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BMO AND HANKEL OPERATORS ON BERGMAN SPACES

KEHE ZHU

Let BMO_∂^p be the space of functions on the open unit ball in $\mathbf{C}^n$ with bounded mean oscillation in the Bergman metric defined using the volume L^p integral (see Introduction for precise definition). This paper studies the structure of BMO_∂^p. In particular, we show how BMO_∂^p depends on p. We also characterize BMO_∂^p in terms of certain Hankel operators acting on weighted Bergman L^p spaces. A parallel study is made on the companion space VMO_∂^p.

1. Introduction. By a well-known theorem of John-Nirenberg [**4**], [**5**], the classical *BMO* of the unit circle is independent of the L^p norm used to define it (usually the L^1 norm is used for the definition of *BMO* on the circle). It is also well known [**12**] that a function f on the circle is in *BMO* if and only if the Hankel operators with symbol f and $\overline{f}$ are both bounded on the Hardy space H^2 of the circle.

A new type of *BMO*, denoted $BMO_\partial(\Omega)$, is introduced in [**1**], [**2**] for any bounded domain Ω in the complex space $\mathbf{C}^n$. The space is defined in terms of the Bergman metric using the L^2 norm with respect to the volume measure. It is proved in [**1**] that an L^2 function f on a bounded symmetric domain Ω is in $BMO_\partial(\Omega)$ if and only if the Hankel operators (defined in terms of the Bergman projection) with symbol f and $\overline{f}$ are both bounded on the Bergman L^2 space.

In this paper we show that *BMO* in the Bergman metric actually depends on the L^p norm used to define it (in contrast with the John-Nirenberg phenomenon). We will precisely describe the dependence of *BMO* in the Bergman metric on p. The *BMO* in the Bergman metric defined using the volume L^p norm will be used to characterize certain bounded Hankel operators acting on weighted Bergman L^p spaces.

We need to introduce some notation in order to state our results precisely. For some technical reasons, we will content ourselves with the open unit ball in $\mathbf{C}^n$. Some of the results and analysis here also hold for bounded symmetric domains (for example, all the results in §2 with some obvious changes).

Let B_n be the open unit ball in $\mathbf{C}^n$ with normalized volume measure $dv(z)$. We will also need the following measures:

$$dv_\alpha(z) = C_\alpha(1 - |z|^2)^\alpha \, dv(z),$$

where $\alpha > -1$ and C_α is a positive normalizing constant so that dv_α is a probability measure. Let $\beta(z, w)$ be the Bergman distance function on B_n. For any $z \in B_n$ and $r > 0$ let

$$D(z, r) = \{w \in B_n : \beta(z, w) < r\}$$

be the Bergman metric ball with center z and radius r. The (normalized) volume of $D(z, r)$ will be denoted by $|D(z, r)|$. For a locally dv integrable function f on B_n we define a function $\hat{f}_r$ on B_n as follows:

$$\hat{f}_r(z) = \frac{1}{|D(z, r)|} \int_{D(z,r)} f(w)\, dv(w), \qquad z \in B_n.$$

$\hat{f}_r(z)$ is the integral mean of f over $D(z, r)$. Fix $r > 0$ and $p \geq 1$, let BMO_r^p denote the space of all locally L^p integrable functions f on B_n such that

$$\|f\|_{r,p} = \sup_{z \in B_n} \left[\frac{1}{|D(z, r)|} \int_{D(z,r)} |f(w) - \hat{f}_r(z)|^p \, dv(w) \right]^{1/p} < +\infty.$$

It is easy to see that BMO_r^p depends on p. For example, if $p < q$ and f is a function with compact support in B_n such that f is in $L^p(B_n, dv)$ but not in $L^q(B_n, dv)$, then f is in BMO_r^p but not in BMO_r^q. In general, $BMO_r^q \subset BMO_r^p$ for $p \leq q$. The inclusion is proper if $p < q$.

Our first result shows that BMO_r^p is independent of r and it tells how BMO_r^p depends on p.

THEOREM A. *BMO_r^p is independent of r. Moreover, a locally L^p integrable function f on B_n belongs to BMO_r^p if and only if $f = f_1 + f_2$, where*

$$\sup_{z \in B_n} \int_{B_n} |f_1 \circ \varphi_z(w)|^p \, dv_\alpha(w) < +\infty$$

for all (for some) $\alpha > -1$ and

$$|f_2(z) - f_2(w)| \leq C(\beta(z, w) + 1)$$

for some constant $C > 0$ and all $z, w \in B_n$. Here φ_z is the canonical involution on B_n described in 2.2 of **[6]**.

By the above theorem, we can write BMO_∂^p for BMO_r^p. The symbol ∂ here stresses the fact that being in BMO_∂^p is essentially a "boundary condition." It follows easily from the above theorem that BMO_∂^p is contained in $L^p(B_n, dv_\alpha)$ for all $\alpha > -1$.

BMO_∂^p can be described in terms of certain Hankel operators acting on weighted Bergman L^p spaces. Recall that for $\alpha > -1$, $dv_\alpha(z) = C_\alpha(1-|z|^2)^\alpha\, dv(z)$, where C_α is a normalizing constant. For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space $L_a^p(dv_\alpha)$ is the subspace of $L^p(B_n, dv_\alpha)$ consisting of holomorphic functions. $L_a^p(dv_\alpha)$ is the closed subspace of $L^p(B_n, dv_\alpha)$ generated by polynomials.

Let P_α denote the orthogonal projection from $L^2(B_n, dv_\alpha)$ onto $L_a^2(dv_\alpha)$. P_α is an integral operator given by

$$P_\alpha f(z) = \int_{B_n} K^{(\alpha)}(z, w) f(w)\, dv_\alpha(w),$$

where

$$K^{(\alpha)}(z, w) = \frac{1}{(1 - \langle z, w\rangle)^{n+1+\alpha}}$$

is the reproducing kernel of $L_a^2(dv_\alpha)$. It is well known [3] that for $p \geq 1$ and $\alpha, \lambda > -1$, P_α is a bounded projection from $L^p(B_n, dv_\lambda)$ onto $L_a^p(dv_\lambda)$ if and only if $p(\alpha+1) > \lambda + 1$.

