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POLES OF EISENSTEIN SERIES ON SL_n INDUCED FROM MAXIMAL PARABOLICS

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The author locates poles for Eisenstein series on algebraic groups $\mathrm{SL}_n(\Delta)$, where $n \in \mathbb{N}$ and Δ is an arbitrary finite dimensional division algebra over a number field. An explicit family of non-holomorphic functions, which include series of arbitrary level, is characterized. Each series $E(z, s)$ is induced from a character on a maximal parabolic. For each $E(z, s)$ in the family, there is an explicit product $\Lambda(s)$ of Γ-functions, L-functions and a polynomial term such that $\Lambda(s)E(z, s)$ has only simple poles in the s variable.

Introduction. Let F be a number field and let Δ be a finite dimensional central division F-algebra. Let ∞ denote the infinite primes of F, and for $\nu \in \infty$, let F_ν denote a completion of F and identify $\Delta_\nu = \Delta \otimes_F F_\nu$ with a matrix ring over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, accordingly. (We refer to ν as a real, complex or quaternionic prime of Δ, respectively.) Let $m, n \in \mathbb{N}$, and consider algebraic groups over F

$$G = \mathrm{SL}_{m+n}(\Delta), \qquad P = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in G \right\}, \tag{1}$$

where in (1) and hereafter we divide $m+n$ square matrices into blocks $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in which a, b, c and d have sizes $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively.

Let $\nu \in \infty$, and identify $G_\nu = G(F_\nu)$ with a matrix group over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ accordingly. Set

$$K_\nu = \{T \in G_\nu : T \cdot {}^t T^\rho = 1_{n+m}\}, \tag{2}$$

where ρ is $1_{\mathbb{R}}$, complex conjugation or the main involution of $\mathbb{H}$, respectively. Then $G_\nu = P(F_\nu) \cdot K_\nu$, and we may define Y_ν on $G_\nu \times \mathbb{C}$ by

$$Y_\nu\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \omega, s \right) = |dt(d)|^{-1}, \tag{3}$$

where dt is the reduced norm from $M_n(\Delta_\nu)$ to $\mathbb{R}$ and $|\ |$ is the standard norm. Put

$$G_\infty = \prod_{\nu\in\infty} G_\nu, \tag{4}$$

$$Y(z) = \prod_{\nu\in\infty} Y(z_\nu) \quad \text{for } z \in G_\infty, s \in \mathbb{C}.$$

For $\Gamma \subseteq G(F)$ a congruence subgroup, the sum

$$E(z, s; \Gamma) = \sum_{\alpha\in(P\cap\Gamma)\backslash\Gamma} Y(\alpha\cdot z)^s, \tag{5}$$

converges for $\mathrm{Re}(s) >> 0$, and has the property that $E(\alpha z, s; \Gamma) = E(z, s; \Gamma)$ for each $\alpha \in \Gamma$. Moreover, the sum has a meromorphic extension to all $s \in \mathbb{C}$. We are seeking precise information on the location and order of poles of such series.

Difficulties arise from considering non-trivial level. The meromorphic nature of an Eisenstein series can be resolved by computing its constant terms with respect to parabolic subgroups. However, as the level increases, so does the number of cusps. More importantly, the required integrations do not fit a clear pattern, even though local integrals at primes away from the level are classified. In this paper, we exploit a different approach.

Instead of working directly with the series $E(z, s; \Gamma)$, we introduce a family of series $E(z, s; \psi, \mathfrak{b})$. Here, $\mathfrak{b}$ is a "level" and ψ is a Hecke character of this conductor. In fact,

(5.a) each series of the form $E(z, s; \Gamma)$ is a finite sum of series $E(z, s; \psi, \mathfrak{b})|\tau$,

(5.b) for each choice of ψ and $\mathfrak{b}$, there is an explicit product $\Lambda(s; \psi, \mathfrak{b})$ of L-functions and Γ-factors such that $\Lambda(s; \psi, \mathfrak{b})\cdot E(z, s; \psi, \mathfrak{b})$ is entire unless ψ is trivial, in which case it may only have simple poles at assigned places.

To find $\Lambda(s; \psi, \mathfrak{b})$, we compute the *entire* Fourier expansion of $E(z, s; \psi, \mathfrak{b})$ with respect to one specific parabolic subgroup. The particular expansion happens to be simple to calculate. Terms indexed by smaller Bruhat cells, which appear in expansions for conjugate parabolics, vanish for the expansion in question. The set of poles of $E(z, s; \psi, \mathfrak{b})$ is the union of the sets of poles for the coefficient functions.

