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THE EULER CLASS FOR "PIECEWISE" GROUPS

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The Euler class is a semiconjugacy invariant of a discrete group G of orientation preserving homeomorphisms of the circle. An element of the second cohomology group of G with integral coefficients, it is often difficult to calculate, but even its nonvanishing seems related to dynamical complexity of G. In this note, we consider a family of discrete groups $\Gamma_{H,S}(p, q)$ of homeomorphisms of the circle, whose definition generalizes that of piecewise linear homeomorphisms. We define an invariant with which one can verify the vanishing of the Euler class in a surprising range of cases. On the other hand, the vanishing of the invariant, together with a simple geometric condition, assures the nonvanishing of the Euler class.

The invariant has a simple "operational" definition, but can also be interpreted as an element of the fundamental group of the classifying space of a certain pseudogroup. We also apply it to the question of the existence of elements of finite order in the groups $\Gamma_{H,S}(p, q)$.

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1. Definitions and results.

1.1. *The groups.* Let H be a group of analytic, orientation preserving homeomorphisms of the real line $\mathbf{R}$, and let S be an H-invariant subset of $\mathbf{R}$. Let $p < q$ be elements of S in the same H-orbit. Let $S^1_{p,q}$ denote the closed interval $[p, q]$ with p and q identified as the basepoint. Define $\Gamma_{H,S}(p, q)$ to be the group of homeomorphisms g of $S^1_{p,q}$ such that there exist $s_i \in S$, $p = s_0 < \cdots < s_n = q$, so that the restriction of g to $[s_i, s_{i+1}]$ agrees pointwise with an element h_i of H. Thus, $\Gamma_{H,S}(p, q)$ is "the group of piecewise-H homeomorphisms of $S^1_{p,q}$, with breakpoints in S."

1.2. *The invariant.* Let $N = N(S)$ denote the normal subgroup of H generated by all elements which fix some point in S. If $p, q \in S$ are in the same H-orbit, the equivalence class in H/N of an element

$h \in H$ such that $h(p) = q$ depends on p and q only. This equivalence class will be called the *manifold class* h *of the circle* $S^1_{p,q}$.

1.3. *A flexibility condition.* We say that a pair (H, S) is *flexible* if the following condition is satisfied: let $a, b, c, d \in S$, $a < b$, $c < d$. Suppose there are $g, h \in H$, with $g \equiv h$ in H/N, such that $g(a) = c$, $h(b) = d$. Then there are $s_i \in S$, $a = s_0 < \cdots < s_n = b$, and $g_i \in H$ so that $g_1(a) = c$, $g_n(b) = d$, and $g_i(s_i) = g_{i+1}(s_i)$, $1 \le i \le n-1$. That is, there is a piecewise-H homeomorphism from $[a, b]$ to $[c, d]$.

We can now state the main results, staying with the notation established above. Suppose that G is a group acting on the circle. By the *rational* Euler class of G we mean the rational reduction in $H^2(G; \mathbf{Q})$ of the (integral) Euler class.

1.4. THEOREM. *If no nonzero power of the manifold class of* $S^1_{p,q}$ *can be written as a product of commutators in* H/N, *then the rational Euler class of* $\Gamma_{H,S}(p, q)$ *vanishes.*

One can restate the hypothesis; it requires that the manifold class be of infinite order in the abelianization of H/N.

1.5. COROLLARY. *If* H/N *is abelian and the manifold class is of infinite order, then the rational Euler class of* $\Gamma_{H,S}(p, q)$ *vanishes.*

A partial converse to 1.4 is the following.

1.6. THEOREM. *Suppose that* (H, S) *is flexible, and that the manifold class of* $S^1_{p,q}$ *is null. Then the Euler class is nonzero, and not divisible in* $H^2(\Gamma_{H,S}(p, q); \mathbf{Z})$.

Regarding elements of torsion, we have the following result.