Given a function f on B_n, let M_f denote the multiplication operator induced by f. For $\alpha > -1$ and f on B_n we define two operators $T_f^{(\alpha)}$ and $H_f^{(\alpha)}$ as follows:

$$T_f^{(\alpha)} = P_\alpha M_f P_\alpha, \quad H_f^{(\alpha)} = (I - P_\alpha) M_f P_\alpha,$$

where I is the identity operator. $T_f^{(\alpha)}$ and $H_f^{(\alpha)}$ are called the Toeplitz and Hankel operator, respectively, with symbol f. Note that these operators are densely defined (but unbounded in general) on $L^p(B_n, dv_\lambda)$ as long as f is in $L^p(B_n, dv_\lambda)$. We can now state our second result.

Theorem B. *Suppose $p \geq 1$, $p(\alpha+1) > \lambda+1 > 0$, and f is in $L^p(B_n, dv_\lambda)$. Then f belongs to BMO_α^p if and only if the two Hankel operators $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are both bounded on $L^p(B_n, dv_\lambda)$.*

When $\alpha = 0$, we will write dv for dv_α, P for P_α, $K(z, w)$ for $K^{(\alpha)}(z, w)$, L_a^p for $L_a^p(dv_\alpha)$, T_f for $T_f^{(\alpha)}$, and H_f for $H_f^{(\alpha)}$. We state two corollaries to Theorem B.

Corollary 1. *If $p > 1$ and $\alpha > -1$, then $f \in BMO_\partial^p$ if and only if $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are both bounded on $L^p(B_n, dv_\alpha)$.*

The above corollary is proved in **[1]** in the special case $p = 2$ and $\alpha = 0$. However, all results in **[1]** are proved in the context of a bounded symmetric domain in $\mathbf{C}^n$.

COROLLARY 2. *If $\alpha > \lambda > -1$, then $f \in BMO_\partial^1$ if and only if $H_f^{(\alpha)}$ and $H_{\bar{f}}^{(\alpha)}$ are both bounded on $L^1(B_n, dv_\lambda)$.*

The above corollary is partially proved in **[11]** in the special case $\alpha = n + 1$ and $\lambda = 0$; but again the setting in **[11]** is a bounded symmetric domain. The projection P_{n+1} is frequently used in the study of the Bergman space L_a^1 (see **[7]**, **[12]**).

A similar study will be made on the corresponding VMO_∂^p and compactness of Hankel operators on $L_a^p(dv_\lambda)$. Holomorphic functions in BMO_∂^p (or VMO_∂^p) are precisely the functions in the Bloch space (or the little Bloch space) of B_n.

In the first version of the paper Theorem B was proved under the additional assumption $p(n+1+\alpha) = 2(n+1+\lambda)$. (The proof was based on a method introduced in **[8]**.) Daniel Luecking read the preprint and found a way of getting around this condition. I am grateful to Professor Luecking for allowing me to use his proof and obtain Theorem B in its present form. I also wish to thank the referee for carefully reading the manuscript and making several useful suggestions for improvement (and in some instances corrections) of the paper. In particular, the referee significantly simplified the proof of Lemma 9 and part of the proof of Theorem 5.

2. The structure of BMO_∂^p. In this section we study the structure of the space BMO_r^p, consisting of functions f on B_n with

$$\sup_{z \in B_n} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - \hat{f}_r(z)|^p \, dv(w) < +\infty,$$

where

$$\hat{f}_r(z) = \frac{1}{|D(z, r)|} \int_{D(z, r)} f(w) \, dv(w)$$

is the dv integral mean of f over $D(z, r)$.

LEMMA 1. *f is in BMO_r^p if and only if there exists a constant $C > 0$ such that for any $z \in B_n$ there is a constant λ_z with*

$$\frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - \lambda_z|^p \, dv(w) \le C.$$

Proof. The "only if" part follows by taking $\lambda_z = \hat{f}_r(z)$. To prove the "if" part, assume that the above inequality holds for all $z \in B_n$. By the triangle inequality for the L^p integral,

$$\begin{aligned}&\left[\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(w)-\hat{f}_r(z)|^p\,dv(w)\right]^{1/p}\\ &\qquad\le\left[\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(w)-\lambda_z|^p\,dv(w)\right]^{1/p}\\ &\qquad\quad+|\hat{f}_r(z)-\lambda_z|.\end{aligned}$$

But

$$\begin{aligned}|\hat{f}_r(z)-\lambda_z|&=\left|\frac{1}{|D(z,r)|}\int_{D(z,r)}(f(w)-\lambda_z)\,dv(w)\right|\\ &\le\left[\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(w)-\lambda_z|^p\,dv(w)\right]^{1/p}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left[\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(w)-\hat{f}_r(z)|^p\,dv(w)\right]^{1/p}\\ &\qquad\le 2\left[\frac{1}{|D(z,r)|}\int_{D(z,r)}|f(w)-\lambda_z|^p\,dv(w)\right]^{1/p},\end{aligned}$$

completing the proof of the lemma. □

For any $r > 0$, let BO_r denote the space of continuous functions f on B_n such that

$$\omega_r(f)(z)=\sup\{|f(z)-f(w)|\colon w\in D(z,r)\}$$

is a bounded function on B_n. $\omega_r(f)(z)$ is the oscillation of f at z in the Bergman metric.

LEMMA 2. *BO_r is independent of r. Moreover, a continuous function f on B_n is in BO_r if and only if there is a constant $C > 0$ such* *that*

$$|f(z) - f(w)| \leq C(\beta(z, w) + 1)$$

for all z *and* w *in* B_n.

