, Y_r) \neq 0$ and the structure (J, ω, ξ) is not integrable. Notice that the distribution Q is not integrable either.

4. Submanifolds of almost tangent manifolds of order k. Let W be a differentiable manifold of dimension $(k+1)(n+1)$ with an almost tangent structure F of order k, that is $F^{k+1} = 0$, $\operatorname{rank} F = k(n+1)$. Let V be a submanifold of W of codimension k and $\imath \colon V \to W$ the imbedding map. Suppose that there exists a vector field ξ on V such that the vector fields $C = F\imath_*\xi, FC, \dots, F^{k-1}C$ along V are linearly independent at each point and nowhere tangent to V. Then the vector field $C - \imath_*\xi$ along V is also nowhere tangent to V and $C - \imath_*\xi, FC, \dots, F^{k-1}C$ are linearly independent at each point of V.

Now, we define a tensor field J of type $(1, 1)$ and k differential 1-forms $\alpha_1, \dots, \alpha_k$ on V by

$$(6)\quad F\imath_*X = \imath_*JX + \alpha_1(X)(C - \imath_*\xi) + \alpha_2(X)FC + \cdots + \alpha_k(X)F^{k-1}C.$$

Applying F to both sides of (6) k times we have

$$\begin{aligned} 0 = {} & \imath_*(J^{k+1}(X) - \alpha_1(J^kX)\xi) + (\alpha_1(J^kX) - \alpha_1(J^{k-1}X))C \\ & + (\alpha_2(J^kX) + \alpha_1(J^{k-1}X) - \alpha_1(J^{k-2}X))FC \\ & + \cdots + (\alpha_k(J^kX) + \alpha_{k-1}(J^{k-1}X) + \cdots + \alpha_1(JX) - \alpha_1(X))F^{k-1}C. \end{aligned}$$

In particular, we have

$$J^{k+1}(X) = \alpha_1(J^{k-1}X)\xi, \qquad \alpha_1(J^kX) = \alpha_1(J^{k-1}X).$$

Putting $\omega = \alpha_1 \circ J^{k-1}$, we get $J^{k+1} = \omega \otimes \xi$. Now, setting $X = \xi$ in (6) we obtain

$$C = \imath_*(J\xi - \alpha_1(\xi)\xi) + \alpha_1(\xi)C + \alpha_2(\xi)FC + \cdots + \alpha_k(\xi)F^{k-1}C.$$

Thus $\alpha_1(\xi) = 1$, $J\xi = \xi$ and $\omega(\xi) = \alpha_1(J^{k+1}\xi) = 1$. Since $\operatorname{rank} F = k(n+1)$, then $\operatorname{rank} J = kn + 1$. Hence, we have proved the following

PROPOSITION 4.1. *Let W be a $(k+1)(n+1)$-dimensional differentiable manifold with an almost tangent structure F of order k and let $(V, \imath)$ a submanifold of W of codimension k. If there exists a vector field ξ on V such that $C = F\imath_*\xi$, $FC, \dots, F^{k-1}C$ are linearly independent at each point and nowhere tangent to V, then V admits an almost s-tangent structure of order k.*

EXAMPLE 4.1 ([20]). The tangent sphere bundle of a Riemannian manifold M inherits an almost s-tangent structure of order 1 from the canonical almost tangent structure of the tangent bundle TM.

EXAMPLE 4.2. (*A submanifold of T^2M.*) Let M be an $(n+1)$-dimensional Riemannian manifold with metric g, T^2M its tangent bundle of order 2 and $TM \oplus TM$ the Whitney sum of the tangent bundle of M with itself. Let $\{U, x_i\}$ be a coordinate neighbourhood of M and (x^i, v^i, w^i) the corresponding coordinates of $TM \oplus TM$, where (v^i, w^i) are the fibre coordinates. On the other hand, let (x^i, y^i, z^i) be the induced coordinates in T^2U, that is, if $j^2\sigma$ is the 2-jet at $0 \in \mathbb{R}$ of a differentiable curve $\sigma: \mathbb{R} \to M$ then

$$x^i(j^2\sigma) = x^i(\sigma(0)), \quad y^i(j^2\sigma) = \frac{d(x^i \circ \sigma)}{dt}(0),$$
$$z^i(j^2\sigma) = \frac{1}{2}\frac{d^2(x^i \circ \sigma)}{dt^2}(0).$$