1.7. THEOREM. *Suppose there is a* $g \in \Gamma_{H,S}(p, q)$ *and* $s \in [p, q] \cap S$ *such that the orbit* $\{g^k(s)\}_{k \in \mathbf{Z}}$ *has* m *elements,* $m < \infty$. *Then the manifold class of* $S^1_{p,q}$ *is an* m*th power in* H/N.

1.8. COROLLARY. *If the manifold class is not divisible in* H/N, $\Gamma_{H,S}(p, q)$ *is torsion free.*

Before stating results about divisibility of the integral Euler class and embeddings of surface groups, let us pause to consider two families of examples.

1.9. *Piecewise linear examples.* Let A be an additive subgroup of the real numbers, and let U be a multiplicative subgroup of the positive real numbers, such that for all $a \in A$, $u \in U$, we have $ua \in A$. Let H be the group of affine transformations of the form $h(x) = ux + a$, $u \in U$, $a \in A$, and let S be the H-orbit of 0, that is A. From ([**Gr1**], 1.13) we extract:

1.10. PROPOSITION. (a) (H, S) *is flexible.*

(b) $H/N \cong A/(u-1)$, *the quotient of* A *by the subgroup generated by all* $ua - a$, $u \in U$, $a \in A$. *If* $h \in H$, $h(x) = ux + a$, *the equivalence class of* h *in* H/N *is that of* a *in* $A/(u-1)$.

For example, take $r \in \mathbf{R}$, $r > 1$, and let $A = Q[r, r^{-1}]$ be the ring of finite rational Laurent series in r. Let $U = \{r^n, n \in \mathbf{Z}\}$. Let $p = 0$, $q = 1$, and consider $\Gamma_{H,A}(0, 1)$. The manifold class of $S^1_{0,1}$ is the equivalence class of $h(x) = x + 1$ in $A/(r^n - 1)$. This class is 0 if r is algebraic and of infinite order if r is transcendental, and the Euler class of $\Gamma_{H,A}(0, 1)$ is nonzero or zero accordingly.

Note that the Euler class of $\Gamma_{H,A}(0, r-1)$ is always nonzero.

1.11. *Piecewise projective examples.* We begin with a slight modification of the original construction. Let G be a group of orientation preserving analytic homeomorphisms of the circle S^1. Let $H = \widetilde{G}$ be the group of homeomorphisms of the universal cover $\widetilde{S}^1$ which cover elements of G. Recall that H is a central extension $\mathbf{Z} \to H \to G$, where the $\mathbf{Z} = \pi_1 S^1$ is identified as the group of covering transformations of $\widetilde{S}^1$ over S^1. Let T denote the positive generator of $\pi_1 S^1$ (that is $Tx > x$, $x \in \widetilde{S}^1$).

Let S be a G-invariant subset of S^1, and $\widetilde{S} \subseteq \widetilde{S}^1$ its inverse image. The manifold class of S^1_{p,T_p}, $p \in \widetilde{S}$, is simply $T \in H/N$; we consider the group $\Gamma_{H,\widetilde{S}}(p, T_p)$.

In particular we will consider subgroups G of $\mathrm{PSL}_2\mathbf{R}$, acting as usual on $S^1 = \mathbf{R} \cup \{\infty\}$ by linear fractional transformations. For example, take $G = \mathrm{PSL}_2 F$, F a subfield of $\mathbf{R}$, and let $S = F \cup \{\infty\}$. The results of [**Gr2**] imply:

1.12. PROPOSITION. (a) $\mathrm{PSL}_2(F, \widetilde{S})$ *is flexible.*

(b) $\mathrm{PSL}_2 F/N$ *is the group with one element.*

As a consequence, the Euler class of $\Gamma_{H,\widetilde{S}}(p, T_p)$ is nonvanishing.

Now set $G = \mathrm{PSL}_2\mathbf{Z}$, and $S = \mathbf{Q} \cup \{\infty\}$ the G-orbit of 0. The group $\Gamma_{H,\widetilde{S}}(p, T_p)$ can be shown (following a remark of Thurston) to be isomorphic to the "remarkable group" studied by Etienne Ghys and Vlad Sergiescu in **[GhS]**. It turns out (**[GhS]**) that H, $\widetilde{S}$ is flexible, and that $\Gamma_{H,\widetilde{S}}(p, T_p)$ has a nonzero Euler class.