$E(z, s; \psi, \mathfrak{b})$ is constructed adelicly. Its precise definition is in §2, and the formula for $\Lambda(s; \psi, \mathfrak{b})$ is given in (4.27). Our main result is Theorem 4.2. Our method is a variation on the program developed by Shimura in **[11]** and **[12]**, and supplemented by this author in **[6]**, to study Eisenstein series for congruence subgroups on symplectic and special unitary groups.

Section 1 states our conventions for going from global to adelic notions of automorphicity. Lemma 1.1 extends **[12**; Lemma 1.4] to a broad class of algebraic groups. Section 2 introduces the series $E(z, s; \psi, \mathfrak{b})$ and proves a summation formula analogous to **[12**; Proposition 2.4]. Section 3 recalls work of Bengtsen **[2]** on confluent hypergeometric functions which arise as local integrals in our case. In §4, we compute Fourier coefficients and determine the relevant factor $\Lambda(s)$. When $\psi = 1$, ζ-functions appear in our formulas and additional simple poles occur.

1. Geometric conventions and a lemma. We begin with conventions on adelization. Let F be a number field. Denote the set of non-archimedean (or "finite") primes of F by $\mathfrak{f}$, and denote the remaining (or "finite") primes by ∞. For ν a prime of F, let F_ν denote a localization of F at ν, and let $|\ |_\nu$ be the normalized absolute value on F_ν. For $R \subseteq F$ a subring (usually the ring of integers of F) and $\wp \in \mathfrak{f}$, let $R_\wp$ denote the closure of R in $F_\wp$. Let $\mathbb{A}$ and $\mathbb{A}_\mathfrak{f}$ denote the rings of F-adeles and of finite F-adeles, respectively.

Let $R \subseteq F$ be the ring of integers of F, and let G be an algebraic group defined over R. For B a commutative R-algebra, denote the B-rational points of G by $G(B)$. We let "G" signify both the algebraic group and the group of "global" points $G(F)$. For $\nu \in \infty \cup \mathfrak{f}$, put $G_\nu = G(F_\nu)$. We also have topological groups

$$\begin{aligned} G_\infty &= \prod_{\nu \in \infty} G_\nu, \\ G_\mathfrak{f} &= G(\mathbb{A}_\mathfrak{f}), \\ G_\mathbb{A} &= G(\mathbb{A}). \end{aligned} \tag{1.1}$$

Identify $G_\mathbb{A}$ with $G_\infty \times G_\mathfrak{f}$. For $\alpha \in G_\mathbb{A}$, we let $\alpha_\wp, \alpha_\infty$ and $\alpha_\mathfrak{f}$ denote the projections of α to $G_\wp$, G_∞ and $G_\mathfrak{f}$, respectively.

Suppose Δ is a finite dimensional central simple F-algebra. There is an order of Δ which is a free R-module; fixing a choice of basis for such an order, we can represent $\mathrm{GL}_n(\Delta)$ for $n \in \mathbb{N}$ as the F-rational points of an algebraic group. We apply the above conventions to $\mathrm{GL}_n(\Delta)$ (as well as any algebraic subgroup of products of such matrix groups). The actual choice of order will not affect our work. For ν a prime of F, put $\Delta_\nu = \Delta \otimes_F F_\nu$; if $\nu \in \mathfrak{f}$, and S is an order of Δ, let $S_\nu = S \otimes_R R_\nu$ denote the closure of S in Δ_ν.

We quickly review the link between adelic and global approaches to automorphic forms.

Let G be a (discrete) group which acts on a topological space $\mathfrak{H}$ on the left. A factor of automorphy (with respect to the action) is a function $j\colon G \times \mathfrak{H} \to \mathbb{C}^*$ such that

(1.2.a) $j(\alpha, z)$ is continuous in the variable $z \in \mathfrak{H}$,

(1.2.b) $j(e, z) = 1$ for all $z \in \mathfrak{H}$ and e the identity of G, and

(1.2.c) $j(\alpha\beta, z) = j(\alpha, \beta \cdot z)j(\beta, z)$ for all $\alpha, \beta \in G$ and $z \in \mathfrak{H}$.

Let $\mathscr{F}$ be the set of functions from $\mathfrak{H}$ to $\mathbb{C}$. For $k \in \mathbb{Z}$, there is a right action by G on $\mathscr{F}$ given by

$$(1.3) \qquad (f|_k\alpha)(z) = f(\alpha \cdot z)j(\alpha, z)^{-k} \quad \text{for all } f \in \mathscr{F}, \ \alpha \in G, \ \text{and } z \in \mathfrak{H}.$$

We write $f|\alpha$ for $f|_k\alpha$ when the context is clear. If Γ is a subgroup of G and $f|\alpha = f$ for all $\alpha \in \Gamma$, then we say that f is Γ-invariant or that f is an automorphic form of weight k with respect to Γ. The space of Γ-invariant forms is denoted by $\mathscr{M}(\Gamma)$. If j is a factor of automorphy, then so is j^k; we only refer to "weight k" to emphasize the factor j rather than simply regard j^k as an abstract factor of automorphy.