Proof. See **[1]**. □

We will simply write BO for BO_r. The initials BO stand for "bounded oscillation." We will put the following semi-norm on BO:

$$\|f\| = \sup\{|f(z) - f(w)| : \beta(z, w) \leq 1\}.$$

Let BA_r^p denote the space of all functions f on B_n with the property that $\widehat{|f|_r^p}(z) \in L^\infty(B_n)$. The initials BA stand for "bounded average." The next lemma describes the functions in BA_r^p.

LEMMA 3. BA_r^p *is independent of* r. *Moreover, the following conditions are all equivalent*:

(1) $f \in BA_r^p$;

(2) $\sup_{z\in B_n} \int_{B_n} |f\circ\varphi_z(w)|^p \, dv_\lambda(w) < +\infty$ *for all (or some)* $\lambda > -1$;

(3) $M_f \colon L_a^p(dv_\lambda) \to L^p(B_n, dv_\lambda)$ *is bounded for all (or some)* $\lambda > -1$.

Proof. See Theorem A in **[10]**. □

We will simply write BA^p for BA_r^p. We will use the following norms on BA^p:

$$\|f\|_\lambda^p = \sup_{z\in B_n} \int_{B_n} |f \circ \varphi_z(w)|^p \, dv_\lambda(w).$$

For $\lambda > -1$ and f on B_n we will write

$$B_\lambda f(z) = \int_{B_n} f \circ \varphi_z(w) \, dv_\lambda(w), \qquad z \in B_n.$$

This is called the Berezin transform of f with respect to the measure dv_λ. It is easy to check that the following change of variable formula holds for all $\lambda > -1$:

$$B_\lambda f(z) = \int_{B_n} f(w) |k_z^{(\lambda)}(w)|^2 \, dv_\lambda(w),$$

where

$$k_z^{(\lambda)}(w) = \frac{K^{(\lambda)}(w, z)}{\sqrt{K^{(\lambda)}(z, z)}}$$

are the normalized reproducing kernels for $L_a^2(dv_\lambda)$.

LEMMA 4. *Suppose r, s, and R are positive constants; then there exists a constant $C > 0$ such that*
(1) $C^{-1} \le |D(z, r)|/|D(w, s)| \le C$,
(2) $C^{-1} \le (1-|z|^2)/|1-\langle z, w\rangle| \le 2(1-|z|^2)/(1-|w|^2) \le C$,
(3) $C^{-1} \le |D(z, r)|/(1-|z|^2)^{n+1} \le C$,
for all $z, w \in B_n$ with $\beta(z, w) \le R$.

Proof. See Lemmas 6 and 8 in [2]. □

We can now prove the main result of this section.

THEOREM 5. *Suppose $r > 0$ and $p \ge 1$. Then the following are equivalent:*
(1) $f \in BMO_r^p$;
(2) $f \in BO + BA^p$;
(3) $\sup_{z\in B_n} \int_{B_n} |f\circ\varphi_z(w) - B_\lambda f(z)|^p \, dv_\lambda(w) < +\infty$ *for all (or some) $\lambda > -1$;*
(4) *For any (or some) $\lambda > -1$, there exists a constant $C > 0$ such that for any $z \in B_n$ there is a constant λ_z with*

$$\int_{B_n} |f\circ\varphi_z(w) - \lambda_z|^p \, dv_\lambda(w) \le C.$$

Proof. (1) ⇒ (2): Since r is arbitrary, it suffices to show that $BMO_{2r}^p \subset BO + BA^p$. Given $f \in BMO_{2r}^p$ and $\beta(z, w) \le r$, we have

$$\begin{aligned} |\hat{f}_r(z) - \hat{f}_r(w)| &\le |\hat{f}_r(z) - \hat{f}_{2r}(z)| + |\hat{f}_{2r}(z) - \hat{f}_r(w)| \\ &\le \frac{1}{|D(z, r)|}\int_{D(z,r)} |f(u) - \hat{f}_{2r}(z)| \, dv(u) \\ &\quad + \frac{1}{|D(w, r)|}\int_{D(w,r)} |f(u) - \hat{f}_{2r}(z)| \, dv(u). \end{aligned}$$

By Lemma 4, $|D(z, r)| \sim |D(w, r)| \sim |D(z, 2r)|$ for all $w \in D(z, r)$. Now the first term above is bounded because of Hölder's inequality, $D(z, r) \subset D(z, 2r)$, and $f \in BMO_{2r}^p$. That the second term above is bounded follows from Hölder's inequality, $D(w, r) \subset D(z, 2r)$, and $f \in BMO_{2r}^p$. This proves that $\hat{f}_r$ belongs to BO_r (and hence BO) if $f \in BMO_{2r}^p$.

Let $g = f - \hat{f}_r$ with $f \in BMO_{2r}^p$, we show that $g \in BA^p$. It is rather easy to see that if f is in BMO_{2r}^p, then f is in BMO_r^p. By the triangle inequality,

$$\begin{aligned}
[\widehat{|g|_r^p}(z)]^{1/p} &= \left[\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(u)-\hat{f}_r(u)|^p\,dv(u)\right]^{1/p} \\
&\le \left[\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(u)-\hat{f}_r(z)|^p\,dv(u)\right]^{1/p} \\
&\quad + \left[\frac{1}{|D(z,r)|}\int_{D(z,r)} |\hat{f}_r(u)-\hat{f}_r(z)|^p\,dv(u)\right]^{1/p} \\
&\le \|f\|_{r,p} + \omega_r(\hat{f}_r)(z).
\end{aligned}$$

Since $\hat{f}_r$ is in BO_r, we see that $g = f - \hat{f}_r$ is in BA^p. Thus we have shown that $f \in BMO^p_{2r}$ implies that $f = \hat{f}_r + (f - \hat{f}_r) \in BO + BA^p$.