The situation changes when we pass to congruence subgroups of $\mathrm{PSL}_2\mathbf{Z}$. Consider $G = \Gamma(M)$, the set of matrices congruent to $\pm I$ modulo $M \in \mathbf{Z}$, $M > 1$. It is known (see **[Sh]**, Chap. 1) that $\Gamma(M)$ is a free group. More precisely, a model for $K(\Gamma(M), 1)$ is a certain Riemann surface with punctures; $\mathbf{Q} \cup \{\infty\}$ breaks up into a finite number of $\Gamma(M)$-orbits, corresponding to the punctures. A small circle about a puncture is the conjugacy class in $\Gamma(M)$ of an element fixing a point in the corresponding orbit. If the surface has nonzero genus, the group is not generated by these stabilizers.

Let S be the union of one or more of these orbits. Since any central extension of a free group is trivial, $H \cong G \times \mathbf{Z}$, and T corresponds to $(e, 1)$; further, $H/N \cong G/N \times \mathbf{Z}$, where N denotes the subgroup of G generated by stabilizers of points in S. Thus if the genus of G is nonzero, the Euler class of $\Gamma_{H,\widetilde{S}}(p, T_p)$ is 0. Or, if the genus of G is zero, but S does not contain at least two of the orbits which comprise $\mathbf{Q} \cup \{\infty\}$, the Euler class of $\Gamma_{H,\widetilde{S}}(p, T_p)$ vanishes.

1.13. *Divisibility and surface groups.* The following rather technical result will be combined with the Milnor-Wood inequality (**[Mi]**, **[W]**) to obtain a result (1.15) on embedding surface groups in the $\Gamma_{H,S}(p, q)$.

1.14. PROPOSITION. *Suppose that H/N is abelian, that the manifold class of $S^1_{p,q}$ is nonzero, and that the rational Euler class of $\Gamma_{H,S}(p, q)$ is nonzero. Then the manifold class of $S^1_{p,q}$ is torsion, and its order in H/N divides the integral Euler class.*

Let $\pi_g = \langle a_i, b_i, i = 1, \ldots, g | \pi[a_i, b_i] = 1\rangle$ be the fundamental group of a compact surface of genus g. Etienne Ghys has asked (**[Gh3]**) for conditions under which π_g may be embedded into certain subgroups of homeomorphisms of the circle.

1.15. COROLLARY. *Suppose that H/N is abelian, that the manifold class of $S^1_{p,q}$ is nonzero, and that there is a homeomorphism $f: \pi_g \to \Gamma_{H,S}(p, q)$ such that the pullback to $H^2(\pi_g; \mathbf{Z})$ of the Euler class is nonzero.*

Then the manifold class is torsion, and its order divides a natural number less than $2g-1$.

Proof. By the assumptions and 1.14, the manifold class is torsion, and its order divides the Euler class of $\Gamma_{H,S}(p, q)$, and hence its order divides the pullback of E in $H^2(\pi_g; \mathbf{Z})$. But the Milnor-Wood inequality asserts that this pullback is at most $\pm(2g-2)$ times the generator of $H^2(\pi_g; \mathbf{Z})$. Consequently, the order of the manifold class divides some natural number less than $2g-1$.

1.16. EXAMPLE. Consider the piecewise linear example (1.10) with $A = \mathbf{Z}[\frac{1}{n}]$, $n \in \mathbf{Z}$, $n > 1$, and $U = \{n^k\}_{k\in\mathbf{Z}}$. Then $H/N \cong \mathbf{Z}/n-1$, and the manifold class of $S^1_{0,1}$ generates. Consequently, for a homomorphism $\pi_t \to \Gamma_{H,S}(S^1_{0,1})$ to pull back a nonzero Euler class, it is necessary that $n-1$ divide one of $2, \dots, 2g-2$. In particular, $n \le 2g$.