Next, consider $\mathbb{G}$ a topological group which acts continuously on a space $\mathfrak{H}$, a fixed element of ι in $\mathfrak{H}$, and G and X subgroups of $\mathbb{G}$. Let C be the stabilizer of ι, and assume that

$$(1.4) \qquad \begin{gathered} \mathbb{G} \text{ acts transitively on } \mathfrak{H}, \\ X \text{ is a closed normal subgroup of } \mathbb{G}, \\ XC = \mathbb{G}, \text{ and} \\ XG \text{ is dense in } \mathbb{G}. \end{gathered}$$

The adelic situation arises as follows: Begin with an algebraic group G_0 over a field F. Put $\mathbb{G} = G_0(\mathbb{A})$, $X = G_\infty$, $G = G_0(F)$. Let K be a maximal compact subgroup of X, and put $\mathfrak{H} = X/K$ and $\iota = K \in \mathfrak{H}$. These choices satisfy (1.4) *provided* that G_0 has a Strong Approximation Property.

In the above context, let j be a continuous factor of automorphy $\mathbb{G} \times \mathfrak{H} \to \mathfrak{H}$. Let $\mathscr{U}$ be the set of open subgroups U of $\mathbb{G}$ which contain X and for which U/X is compact. Define a congruence subgroup of G to be any subgroup $\Gamma = G \cap U$ where $U \in \mathscr{U}$. For $U \in \mathscr{U}$, let $\mathscr{L}(U)$ be the set of continuous functions $F\colon \mathbb{G} \to \mathbb{C}$ such that

$$(1.5) \qquad F(gx\omega) = F(x)j(\omega, \iota)^{-1} \qquad \text{for all } g \in G,\ x \in \mathbb{G}, \text{ and } \omega \in C \cap U.$$

There is a bijective correspondence between $\mathscr{L}(U)$ and the set of automorphic functions with respect to $\Gamma = G \cap U$ determined by the property that

$$(1.6) \qquad \begin{gathered} F \in \mathscr{L}(U) \longleftrightarrow f \in \mathscr{M}(\Gamma) \\ F(x) = f(x \cdot \iota)j(x, \iota)^{-1} \quad \text{for all } x \in U. \end{gathered}$$

If Γ is a congruence subgroup, then the only $U \in \mathscr{U}$ for which $\Gamma = G \cap U$ is $U = \overline{X\Gamma}$.

Let $\mathscr{L} = \bigcup_{U \in \mathscr{U}} \mathscr{L}(U)$ and $\mathscr{M} = \bigcup_{U \in \mathscr{U}} \mathscr{M}(G \cap U)$. The identifications in (1.6) determine a bijection Θ between $\mathscr{L}$ and $\mathscr{M}$. There is a right action by G on $\mathscr{M}$ given by $|_1$, and this induces an action on $\mathscr{L}$. Suppose that $\mathbb{G} = X \times H$ for some group H; in the adelic case, put $H = G_0(\mathbb{A}_\mathrm{f})$. Then for $f \in \mathscr{M}$, $\alpha \in G$, and $F = \Theta(f)$, the function $\Theta(f|\alpha)$ is $x \mapsto F(x\alpha_H^{-1})$ where α_H is the projection of α into H.

The elementary but useful lemmas [**12**; Lemma 1.4] and [**12**; Proposition 2.4] can be generalized. Indeed, we now formulate a version which applies to most standard parabolics in a reductive group.

Define a norm-accessible tuple to be datum $(F, \{K_1, \dots, K_k\}, M, C, \{N_1, \dots, N_k\})$ where

(1.7.a) F is a number field and, for each j, K_j/F is a finite extension field,

(1.7.b) M is a reductive group defined over F,

(1.7.c) C is a compact open subset of M_f,

(1.7.d) for $1 \le j \le k$, $N_j \colon M \to K_j^*$ is a group homomorphism defined over F (where K_j^* is regarded as a torus over F), and the following properties are satisfied: Define $K^* = \prod_j K_j^*$ and let $N = N_1 \times \cdots \times N_m$ be the diagonal map into K^*. We require that

(1.8.a) $N(M(F)) = \prod_j N_j(M(F))$,

(1.8.b) $\mathrm{Ker}(N)$ is a semi-simple simply-connected algebraic group and $\mathrm{Ker}(N)_\infty$ is not compact,

(1.8.c) $N(C)$ is the largest compact-open subgroup of $(K^*)_\mathrm{f}$.