(2) $\Rightarrow$ (3): Fix $\lambda > -1$ and write $\|\ \|_p$ for $\|\ \|_{L^p(dv_\lambda)}$. First note that

$$\|f\circ\varphi_z - B_\lambda f(z)\|_p \le \|f\circ\varphi_z\|_p + |B_\lambda f(z)| \le 2\|f\circ\varphi_z\|_p.$$

By Lemma 3, $\|f\circ\varphi_z - B_\lambda f(z)\|_p$ is bounded in z if f is in BA^p. On the other hand,

$$\begin{aligned}
\|f\circ\varphi_z - B_\lambda f(z)\|_p^p &= \int_{B_n} |f\circ\varphi_z(w) - B_\lambda f(z)|^p\,dv_\lambda(w) \\
&\le \int_{B_n}\int_{B_n} |f\circ\varphi_z(w) - f\circ\varphi_z(u)|^p\,dv_\lambda(w)\,dv_\lambda(u).
\end{aligned}$$

If $f \in BO$, then Lemma 2 shows that there is a constant $C > 0$ such that $|f(z) - f(w)| \le C(\beta(z,w)+1)$ for all $z, w \in B_n$. This, along with the Möbius invariance of the Bergman metric, implies that

$$\|f\circ\varphi_z - B_\lambda f(z)\|_p^p \le C^p \int_{B_n}\int_{B_n} (\beta(w,u)+1)^p\,dv_\lambda(w)\,dv_\lambda(u).$$

The right side of the above inequality is a finite constant; this follows from the triangle inequality $\beta(w,u) \le \beta(0,w) + \beta(0,u)$ and the following explicit formula for the Bergman distance:

$$\beta(0,z) = \left(\frac{n+1}{8}\right)^{1/2} \log\frac{1+|z|}{1-|z|}.$$

We see that $f \in BO$ implies that $\|f\circ\varphi_z - B_\lambda f(z)\|_p$ is bounded in z.

The proof of the equivalence of (3) and (4) is similar to that of Lemma 1. We omit the details.

(3) $\Rightarrow$ (1): By Lemma 4, there is a constant $C > 0$ such that

$$1 \leq CC_\lambda |D(z, r)|\, |k_z^{(\lambda)}(w)|^2 (1 - |w|^2)^\lambda$$

for all $z \in B_n$ and $w \in D(z, r)$. It follows that

$$\begin{aligned}
&\frac{1}{|D(z, r)|} \int_{D(z,r)} |f(w) - B_\lambda f(z)|^p \, dv(w) \\
&\quad \leq C \int_{D(z,r)} |f(w) - B_\lambda f(z)|^p |k_z^{(\lambda)}(w)|^2 \, dv_\lambda(w) \\
&\quad \leq C \int_{B_n} |f \circ \varphi_z(w) - B_\lambda f(z)|^p \, dv_\lambda(w).
\end{aligned}$$

The desired result now follows from Lemma 1. □

Theorem 5 shows that BMO_r^p is independent of the radius r. We will write BMO_∂^p for BMO_r^p. A canonical semi-norm on BMO_∂^p is

$$\|f\|_\lambda = \sup_{z \in B_n} \|f \circ \varphi_z - B_\lambda f(z)\|_{L^p(dv_\lambda)},$$

where $\lambda > -1$. It is easy to check that the above semi-norm is complete and invariant under Möbius transformations.

Corollary 6. *If $\lambda > -1$ and $f \in BMO_\partial^p$, then $B_\lambda f \in BO$ and $f - B_\lambda f \in BA^p$.*

Proof. By Lemma 4, we can choose a constant $C > 0$ such that

$$1 \leq CC_\lambda |D(z, r)|\, |k_z^{(\lambda)}(w)|^2 (1 - |w|^2)^\lambda$$

for all $z \in B_n$ and $w \in D(z, r)$. It follows that

$$\begin{aligned}
|B_\lambda f(z) - \hat{f}_r(z)| &\leq \frac{1}{|D(z, r)|} \int_{D(z,r)} |f(w) - B_\lambda f(z)| \, dv(w) \\
&\leq C \int_{D(z,r)} |f(w) - B_\lambda f(z)|\, |k_z^{(\lambda)}(w)|^2 \, dv_\lambda(w) \\
&\leq C \int_{B_n} |f \circ \varphi_z(w) - B_\lambda f(z)| \, dv_\lambda(w) \\
&\leq C \|f \circ \varphi_z - B_\lambda f(z)\|_{L^p(dv_\lambda)}.
\end{aligned}$$

This shows that $B_\lambda f - \hat{f}_r$ is bounded on B_n if f is in BMO_∂^p. Since bounded continuous functions are both in BO and BA^p, the desired result now follows from the proof of the implication (1) $\Rightarrow$ (2) in the above theorem. □

REMARK. It follows from the proof of the above corollary that if $f \in BMO_\partial^p$, then $B_\lambda f(z) - B_\alpha f(z)$ is bounded on B_n for all α, $\lambda > -1$. This, together with Theorem 5, easily implies that $f \in BMO_\partial^p$ if and only if

$$\sup_{z \in B_n} \int_{B_n} |f \circ \varphi_z(w) - B_\alpha f(z)|^p \, dv_\lambda(w) < +\infty$$

for all (or some) α, $\lambda > -1$ (not necessarily the same!).

Recall that the Bloch space $\mathscr{B}(B_n)$ of B_n consists of holomorphic functions f on B_n such that

$$(1 - |z|^2) \frac{\partial f}{\partial z_k}(z) \qquad (1 \leq k \leq n)$$

are bounded on B_n. The little Bloch space $\mathscr{B}_0(B_n)$ of B_n is the space of all holomorphic functions f on B_n such that

$$(1 - |z|^2) \frac{\partial f}{\partial z_k}(z) \to 0 \qquad (|z| \to 1^-)$$

for all $1 \leq k \leq n$. See **[9]** for the theory of Bloch functions in several complex variables.