1.17. *Organization.* In the next section we give proofs of the results above, save 1.6. In §3 we interpret the manifold class as an element of the fundamental group of a certain pseudogroup, and prove 1.6.

2. The homomorphism C. After introducing some notation, we enunciate a Lemma 2.1 and use it to prove various results. Then the lemma is proved. We continue with the notation already established, and consider an H, S and $p, q \in S$, $p < q$.

Let $\widetilde{S}^1$ denote the universal cover of $S^1_{p,q}$, and let T denote the positive generator of the group of covering transformations of $S^1_{p,q}$. Let $\widetilde{\Gamma}_{H,S}(p, q)$ denote the group of lifts of elements of $\Gamma_{H,S}(p, q)$ to homeomorphisms of $\widetilde{S}^1$. Choose a lift $\tilde{p}$ of p. Let $\widetilde{S}$ denote the set of lifts of points in S to $\widetilde{S}^1$.

2.1. LEMMA. *There is a homomorphism* $C\colon \widetilde{\Gamma}_{H,S} \to H/N$ *such that*

(i) $C(T)$ *is the manifold class of* $S^1_{p,q}$.

(ii) $C(g_1) = C(g_2)$ *if* $g_1(s) = g_2(s)$, $s \in \widetilde{S}$.

Proof of 1.4. Suppose that the rational Euler class of $\Gamma_{H,S}(p, q)$ is nonzero. Then there are $a_i, b_i \in \Gamma_{H,S}(p, q)$, $i = 1, \dots, k$, such that, choosing lifts $\tilde{a}_i, \tilde{b}_i \in \widetilde{\Gamma}_{H,S}(p, q)$, we have $\prod_{i=1}^k [\tilde{a}_i, \tilde{b}_i] = T^m$, $m \ne 0$. Applying the homomorphism C of 2.1, we see that the mth power of the manifold class is a product of commutators.

Proof of 1.7. Suppose that $g \in \Gamma_{H,S}(p, q)$, and that for $s \in S \cap [p, q]$, $\{g^k(s)\}_{k\in\mathbf{Z}}$ has cardinality $m < \infty$. Possibly replacing g with a power of g, there is a lift $\tilde{g} \in \widetilde{\Gamma}_{H,S}(p, q)$, and a lift $\tilde{s} \in S$, such that $\tilde{g}^m(\tilde{s}) = T\tilde{s}$. Applying 2.1 (ii), $C(\tilde{g})^m$ is the manifold class of $S^1_{p,q}$.

Proof of 1.14. Since the rational Euler class is nonzero, the integral Euler class is nonzero, and further, not torsion. Consequently, there is some least $m \neq 0$ such that the mth power of the manifold class is a product of commutators in H/N. The latter being abelian, this mth power is zero. Thus, viewed as a homomorphism from $H_2(\Gamma_{H,S}(p, q); \mathbf{Z})$ to $\mathbf{Z}$, the image of the Euler class is contained in $m\mathbf{Z}$. Thus, the Euler class is divisible by m. □

We now construct the homomorphism C, and prove Lemma 2.1. As it happens, the natural domain of C is a certain collection of homeomorphisms between open subsets of the line. Namely, let $g: U \to V$ be a homeomorphism between open subsets of the line. Then $g \in \Gamma_{H,S}$, and we say g *is piecewise-H with breakpoints in* S if there exists a subset X of $U \cap S$, X discrete in U, such that for any connected component K of $U - X$, the restriction of g to K agrees pointwise with an element h_K of H. We call X the set of *breakpoints* of g. The set of homeomorphisms $\Gamma_{H,S}$ is a pseudogroup ([**Ha**]); it is closed under restriction to open subsets, taking inverses, and (where defined) composition.