($(K^*)_\mathrm{f}$ is the product of idele groups of the K_j, and each of these has a unique maximal compact-open subgroup which consists of the product of unit groups of local integer rings.) Let I_j be the group of fractional K_j-ideals, and let L_j be the subgroup of principal ideals generated by elements in $N_j(M(F))$. Define $\mathrm{cl}(M) = \prod_j(I_j/L_j)$ and define $\theta\colon M_{\mathbb{A}} \to \mathrm{cl}(M)$ to be the product of the maps θ_j which, for each j, maps $x \in M_{\mathbb{A}}$ to the ideal determined by the finite part of $N_j(x)$.

LEMMA 1.1. *Let* $(F, \{K_1, \ldots, K_k\}, M, C, \{N_1, \ldots, N_k\})$ *be a norm-accessible tuple, and let* $\theta\colon M \to \mathrm{cl}(M)$ *be the function defined previously. Let* G *be a reductive group,* P *a parabolic subgroup and* C_0 *be a compact open subgroup of* $G_{\mathfrak{f}}$. *Assume*

(1.9.a) $G_{\mathfrak{f}} = P_{\mathfrak{f}}C_0$,

(1.9.b) $P = MU$ *where* U *is the unipotent radical and* M *is a Levi factor,*

(1.9.c) $M_{\mathfrak{f}} \cap C_0 = C$.

Then

(A) *For each* $1 \leq j \leq m$ *and each* $\wp$ *a finite prime of* K_j, *there is a function* $\varepsilon_{j,\wp}$ *on* $G_{\mathfrak{f}}$ *such that for* $m \in M_{\mathfrak{f}}$, $u \in U_{\mathfrak{f}}$ *and* $\omega \in C_0$, $\varepsilon_{j,\wp}(mu\omega) = |N_j(m_\wp)|_\wp$ *where* $|\ |_\wp$ *is the normalized valuation at* $\wp$.

(B) *There is a function* $\hat{\theta}\colon G_{\mathfrak{f}} \to \mathrm{cl}(M)$ *such that for* $m \in M_{\mathfrak{f}}$, $u \in U_{\mathfrak{f}}$ *and* $\omega \in C_0$, $\hat{\theta}(mu\omega) = \theta(m)$. *Moreover,* $\hat{\theta}$ *factors to an injection on* $P(F)\backslash G_{\mathfrak{f}}/C_0$.

Proof. The only non-triviality is the claim that the factored map of $\hat{\theta}$ to $P(F)\backslash G_{\mathfrak{f}}/C_0$ is injective. Let $x, y \in G_{\mathfrak{f}}$ so $\hat{\theta}(x) = \hat{\theta}(y)$, and we must show that $x \in P(F)yC_0$. By hypothesis (1.7.a) and definition of $\mathrm{cl}(M)$, there exists $m \in M(F)$ such that $\varepsilon_{j,\wp}(mx) = \varepsilon_{j,\wp}(y)$ for each j and each $\wp \in \mathfrak{f}$; thus, we may assume that $\varepsilon_{j,\wp}(x) = \varepsilon_{j,\wp}(y)$ for each j and $\wp$. Express

$$(1.10)\quad x = mu\omega \quad\text{and}\quad y = nv\tau,\qquad \text{where } m, n \in M_{\mathfrak{f}},\ u, v \in U_{\mathfrak{f}},\ \omega, \tau \in C_0.$$

For each j, N_j extends to an algebraic homomorphism $P \to K_j^*$ defined over F by $N_j(U) = \{1\}$. By assumption, there is $c \in C = M_{\mathfrak{f}} \cap C_0$ for which $N_j(xc) = N_j(n)$ for every j. Observe that $xc = (mc)(c^{-1}uc)(c^{-1}\omega c)$. Without changing double cosets, we may assume $x = mu$, $y = nv$ and $N_j(m) = N_j(n)$ for every j.