THEOREM 7. *Let $H(B_n)$ denote the space of all holomorphic functions in B_n. Then $BMO_\partial^p \cap H(B_n) = \mathscr{B}(B_n)$ for all $p \geq 1$.*

Proof. It is shown in **[1]** that $BO \cap H(B_n) = \mathscr{B}(B_n)$. Thus $\mathscr{B}(B_n) \subset BMO_\partial^p \cap H(B_n)$. On the other hand, if f is a holomorphic function in BMO_∂^p, then $B_\alpha f = f$ for all $\alpha > -1$ and hence f is in $BO \cap H(B_n) = \mathscr{B}(B_n)$ by Corollary 6. □

REMARK. The dependence of BMO_∂^p on p is on the "bounded part" of BMO_∂^p, BA^p; the "smooth part" of BMO_∂^p, BO, is independent of p. In this sense, the dependence of BMO_∂^p on p is not heavy.

3. Bounded Hankel operators on Bergman spaces. This section is devoted to the proof of Theorem B. Recall that for any $\lambda > -1$,

$$k_z^{(\lambda)}(w) = \frac{(1 - |z|^2)^{(n+1+\lambda)/2}}{(1 - \langle w, z \rangle)^{n+1+\lambda}}, \qquad z, w \in B_n,$$

are the normalized reproducing kernels of $L_a^2(dv_\lambda)$. For any $p \geq 1$, $(k_z^{(\lambda)})^{2/p}$ are unit vectors in $L^p(B_n, dv_\lambda)$.

LEMMA 8. *For* $-1 < s + \lambda < \alpha$ *there exists a constant* $C > 0$ *such that*

$$\int_{B_n} \frac{\beta(z, w)(1 - |w|^2)^s}{|1 - \langle z, w\rangle|^{n+1+\alpha}} \, dv_\lambda(w) \le C(1 - |z|^2)^{s+\lambda-\alpha}$$

for all $z \in B_n$.

Proof. Given $t > -1$ and $c < 0$ we can choose a positive integer k such that

$$t + \frac{c}{k} > -1, \qquad c - \frac{c}{k} < 0.$$

By the explicit formula for β we can find a constant $C_1 > 0$ satisfying

$$\beta(0, w) \le C_1(1 - |w|^2)^{c/k}, \qquad w \in B_n.$$

Using 1.4.10 of **[6]** we see that there exists a constant $C > 0$ with

$$\int_{B_n} \frac{\beta(0, w)(1 - |w|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+c}} \, dv(w)$$

$$\le C_1 \int_{B_n} \frac{(1 - |w|^2)^{t+c/k}}{|1 - \langle z, w\rangle|^{n+1+(t+c/k)+(c-c/k)}} \, dv(w) \le C$$

for all $z \in B_n$. The desired result now follows easily from the change of variables $w \mapsto \varphi_z(w)$. □

LEMMA 9. *Let* T *and* S *be the operators defined by*

$$Tf(z) = \int_{B_n} \frac{\beta(z, w)(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^{n+1+\alpha}} f(w) \, dv(w),$$

$$Sf(z) = \int_{B_n} \frac{(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^{n+1+\alpha}} f(w) \, dv(w).$$

Then T *and* S *are both bounded on* $L^p(B_n, dv_\lambda)$ *provided that* $p(\alpha + 1) > \lambda + 1 > 0$.

Proof. We prove the boundedness of T. The boundedness of S can be proved similarly (see **[3]**).

The case $p = 1$ follows directly from Fubini's theorem and Lemma 8. So we assume $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Write

$$Tf(z) = \frac{1}{C_\lambda} \int_{B_n} \frac{\beta(z, w)(1 - |w|^2)^{\alpha-\lambda}}{|1 - \langle z, w\rangle|^{n+1+\alpha}} f(w) \, dv_\lambda(w).$$

By Schur's theorem [12], T is bounded on $L^p(B_n, dv_\lambda)$ if we can find a number σ and a positive constant $C > 0$ such that the function $h(z) = (1-|z|^2)^\sigma$ satisfies

$$\int_{B_n} \frac{\beta(z, w)(1-|w|^2)^{\alpha-\lambda}}{|1-\langle z, w\rangle|^{n+1+\alpha}} h(w)^q \, dv_\lambda(w) \le C h(z)^q$$

for all $z \in B_n$ and

$$\int_{B_n} \frac{\beta(z, w)(1-|w|^2)^{\alpha-\lambda}}{|1-\langle z, w\rangle|^{n+1+\alpha}} h(z)^p \, dv_\lambda(z) \le C h(w)^p$$

for all $w \in B_n$. By Lemma 8 this is possible provided that

$$-1 < \alpha + q\sigma < \alpha, \quad -1 < \lambda + p\sigma < \alpha,$$

or

$$-\frac{\alpha+1}{q} < \sigma < 0, \quad \frac{\lambda+1}{p} < \sigma < \frac{\alpha-\lambda}{p}.$$

Clearly such a number σ exists if and only if

$$\left(-\frac{\lambda+1}{p}, \frac{\alpha-\lambda}{p}\right) \cap \left(-\frac{\alpha+1}{q}, 0\right)$$

is nonempty. The desired result now follows easily from the assumption $p(\alpha+1) > \lambda+1 > 0$ and the observation that $(A, B) \cap (C, D)$ is nonempty when $C < B$ and $A < D$. □

LEMMA 10. *For each $p \ge 1$ and $\lambda > -1$ there exists a constant $C > 0$ such that*

$$\int_{B_n} |f(z)|^p \, dv_\lambda(z) \le C \int_{B_n} |u(z)|^p \, dv_\lambda(z)$$

for all holomorphic functions f on B_n with $f(0) = 0$, where u is the real part of f.