2.2. Lemma. *Let* $g \in \Gamma_{H,S}$ *with connected domain* U. *Let* X *be the set of breakpoints of* g, *and let* $h_K \in H$ *denote the restriction of* g *to a component* K *of* $U - X$. *Then the equivalence class of* h_K *in* H/N *is independent of the component* K.

Proof. We must show that if K, K' are two components of $U - X$, then $h_K = h_{K'}$ in H/N. Since X is discrete in U, we can assume that K and K' are adjacent, that is $\overline{K} \cap \overline{K'} = s \in S$. Then $h_K(s) = h_{K'}(s)$, since g is continuous. Thus $h_K h_{K'}^{-1}(s) = s$, so that $h_K h_{K'}^{-1} \in N$.

2.3. Definition. Let $g \in \Gamma_{H,S}$, with connected domain. We denote by $\sigma(g)$ the common value of the h_K in H/N.

2.4. Lemma. *Let* $g_i: U_i \to V_i$, $i = 1, 2$, $g_i \in \Gamma_{H,S}$, *with* U_i *connected. If* $V_1 \subseteq U_2$, *then* $\sigma(g_2 \circ g_1) = \sigma(g_2)\sigma(g_1)$.

Proof. Left to the reader.

2.5. LEMMA. *Let* $g_1, g_2 \in \Gamma_{H,S}$ *with connected domains whose intersection contains a point* $s \in S$. *If* $g_1(s) = g_2(s)$, *then* $\sigma(g_1) = \sigma(g_2)$.

Proof. Restricting to subsets if necessary, we can assume that g_1 and g_2 have a common range V which contains $g_1(s) = g_2(s)$. Thus $g_1^{-1}g_2$ is well-defined, and $g_1^{-1}g_2(s) = s$. Consequently, the restriction of $g_1^{-1}g_2$ to a small interval to the right of s agrees with an element of N, and so $\sigma(g_1) = \sigma(g_2)$.

Proof of 2.1. Pick $h \in H$ so that $h(p) = q$. Let $U = \bigcup_{n\in\mathbf{Z}} h^n[p, q]$. Then U is a connected open subset of $\mathbf{R}$ on which the infinite cyclic group $\{h^k\}_{k\in\mathbf{Z}}$ acts, properly discontinuously.

Choose a lift $L\colon [p, q] \to \widetilde{S}^1$, that is, a continuous map such that the diagram

$$\begin{array}{ccc} & \overset{L}{\nearrow} & \widetilde{S}^1 \\ & & \downarrow \\ [p, q] & \longrightarrow & S^1_{p,q} = [p, q]/\!\sim \end{array}$$

commutes, and such that $L(p) = \tilde{p}$. There is a unique homeomorphism $f\colon \widetilde{S}^1 \to U$ such that $fL(x) = x$, $x \in [p, q]$, and such that $f \circ T = h \circ f$. Indeed, every $x \in \widetilde{S}^1$ can be written uniquely as $T^k L(y)$, for some $y \in [p, q)$; we define $f(x) = h^k(y)$.

It now follows that if $g \in \widetilde{\Gamma}_{H,S}(p, q)$ then $fgf^{-1} \in \Gamma_{H,S}$. Define $C(g) = \sigma(fgf^{-1})$. Then $C(T) = \sigma(fTf^{-1}) = h$, and 2.5 implies that $C(g)$ depends only on $g(s)$, $s \in \widetilde{S}$.

3. Haefliger structures on the circle. In this section we describe a general context for the results of this paper. We use the language of pseudogroups and their classifying spaces as developed by André Haefliger ([**Ha**]). A result of Dusa McDuff, ([**McD**]) as presented by Etienne Ghys and Vlad Sergiescu ([**GhS**]) is applied to prove 1.6.

Let Γ be a pseudogroup of orientation preserving homeomorphisms between open subsets of the line. We shall assume:

3.1. *Assumption.* Every germ in Γ admits an extension to a homeomorphism of the real line which is in Γ.