Let N be the homomorphism of (1.8). Now $xy^{-1} \in \mathrm{Ker}(N)_{\mathfrak{f}}U_{\mathfrak{f}}$ and $\mathrm{Ker}(N)$ and U are algebraic groups for which the global points are dense in the finite adelic points. The set yC_0y^{-1} is open, and so there exists $p \in P(F)$ and $\delta \in yC_0y^{-1}$ such that $xy^{-1} = p\delta$. Consequently, $x \in P(F)YC_0$. □

In [12], Lemma 1.4 is given for (among other cases) F a totally real field, $G = \mathrm{Sp}(n, F)$, P the subgroup of matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ where a, b and d are $n\times n$ blocks, M is the Levi factor of P and N is the homomorphism $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \to \det(d)$. Now suppose F is a number field, Δ a finite dimensional simple F-algebra, and $r_1, \dots, r_k \in \mathbb{N}$. Put $r = \sum_{j=1}^k r_j$, $G = \mathrm{SL}_r(\Delta)$ and let P be the subgroup of matrices of the form

$$\begin{pmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & a_k \end{pmatrix} \quad \text{where } a_j \in \mathrm{GL}_{rj}(\Delta) \text{ for each } j. \tag{1.11}$$

Let M be the subset of P of matrices whose over-diagonal blocks are all 0. For each $j \in \{1, \dots, k\}$, let N_j be the function on P which maps a matrix as given in (1.11) to $\det(a_j)$. If

$$\text{either } r_j > 1 \text{ for all } j \quad \text{or} \quad \Delta \text{ splits at each } \nu \in \infty, \tag{1.12}$$

then any subset of $k-1$ of the functions $\{N_j\}$ determines a norm-accessible tuple. In this paper, we only work in the latter case when $k = 2$.

2. Eisenstein series from maximal parabolic subgroups of SL_m. Fix a number field F, and let $\mathfrak{f}$ and ∞ be the sets of finite and infinite primes of F, respectively. Denote the ring of integers of F by R. Let Δ be a finite-dimensional central division F-algebra and let S be a choice of maximal order of Δ. Hereafter we use the conventions of §1. Assign to each $\nu \in \infty$ a ring isomorphism from Δ_ν with $M_a(\mathbb{R})$, $M_a(\mathbb{C})$ or $M_a(\mathbb{H})$, accordingly. We freely identify matrices over Δ_ν with matrices over the appropriate algebra $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

For $m, n \in \mathbb{N}$, let

$$G_{m,n} = \mathrm{SL}_{m+n}(\Delta), \tag{2.1}$$

where equality is both as an algebraic group and as F-rational points.

Whenever we express an a matrix in $G_{m,n}$ as $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, it is to be understood that a, b, c and d have dimensions $m\times m$, $m\times n$, $n\times m$

and $n \times n$, respectively. Also put

$$
\begin{aligned}
(2.2) \qquad P_{m,n} &= \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_{m,n} \right\}, \\
P^-_{m,n} &= \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in G_{m,n} \right\}, \\
U_{m,n} &= \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in G_{m,n} \right\}, \\
D_{m,n} &= \{(y_1, y_2) \in \mathrm{GL}_m(\Delta) \times \mathrm{GL}_n(\Delta) \colon dt(y_1)dt(y_2) = 1\},
\end{aligned}
$$

where dt is the reduced norm function. When dealing with $y \in D_{m,n}$, we use the notation $y = (y_1, y_2)$. Also define algebraic functions

$$
\begin{aligned}
(2.3) \qquad & N \colon D_{m,n} \to \mathrm{GL}_1(F) && \text{by } (y_1, y_2) \mapsto dt(y_2), \\
& \omega \colon D_{m,n} \to G_{m,n} && \text{by } (y_1, y_2) \mapsto \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}, \\
& \tau \colon M_{n,m}(\Delta) \to G_{m,n} && \text{by } x \mapsto \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}.
\end{aligned}
$$

Extend N to characters on $P_{m,n}$ and $P^-_{m,n}$ by setting it to be trivial on unipotent matrices.

For the rest of this section, we fix a choice of $m, n \in \mathbb{N}$ and omit subscripts. We make the restriction that

(2.4) *Standing Hypothesis*: If there is $\nu \in \infty$ so $\Delta_\nu \approx \mathbb{H}$, then neither m nor n is 1.

For $\nu \in \infty \cup \mathfrak{f}$, let

$$
\begin{aligned}
(2.5) \qquad C_\nu &= \{x \in M_{m+n}(\Delta_\nu) \colon x \cdot {}^t x^\rho = 1\} && \text{if } \nu \in \infty, \\
C_\nu &= \mathrm{SL}_{m+n}(S_\nu) && \text{if } \nu \in \mathfrak{f},
\end{aligned}
$$

where ρ is the involution $\mathrm{id}_{\mathbb{R}}$, complex conjugation or the main involution of $\mathbb{H}$, if Δ_ν is a matrix over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, respectively. Put $C_\infty = \prod_{\nu \in \infty} C_\nu$ and $C_{\mathfrak{f}} = \prod_{\wp \in \mathfrak{f}} C_\wp$. Put $\mathfrak{H} = C_\infty / G_\infty$, $\imath = C_\infty$ in $\mathfrak{H}$, and let $G_{\mathbb{A}}$ act on $\mathfrak{H}$ by $\alpha \cdot z C_\infty = (\alpha_\infty z) C_\infty$.