Proof. By using a limit argument, we may as well assume that f is holomorphic in a neighborhood of B_n. Choose α so that $p(\alpha+1) > \lambda + 1 > 0$; then P_α is bounded on $L^p(B_n, dv_\lambda)$. Thus there is a constant $C > 0$ such that

$$\int_{B_n} |P_\alpha u(z)|^p \, dv_\lambda(z) \le C \int_{B_n} |u(z)|^p \, dv_\lambda(z)$$

for all functions u on B_n. Now if u is the real part of a holomorphic function f on B_n with $f(0) = 0$, then

$$u = \frac{f + \overline{f}}{2},$$

and hence

$$P_\alpha u = \frac{f + P_\alpha \overline{f}}{2} = \frac{f + \overline{f}(0)}{2} = \frac{f}{2}.$$

It follows that

$$\begin{aligned}\int_{B_n} |f(z)|^p \, dv_\lambda(z) &= 2^p \int_{B_n} |P_\alpha u(z)|^p \, dv_\lambda(z) \\ &\leq 2^p C \int_{B_n} |u(z)|^p \, dv_\lambda(z),\end{aligned}$$

completing the proof of Lemma 10. □

We can now prove the main result of this section. Recall that for $\alpha > -1$ and f on B_n, the Hankel operator $H_f^{(\alpha)}$ is defined by $H_f^{(\alpha)} = (I - P_\alpha) M_f P_\alpha$, where I is the identity operator, M_f is the multiplication operator induced by f, and P_α is the orthogonal projection from $L^2(B_n, dv_\alpha)$ onto $L_a^2(dv_\alpha)$.

THEOREM 11. *Suppose $p \geq 1$ and $p(\alpha + 1) > \lambda + 1 > 0$. Then a function f on B_n belongs to BMO_∂^p if and only if the Hankel operators $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both bounded on $L^p(B_n, dv_\lambda)$.*

Proof. First assume that $f \in BMO_\partial^p$. We show that $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both bounded on $L^p(B_n, dv_\lambda)$. Since $BMO_\partial^p = BO + BA^p$ (by Theorem A) and $H_f^{(\alpha)}$ depends on f linearly, it suffices to show that $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are bounded on $L^p(B_n, dv_\lambda)$ for $f \in BO$ and $f \in BA^p$.

If $f \in BA^p$, then Lemma 3 implies that M_f and $M_{\overline{f}}$ are both bounded as operators from $L_a^p(dv_\lambda)$ into $L^p(B_n, dv_\lambda)$. Since $p(\alpha + 1) > \lambda + 1 > 0$, P_α is a bounded projection from $L^p(B_n, dv_\lambda)$ onto $L_a^p(dv_\lambda)$. Thus $H_f^{(\alpha)} = (I - P_\alpha) M_f P_\alpha$ and $H_{\overline{f}}^{(\alpha)} = (I - P_\alpha) M_{\overline{f}} P_\alpha$ are both bounded on $L^p(B_n, dv_\lambda)$.

On the other hand, the integral formula for P_α gives

$$H_f^{(\alpha)} g(z) = \int_{B_n} (f(z) - f(w)) K^{(\alpha)}(z, w) g(w) \, dv_\alpha(w), \qquad g \in L_a^p(dv_\lambda).$$

Since P_α maps $L^p(B_n, dv_\lambda)$ boundedly onto $L_a^p(dv_\lambda)$, the boundedness of $H_f^{(\alpha)} : L^p(B_n, dv_\lambda) \to L^p(B_n, dv_\lambda)$ is equivalent to the boundedness of $H_f^{(\alpha)} : L_a^p(dv_\lambda) \to L^p(B_n, dv_\lambda)$. Now if $f \in BO$, then

Lemma 2 shows that there is a constant $C > 0$ such that

$$|f(z) - f(w)| \leq C(\beta(z, w) + 1)$$

for all z and w in B_n. It follows that

$$\begin{aligned} |H_f^{(\alpha)} g(z)| &\leq C \int_{B_n} (\beta(z, w) + 1)|K^{(\alpha)}(z, w)||g(w)|\, dv_\alpha(w) \\ &= C C_\alpha \int_{B_n} \frac{(\beta(z, w) + 1)(1 - |w|^2)^\alpha}{|1 - \langle z, w\rangle|^{n+1+\alpha}} |g(w)|\, dv(w) \end{aligned}$$

for all $g \in L_a^p(dv_\lambda)$ and $z \in B_n$. It follows from Lemma 9 that $H_f^{(\alpha)}$ is bounded on $L^p(B_n, dv_\lambda)$. Similarly, $f \in BO$ implies that $H_{\bar f}^{(\alpha)}$ is bounded on $L^p(B_n, dv_\lambda)$. Thus we have proved that $f \in BMO_\partial^p$ implies that both $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are bounded on $L^p(B_n, dv_\lambda)$.

Next we assume that $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are bounded on $L^p(B_n, dv_\lambda)$. We show that $f \in BMO_\partial^p$. By considering the real and imaginary parts of f, we may as well assume that f is real-valued. Since $(k_z^{(\lambda)})^{2/p}$ are unit vectors in $L^p(B_n, dv_\lambda)$, there is a constant $C > 0$ such that

$$\|H_f^{(\alpha)}(k_z^{(\lambda)})^{2/p}\| \leq C$$

for all $z \in B_n$. $\|\ \|$ in this paragraph always means the norm in $L^p(B_n, dv_\lambda)$. Using the definition of $H_f^{(\alpha)}$, we have

$$\|(I - P_\alpha)(f(k_z^{(\lambda)})^{2/p})\| \leq C.$$

Note that each $k_z^{(\lambda)}$ is a nonzero holomorphic function on B_n and $|k_z^{(\lambda)}(w)|^2$ is the Jacobian determinant of the change of variable $w \mapsto \varphi_z(w)$ with respect to the measure dv_λ. Thus for each $z \in B_n$ there is a holomorphic function g_z on B_n with

$$\|f \circ \varphi_z - g_z\| \leq C.$$

In fact, g_z can be chosen as follows:

$$g_z(w) = P_\alpha(f(k_z^{(\lambda)})^{2/p})(\varphi_z(w))(k_z^{(\lambda)})^{-2/p}(\varphi_z(w)).$$