We can think of Γ as its space of germs, with the sheaf topology. Then 3.1 implies that $\pi_0\Gamma$ inherits the structure of a group from Γ. The construction of the homomorphism C in §2 proves:

3.2. PROPOSITION. $\pi_0\Gamma_{H,S} \cong H/N$. *Indeed, the function* σ *of* 2.3 *gives the isomorphism.*

Associated to a pseudogroup Γ is its classifying space $B\Gamma$, and it follows from Assumption 3.1 that $\pi_1 B\Gamma \cong \pi_0\Gamma$. Now, $\pi_1 B\Gamma$ is identified in the theory of Haefliger ([**Ha**]) with the set of homotopy classes of Γ-structures on S^1 with a distinguished basepoint. Indeed, given $\underline{h} \in \pi_0\Gamma$ represented by $h \in \Gamma$, such that for some $p \in \mathbf{R}$, $q = h(p) > p$, we can construct a Γ-structure on S^1 corresponding to $\underline{h} \in \pi_0\Gamma = \pi_1 B\Gamma$ by "gluing p to q using h" as described below. Hence, what we defined in 1.2 as the manifold class of $S^1_{p,q}$ corresponds, via 3.2, to the element of $\pi_1 B\Gamma$ which $S^1_{p,q}$, with basepoint $p \sim q$ and its $\Gamma_{H,S}$-structure, represents.

We now make precise the "gluing" construction referred to above. Pick a global extension of h in Γ, to a homeomorphism of $\mathbf{R}$ which we will also denote $h \in \Gamma$ (using 3.1). Let $U = \bigcup_{n\in\mathbf{Z}} h^n[p, q]$. This is a connected open interval in $\mathbf{R}$, on which the infinite cyclic group $C_n = \{h^k\}_{k\in\mathbf{Z}}$ acts properly discontinuously. We define S^1_h as U modulo the action of C_h. Then S^1_h, with basepoint p, is a circle with Γ-structure corresponding to $\underline{h} \in \pi_0\Gamma$.

Let $\Gamma|_U$ denote the pseudogroup Γ restricted to U. The group C_h acts on the space $\Gamma|_U$ on the left and the right, by composition, and the quotient, denoted Γ_h, is the sheaf of locally-Γ homeomorphisms between open subsets of S^1_h.

Let $\Gamma(S^1_h)$ be the group of global sections of Γ_h over S^1_h, such that composition with the target map gives a homeomorphism of S^1_h. Then $\Gamma(S^1_h)$ is the group of locally-Γ homeomorphisms of S^1_h. When $\Gamma = \Gamma_{H,S}$, $\Gamma(S^1_h)$ is precisely the group $\Gamma_{H,S}(p, q)$. We generalize 1.4 as follows.

3.3. PROPOSITION. *If no nonzero power of* $\underline{h}$ *is a commutator in* $\pi_0\Gamma$, *then the rational Euler class of* $\Gamma(S^1_h)$ *vanishes.*

Proof (*Sketch*). As in §2, we must define a homomorphism $C: \widetilde{\Gamma}(S^1_h) \to \pi_0\Gamma$, where $\widetilde{\Gamma}(S^1_h)$ is the group of homeomorphisms of $\widetilde{S}^1_h$ which cover elements of $\Gamma(S^1_h)$. Having constructed S^1_h as a quotient of $U = \widetilde{S}^1$, we see that $\widetilde{\Gamma}(S^1_h)$ is the subgroup of the group $\Gamma(U)$ of Γ-homeomorphisms of U, consisting of elements which commute with h. Then C is just the composition $\widetilde{\Gamma}(S^1_h) \to \Gamma(U) \to \pi_0\Gamma$. The rest of the proof is as in §2.

In order to prove 1.6, we will invoke a theorem of Dusa McDuff **[McD]** to provide a homological model of the Eilenberg-Mac Lane spaces $B\Gamma_{H,S}(p, q)$, when (H, S) is flexible. We begin by recalling some properties of the free loop space.