For an R-ideal $\mathfrak{b}$, define for each $\wp \in \mathfrak{f}$

$$
\begin{aligned}
(2.6) \qquad U_0(\mathfrak{b})_\wp &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_{m+n}(S_\wp) \colon c \equiv 0 \bmod (\mathfrak{b} S_\wp) \right\}, \\
U_u(\mathfrak{b})_\wp &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U_0(\mathfrak{b})_\wp \colon dt(d) \equiv 1 \bmod (\mathfrak{b} R_\wp) \right\}, \\
U(\mathfrak{b})_\wp &= \{\alpha \in \mathrm{SL}_{m+n}(S_\wp) \colon \alpha \equiv 1_{m+n} \bmod (\mathfrak{b} S_\wp)\}.
\end{aligned}
$$

Put $U_0(\mathfrak{b}) = G_\infty \times \prod_{\wp\in\mathfrak{f}} U_0(\mathfrak{b})_\wp$, $U_u(\mathfrak{b}) = G_\infty \times \prod_{\wp\in\mathfrak{f}} U_u(\mathfrak{b})_\wp$ and $U(\mathfrak{b}) = G_\infty \times \prod_{\wp\in\mathscr{F}} U(\mathfrak{b})_\wp$, and put $\Gamma_0(\mathfrak{b}) = G \cap U_0(\mathfrak{b})$, $\Gamma_u(\mathfrak{b}) = G \cap U_u(\mathfrak{b})$, and $\Gamma(\mathfrak{b}) = G \cap U(\mathfrak{b})$. A subgroup $\Gamma \subseteq G$ is a congruence subgroup (in the sense of §1) if and only if there exists an ideal $\mathfrak{b}$ such that $\Gamma(\mathfrak{b})$ is a subgroup of finite index in Γ.

We have the elementary

LEMMA 2.1. *Let $\mathfrak{b}$ be an integral ideal of F. Then the injection map induces an isomorphism*

$$(P \cap \Gamma(\mathfrak{b}))\backslash\Gamma(\mathfrak{b}) \approx (P \cap \Gamma_u(\mathfrak{b}))\backslash\Gamma_u(\mathfrak{b}). \tag{2.7}$$

Proof. It suffices to show that $(P \cap \Gamma_u(\mathfrak{b}))\alpha \cap \Gamma(\mathfrak{b}) \neq \varnothing$ for a given $\alpha = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_u(\mathfrak{b})$. There is an element $\hat{d} \in \mathrm{SL}_n(\Delta_{\mathbb{A}})$ such that for each $\wp \in \mathfrak{f}$, $\hat{d}_\wp \in \mathrm{SL}_n(S_\wp)$ and $\hat{d}_\wp \equiv d_\wp \bmod (\mathfrak{b}S_\wp)$. By Strong Approximation, there is $d' \in \mathrm{SL}_n(S)$ so that $d'_\wp \equiv \hat{d}_\wp \bmod (\mathfrak{b}S_\wp)$. Replacing α with $\omega(a', d')^{-1}\alpha$, we may assume that a and d are congruent to identity matrices $\bmod (\mathfrak{b})$. But now $\left[\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix}\right]\alpha \in \Gamma(\mathfrak{b})$. □

It is routinely verified that G, P, D, $C_\mathfrak{f}$ and N satisfy the hypotheses of Lemma 1.1. For $\nu \in \infty \cup \mathfrak{f}$, define ε_ν on G_ν by the condition that $\varepsilon_\nu(y\omega) = |N(y)|_\nu$ for $y \in P_\nu$ and $\omega \in C_\nu$. For $\alpha \in G_\mathbb{A}$ and $z \in \mathfrak{H}$, put

$$\begin{aligned}
(2.8)\quad \varepsilon(\alpha) &= \prod_{\nu\in\infty\cup\mathfrak{f}} \varepsilon_\nu(\alpha), \\
Y(z) &= \prod_{\nu\in\infty} \varepsilon_\nu(\alpha_0)^{-1} \quad \text{for each } \alpha_0 \in G_\mathbb{A} \text{ such that } \alpha_0 \cdot \imath = z, \\
J(\alpha, z) &= Y(z)/Y(\alpha \cdot z).
\end{aligned}$$