Since f is real-valued, $\|f \circ \varphi_z - g_z\| \leq C$ implies that

$$\|\operatorname{Im} g_z\| \leq C, \qquad z \in B_n.$$

By Lemma 10, there exists another constant $M > 0$ such that

$$\|g_z - g_z(0)\| \leq M$$

for all $z \in B_n$. It follows from the triangle inequality that

$$\|f \circ \varphi_z - g_z(0)\| \le C + M$$

for all $z \in B_n$. By the equivalence of (1) and (4) in Theorem 5, we have $f \in BMO_\partial^p$. This completes the proof of Theorem 11. □

COROLLARY 12. *For any* $p > 1$ *and* $\alpha > -1$ *we have* $f \in BMO_\partial^p$ *if and only if* $H_f^{(\alpha)}$ *and* $H_{\bar f}^{(\alpha)}$ *are both bounded on* $L^p(B_n, dv_\alpha)$.

Proof. This follows from Theorem 11 by setting $\alpha = \lambda$. □

COROLLARY 13. *Suppose* $\alpha > \lambda > -1$. *Then* $f \in BMO_\partial^1$ *if and only if* $H_f^{(\alpha)}$ *and* $H_{\bar f}^{(\alpha)}$ *are both bounded on* $L^1(B_n, dv_\lambda)$.

Proof. This is just the special case $p = 1$ in Theorem 11. □

COROLLARY 14. *Suppose* $p \ge 1$, $p(\alpha + 1) > \lambda + 1 > 0$, *and* f *is holomorphic in* B_n. *Then* $H_{\bar f}^{(\alpha)}$ *is bounded on* $L^p(B_n, dv_\lambda)$ *if and only if* $f \in \mathscr{B}$, *the Bloch space of* B_n.

Proof. This follows from Theorems 7 and 11 and the fact that $H_f^{(\alpha)} = 0$ if f is holomorphic. □

4. VMO_∂^p **and compact Hankel operators.** In this section we study the companion space VMO_∂^p and its relationship to compact Hankel operators on $L^p(B_n, dv_\lambda)$. When $1 < p < +\infty$, any two reasonable definitions for compact operators on $L^p(B_n, dv_\lambda)$ are equivalent. However, when $p = 1$, the space $L^1(B_n, dv_\lambda)$ is no longer reflexive, and hence the definition for compact operators on $L^1(B_n, dv_\lambda)$ will surely make a difference. In this section we will think of the Hankel operators $H_f^{(\alpha)}$ as acting on the Bergman spaces $L_a^p(dv_\lambda)$. We first clarify the notion of compact operators on $L_a^p(dv_\lambda)$.

When $1 < p < +\infty$, $L_a^p(dv_\lambda)$ is the dual of $L_a^q(dv_\lambda)$. $L_a^1(B_n, dv_\lambda)$ is the dual of the little Bloch space $\mathscr{B}_0$ (see [3]). The dualities just mentioned are given by

$$\langle f, g\rangle = \lim_{r\to 1^-} \int_{B_n} f(rz)\overline{g(rz)}\, dv_\lambda(z).$$

(The above limit can be taken inside the integral when $1 < p < +\infty$.) For $1 \le p < +\infty$, we equip $L_a^p(dv_\lambda)$ with the weak-star topology induced by the above dualities. We say that a linear operator

$T: L_a^p(dv_\lambda) \to L^p(B_n, dv_\lambda)$ is compact if $Tf_n \to 0$ (in norm) in $L^p(B_n, dv_\lambda)$ whenever $f_n \to 0$ in the weak-star topology of $L_a^p(dv_\lambda)$. If $1 < p < +\infty$, then the weak-star compactness defined above is equivalent to the usual compactness of operators on Banach spaces.

LEMMA 15. *For any* $1 \le p < +\infty$ *and* $\lambda > -1$, *a sequence* $\{f_n\}$ *in* $L_a^p(dv_\lambda)$ *converges to zero in the weak-star topology if and only if* $f_n(z) \to 0$ *uniformly on compact sets and* $\|f_n\|_{L^p(dv_\lambda)} \le C$ *for some constant* $C > 0$ *and all* $n \ge 1$.

Proof. The proof is similar to that of Lemma 11 in **[11]**. □

COROLLARY 16. *For any* $p \ge 1$ *and* $\lambda > -1$ *we have* $(k_z^{(\lambda)})^{2/p} \to 0$ $(|z| \to 1^-)$ *in the weak-star topology of* $L_a^p(dv_\lambda)$.

Proof. This follows directly from Lemma 15. □

For any $p \ge 1$ and $r > 0$, let VMO_r^p denote the subspace of BMO_∂^p consisting of functions f such that

$$\lim_{|z|\to 1^-} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - \hat{f}_r(z)|^p \, dv(w) = 0.$$

To describe the structure of VMO_r^p, we introduce two subspaces of VMO_r^p.

Recall that

$$\omega_r(f)(z) = \sup\{|f(z) - f(w)| : w \in D(z, r)\}$$

is the oscillation of a continuous function f at z in the Bergman metric. Let VO_r be the space of all continuous functions f on B_n such that $\omega_r(f)(z) \to 0$ $(|z| \to 0)$. The initials VO here stand for vanishing oscillation.

LEMMA 17. VO_r *is independent of* r. *Moreover,* VO_r *is the closure of* $\mathbf{C}(\overline{B_n})$ *in* BO, *where* $\mathbf{C}(\overline{B_n})$ *is the space of all functions on* B_n *which are continuous up to the boundary of* B_n.