If ∇ is the Riemannian connection of g, there exists a diffeomorphism ([5], [18]) $\varphi: T^2M \to TM \oplus TM$ given by

$$\varphi(j^2\sigma) = (\dot\sigma(0), (\nabla_{\dot\sigma(0)}\dot\sigma)(0))$$

where $\dot\sigma$ is the tangent vector field to σ. In coordinates

$$T^2U \xrightarrow{\varphi} TU \oplus TU$$
$$(x^i, y^i, z^i) \mapsto (x^i, y^i, 2z^i + y^j y^k \Gamma^i_{jk}),$$

Γ^i_{jk} are the local components of the Riemannian connection.

Let F be the standard almost tangent structure of order 2 on T^2M, given by

$$F\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad F\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial z^i}, \quad F\left(\frac{\partial}{\partial z^i}\right) = 0.$$

By means of the diffeomorphism φ, the tensor field F defines an almost tangent structure $\overline{F}$ of order 2 on $TM \oplus TM$, which is given by

$$\overline{F}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial v^i} + 2v^k\Gamma^j_{ik}\frac{\partial}{\partial w^j},$$
$$\overline{F}\left(\frac{\partial}{\partial v^i}\right) = 2\frac{\partial}{\partial w^i}, \quad \overline{F}\left(\frac{\partial}{\partial w^i}\right) = 0.$$

Let us consider the tangent sphere bundle V_1M of M, and let V_2M be the subbundle of the fibre product $V_1M \oplus V_1M$, whose fibre at each point $x \in M$ is

$$(V_2M)_x = \{(u, v) \in (V_1M)_x \oplus (V_1M)_x \mid g(u, v) = 0\}.$$

We consider $\overline{V} = V_1M \oplus V_1M - V_2M$, which is the submanifold of codimension 2 of $TM \oplus TM$ defined by the conditions

$$g_{ij}v^iv^j = g_{ij}w^iw^j = 1, \qquad g_{ij}v^iw^j \neq 0.$$

Let $\imath: \overline{V} \to TM \oplus TM$ be the imbedding map. We define a vector field $\overline{\xi}$ on $\overline{V}$ by

$$\imath_*\overline{\xi} = v^i \left(\frac{\partial}{\partial x^i}\right)^H$$

where $(\partial/\partial x^i)^H$ is the horizontal lift of $\partial/\partial x^i$ to $TM \oplus TM$ with respect to the Riemannian connection of M; that is

$$\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - \Gamma^j_{ik}v^k\frac{\partial}{\partial v^j} - \Gamma^j_{ik}w^k\frac{\partial}{\partial w^j}.$$

The vector fields

$$C = \overline{F}\imath_*\overline{\xi} = v^i\frac{\partial}{\partial v^i} \quad \text{and} \quad \overline{F}C = 2v^i\frac{\partial}{\partial w^i}$$

are nowhere tangent to $\overline{V}$ and they are linearly independent at each point of $\overline{V}$. By Proposition 4.1, $\overline{V}$ inherits an almost tangent structure $(\overline{J}, \overline{\omega}, \overline{\xi})$ of order 2, which is given by

$$\overline{F}\imath_*X = \imath_*\overline{J}X + \alpha_1(X)(C - \imath_*\overline{\xi}) + \alpha_2(X)FC, \qquad \overline{\omega}(X) = \alpha_1(\overline{J}X).$$

Hence $(J = \varphi_*^{-1}\overline{J}\varphi_*,\ \omega = \varphi^*\overline{\omega},\ \xi = \varphi_*^{-1}\overline{\xi})$ is an almost s-tangent structure of order 2 on the submanifold $V = \varphi^{-1}(\overline{V})$ of T^2M.

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