Note that $J(\alpha, z) = \varepsilon(\alpha)$ if $\alpha \in P_\infty$. For Γ a congruence subgroup of G define

$$\begin{aligned}
E(z, s; \Gamma) &= Y(z)^s \sum_{\alpha\in(P\cap\Gamma)\backslash\Gamma} J(\alpha, z)^{-s} \\
&= \sum_{\alpha\in(P\cap\Gamma)\backslash\Gamma} Y(\alpha \cdot z)^s,
\end{aligned} \tag{2.9}$$

where $z \in \mathfrak{H}$ and $s \in \mathbb{C}$. More precisely, the summation on the right is known to converge absolutely and uniformly on compact subsets of $\mathfrak{H} \times \mathbb{C}$ on which $\mathrm{Re}(s)$ is sufficiently large; the corresponding function has a meromorphic continuation to all of $\mathfrak{H} \times \mathbb{C}$. When dealing with such sums, we prove results by performing formal manipulations which are valid where convergence is absolute and uniform, and then deduce our claims from uniqueness of meromorphic continuation.

Clearly $E(z, s; \Gamma)$ is automorphic in z with respect to the Γ and the trivial factor of automorphy $j(\alpha, z) = 1$. Using the conventions of §1 with $\mathbb{G} = G_{\mathbb{A}}$ acting on $\mathfrak{H}$ and $X = G_\infty$, we establish a bijection from automorphic forms on $\mathfrak{H}$ with respect to subgroups of G with automorphic forms on $G_{\mathbb{A}}$. Let $E^*(x, s; \Gamma)$ denote the adelic form of $E(z, s; \Gamma)$.

Next, we define a family of adelic functions. For a Hecke character on ψ on $(F^*)_{\mathbb{A}}$, we denote by $\psi_\wp$, ψ_∞, and $\psi_{\mathfrak{f}}$ the restrictions of ψ to subgroups $F^*_\wp$, F^*_∞ and $F^*_{\mathfrak{f}}$, respectively. For an ideal $\mathfrak{b}$, define

(2.10) $\mathscr{X}(\mathfrak{b})$ is the group of Hecke characters ψ such that

(2.10.a) the conductor of ψ divides $\mathfrak{b}$,

(2.10.b) $\{\psi_\infty \circ dt\}(\Delta^*_\infty) = \{1\}$.

For each $\psi \in \mathscr{X}(\mathfrak{b})$, define $\varepsilon_\psi = \varepsilon_{\psi, \mathfrak{b}}$ on $G_{\mathbb{A}} \times \mathbb{C}$ by

$$(2.11)\quad \varepsilon_\psi(y\omega, s) = \varepsilon(x)^{-s}\psi^{-1}(Ny)\prod_{p|\mathfrak{b}}\psi_\wp^{-1}(dt(d_\wp))$$
$$\text{for } y \in P_{\mathbb{A}} \text{ and } \omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U_0(\mathfrak{b}) \cap C,$$
$$\varepsilon_\psi(x, s) = 0 \quad \text{for } x \notin P_{\mathbb{A}} \cdot U_0(\mathfrak{b}).$$

If $p \in P_F$, $x \in G_{\mathbb{A}}$ and $s \in \mathbb{C}$, then $\varepsilon_\psi(px, s) = \varepsilon_\psi(x, s)$. We define

$$(2.12)\qquad E^*(x, s; \psi, \mathfrak{b}) = \sum_{\alpha \in P\backslash G} \varepsilon_{\psi, \mathfrak{b}}(\alpha x, s).$$

Denote the restriction of $\varepsilon_{\psi, \mathfrak{b}}$ to $G_\wp \times \mathbb{C}$ by $\varepsilon_{\psi, \mathfrak{b}, \wp}$, and then

$$(2.13)\qquad \varepsilon_{\psi, \mathfrak{b}}(x, s) = \prod_{\wp \in \infty \cup \mathfrak{f}} \varepsilon_{\psi, \mathfrak{b}, \wp}(x_\wp, s).$$

When meaning is clear from context, we omit the subscripts. By the theorem of Langlands [**9**] (see also the formulation by Arthur [**1**]), the sum (2.12) converges for $\mathrm{Re}(s)$ sufficiently large and the function E^* has a meromorphic continuation to $G_{\mathbb{A}} \times \mathbb{C}$. Let $E(z, s; \psi, \mathfrak{b})$ denote the form on $\mathfrak{H}$ corresponding to $E^*(x, s; \psi, \mathfrak{b})$. Each function $E(z, s; \Gamma)$ is a finite sum of terms $E(z, s; \psi, \mathfrak{b})|\tau$. In fact

THEOREM 2.2 (*Context of* §1). *Let* F, Δ, m, n *etc., be given. Let* Γ *be a congruence subgroup of* G.