Proof. The proof here is similar to that in the case $n = 1$ given in §7.2 of **[12]**. We omit the details here. □

We will simply write VO for VO_r.

For $r > 0$ and $p \ge 1$, let VA_r^p denote the space of functions f on B_n such that $\widehat{|f|}_r^p(z) \to 0$ $(|z| \to 1^-)$. The initials VA stand for vanishing average. We have

LEMMA 18. *VA_r^p is independent of r. Moreover, VA_r^p is the closure in BA^p of the set of functions with compact support in B_n.*

Proof. The proof is similar to that of the special case $p = 2$ and $n = 1$ given in §7.2 of **[12]**. We omit the details here. See also **[10]**. $\square$

The above lemma enables us to write VA^p for VA_r^p. Note that a similar version of Lemma 3 can also be proved for VA^p. We can now describe the structure of the space VMO_r^p.

THEOREM 19. *The space VMO_r^p is independent of r. Moreover, the following conditions are equivalent:*

(1) $f \in VMO_r^p$;

(2) $f \in VO + VA^p$;

(3) *For any $\varepsilon > 0$, there exists a constant $\delta \in (0, 1)$ such that for any $\delta < |z| < 1$ in B_n there is a constant λ_z with*

$$\frac{1}{|D(z, r)|}\int_{D(z,r)} |f(w) - \lambda_z|^p \, dv(w) < \varepsilon;$$

(4) $\lim_{|z|\to 1^-} \|f \circ \varphi_z - B_\lambda f(z)\|_{L^p(dv_\lambda)} = 0$ *for all (or some) $\lambda > -1$;*

(5) *For any $\varepsilon > 0$ and $\lambda > -1$, there exists $\delta \in (0, 1)$ such that for any $\delta < |z| < 1$ in B_n there is a constant λ_z with $\|f\circ\varphi_z - \lambda_z\|_{L^p(dv_\lambda)} < \varepsilon$.*

Proof. The proof for the theorem is similar to that of the corresponding statements for BMO_∂^p in §2. We omit the details here. $\square$

We will write VMO_∂^p for VMO_r^p. By the above theorem and Lemmas 17 and 18, VMO_∂^p is the subspace of BMO_∂^p generated by functions in $\mathbf{C}(\overline{B_n})$ and functions with compact support.

THEOREM 20. *Suppose $p \geq 1$ and $p(\alpha + 1) > \lambda + 1 > 0$. Then a function f on B_n belongs to VMO_∂^p if and only if $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both compact on $L_a^p(dv_\lambda)$.*

Proof. It is easy to show that $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both compact on $L^p(B_n, dv_\lambda)$ if $f \in \mathbf{C}(\overline{B_n})$. Also if $f \in BA^p$ has compact support in B_n, then $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are both compact on $L^p(B_n, dv_\lambda)$. Since VMO_∂^p is generated by $\mathbf{C}(\overline{B_n})$ and functions with compact support, we see that $H_f^{(\alpha)}$ and $H_{\overline{f}}^{(\alpha)}$ are compact on $L^p(B_n, dv_\lambda)$ for $f \in VMO_\partial^p$.

Conversely, if $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are both compact on $L^p(B_n, dv_\lambda)$, then

$$\|H_f^{(\alpha)}(k_z^{(\lambda)})^{2/p}\| \to 0, \quad \|H_{\bar f}^{(\alpha)}(k_z^{(\lambda)})^{2/p}\| \to 0 \qquad (|z| \to 1^-)$$

since $(k_z^{(\lambda)})^{2/p} \to 0$ $(|z| \to 1^-)$ in the weak-star topology of $L_a^p(dv_\lambda)$, where $\| \ \|$ denotes the norm in $L^p(B_n, dv_\lambda)$. By the second part of the proof of Theorem 11, we have $\|f \circ \varphi_z - g_z(0)\| \to 0$ $(|z| \to 1^-)$. By Theorem 19, $f \in VMO_\partial^p$, completing the proof of Theorem 20. □

The following two corollaries are immediate consequences of Theorem 20.

COROLLARY 21. *If $p > 1$ and $\alpha > -1$, then $f \in VMO_\partial^p$ if and only if $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are both compact in $L^p(B_n, dv_\alpha)$.*

COROLLARY 22. *If $\alpha > -\lambda > -1$, then $f \in VMO_\partial^1$ if and only if $H_f^{(\alpha)}$ and $H_{\bar f}^{(\alpha)}$ are compact in $L^1(B_n, dv_\lambda)$.*

Finally, we show that holomorphic functions in VMO_∂^p are precisely the functions in the little Bloch space. Recall that $H(B_n)$ is the space of all holomorphic functions on B_n and $\mathscr{B}_0(B_n)$ is the little Bloch space of B_n.

LEMMA 23. $VMO_\partial^p \cap H(B_n) = \mathscr{B}_0(B_n)$.

Proof. It is shown in [1] that $VO \cap H(B_n) = \mathscr{B}_0(B_n)$. Thus $\mathscr{B}_0(B_n) \subset VMO_\partial^p \cap H(B_n)$. On the other hand, it is easy to see that if $f \in VMO_\partial^p$, then $B_\alpha f \in VO$ for all $\alpha > -1$ (see the proof of Corollary 6). Thus if f is a holomorphic function in VMO_∂^p, then $f = B_\alpha f \in VO$, and hence $f \in \mathscr{B}_0(B_n)$. □

COROLLARY 24. *Suppose $p \geq 1$, $p(\alpha + 1) > \lambda + 1 > 0$, and f is holomorphic in B_n. Then $H_{\bar f}^{(\alpha)}$ is compact on $L_a^p(dv_\lambda)$ if and only if $f \in \mathscr{B}_0(B_n)$.*

Proof. This follows from Lemma 23 and Theorem 20. □

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STATE UNIVERSITY OF NEW YORK
ALBANY, NY 12222