3.4. REMARKS ON LX. Suppose that X is a connected and simply connected space. Let ΛX denote the space of maps from S^1 to X, without regard to basepoint. The group $\mathrm{Homeo}^+ S^1$ of orientation preserving homeomorphisms of S^1 acts on ΛX on the left (composition with the inverse) and we denote by LX the homotopy quotient $(E\,\mathrm{Homeo}^+ S^1) \times_{\mathrm{Homeo}^+ S^1} (\Lambda X)$. Then LX is functorial in X. The evident map $X \to *$ induces a map $LX \to L^* = B\,\mathrm{Homeo}^+ S^1$, and thus we can pull the Euler class of $B\,\mathrm{Homeo}^+ S^1$ back to an element of $H^2(LX; \mathbf{Z})$, which we call the Euler class of LX.

3.5. LEMMA. *If X is connected and simply connected, the Euler class of LX is nonzero and not divisible.*

Proof. Apply the construction L to the inclusion and retraction of a point in X. We obtain $B\,\mathrm{Homeo}^+ S^1 \to LX \to B\,\mathrm{Homeo}^+ S^1$ whose composition is homotopic to the identity, whence the lemma.

3.6. *Germ-connectedness and flexibility.* In **[McD]** Dusa McDuff proved that there is a map $f: B\,\mathrm{Diff}^+ S^1 \to LB\Gamma^\infty$, where $\mathrm{Diff}^+ S^1$ is the discrete group of orientation preserving diffeomorphisms of the circle and Γ^∞ is the pseudogroup of orientation preserving diffeomorphisms between open subsets of $\mathbf{R}$, such that f induces an isomorphism in homology, and further, such that the pullback by f of the Euler class of $LB\Gamma^\infty$ is the Euler class of $\mathrm{Diff}^+ S^1$. (The latter from the construction of f as a quotient of the map between certain circle bundles). It turns out, following the careful exposition in **[GhS]**, that her result holds for a wider class of pseudogroups.

3.7. DEFINITION (**[Gr 1]**). Let Γ be a pseudogroup of orientation preserving homeomorphisms between open subsets of $\mathbf{R}$. We say that Γ is *germ-connected to the identity* if, given germs in Γ, $g: a \to c$, $h: b \to d$ with $a < b$, $c < d$, there is an $s \in \Gamma$, whose connected domain contains both a and b, such that $s|_a \equiv g$, $s|_b \equiv h$.

3.8. REMARK. Evidently, $\Gamma_{H,S}$ is germ-connected to the identity if and only if $H = N$ and (H, S) is flexible.

Let Γ be a pseudogroup of orientation preserving homeomorphisms on $\mathbf{R}$. A Γ-*circle* is a manifold homeomorphic to S^1, but with charts and gluing data from Γ; the Γ-structures S^1_h constructed above are examples. Not every Γ-structure, in general, is homotopic to a Γ-circle. However, if Γ is germ-connected to the identity, then any two Γ-circles are Γ-homeomorphic (in the obvious sense). Thus, the group $\Gamma(S^1)$ of Γ-homeomorphisms of a Γ-circle is well defined. The careful argument in ([**GhS**], §2) proves:

3.9. PROPOSITION ([**GhS**], 2.11). *Suppose that Γ is a pseudogroup of homeomorphisms between open subsets of* $\mathbf{R}$, *and that Γ is germ-connected to the identity. There is a map $B\Gamma(S^1) \to LB\Gamma$ inducing isomorphism on homology, and pulling the Euler class of $LB\Gamma$ back to that of the group $\Gamma(S^1)$.*

We conclude with the proof of 1.6.

Proof of 1.6. Since the manifold class of $S^1_{p,q}$ is null, the group $\Gamma_{H,S}(p, q)$ exists and is a subgroup of $\Gamma_{H,S}(p, q)$. It suffices to show that the Euler class of $\Gamma_{N,S}(p, q)$ is nonzero and not divisible. Since (H, S) is flexible, $\Gamma_{N,S}$ is germ connected to the identity by 3.8 and we are done, by 3.9.

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