(A) *If* $\Gamma = \Gamma(\mathfrak{b})$ *for some ideal* $\mathfrak{b}$, *then*

$$(2.14)\qquad |\mathscr{X}|E(z, s; \Gamma) = \sum_{\psi \in \mathscr{X}} E(z, s; \psi, \mathfrak{b}),$$

where $\mathscr{X} = \mathscr{X}(\mathfrak{b})$.

(B) *If* $\Gamma' \subseteq \Gamma$ *is another congruence subgroup, then*

$$(2.15) \qquad r \cdot E(z, s; \Gamma) = \sum_{\tau \in \Gamma' \backslash \Gamma} E(z, s; \Gamma')|\tau,$$

where $r = [P \cap \Gamma : P \cap \Gamma']$.

Proof. It suffices to prove the equations formally. Statement (B) follows trivially from reordering sums, and we omit proof. Put $\mathscr{X} = \mathscr{X}(\mathfrak{b})$. For $\wp \in \mathfrak{f}$, let $U_\wp$ be the unit group of $R_\wp$.

Let $y \in P_{\mathbb{A}}$ and $\omega = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in U_0(\mathfrak{b}) \cap C$, and fix $x = y\omega$. The value $\varepsilon_\psi(x, s)$ is a product of a term dependent only on s and a factor which is multiplicative with respect to ψ in the group $\mathscr{X}$. By elementary character theory, $\sum_{\psi \in \mathscr{X}} \varepsilon_\psi(x, s)$ is 0 unless $(Ny)\prod_{\wp|\mathfrak{b}} dt(d_\wp)$ lies in the subgroup of $F_{\mathbb{A}}^*$ which is generated by $dt(\Delta_\infty^*)$, F, and

$$(2.16) \qquad V_{\mathfrak{b}} = \{\alpha \in F_{\mathfrak{f}}^* : \alpha_\wp \in (1 + \mathfrak{b}R_\wp) \cap U_\wp \text{ for each } \wp \in \mathfrak{f}\}.$$

Suppose the sum is non-zero. Now $dt(d)$ is a $\wp$-adic unit at each $\wp|\mathfrak{b}$. Thus, we may express $Ny = abc$ for $a \in dt(\Delta_\infty^*)$, $b \in F$ and $c \in \prod_{\wp \in \mathfrak{f}} U_\wp$. It follows that $b_\infty \in dt(\Delta_\infty^*)$. By Eichler's Theorem on norms, b must be a global norm. By Lemma 1.1, there is a global $g \in P_F$ for which $gx \in P_\infty C$. Rewrite $gx = y_1\omega_1$ where now $y_1 \in P_\infty$ and $\omega_1 \in C$, and let d_1 be the corresponding submatrix of ω_1. Then $\sum_{\psi \in \mathscr{X}} \varepsilon_\psi(gx, s) = \sum_{\psi \in \mathscr{X}} \varepsilon_\psi(x, s)$, so $(\prod_{\wp|\mathfrak{b}} dt(d_1)_\wp)$ is in $dt(\Delta_\infty^*) \cdot F \cdot U_{\mathfrak{b}}$. Consequently, there is another global parabolic h such that $hx \in U_u(\mathfrak{b})$. We may now conclude that

(2.17) for $x \in U_u(\mathfrak{b})$ and $\alpha \in G$, $\sum_{\psi \in \mathscr{X}} \varepsilon_\psi(\alpha x, s) \neq 0$ only if $\alpha \in P_F \cdot \Gamma_u(\mathfrak{b})$.

Thus, for $x \in U_u(\mathfrak{b})$,

$$(2.18) \qquad \sum_{\psi \in \mathscr{X}} E^*(x, s; \psi, \mathfrak{b}) = \sum_{\alpha \in P \backslash P \cdot \Gamma_u(\mathfrak{b})} |\mathscr{X}| \cdot \varepsilon(\alpha x, s).$$

By Lemma 2.1, injection induces an isomorphism $(P \cap \Gamma(\mathfrak{b})) \backslash \Gamma(\mathfrak{b}) \to P \backslash P \cdot \Gamma_u(\mathfrak{b})$. Moreover, if $\alpha \in \Gamma(\mathfrak{b})$ and $x \in U_u(\mathfrak{b})$, then $\varepsilon(\alpha x, s) = Y(\alpha x, s)$. The right-hand side becomes the adelic version of $|\mathscr{X}| \cdot E(z, s; \Gamma(\mathfrak{b}))$. □

Another useful characterization is

LEMMA 2.3 (*Context of* §1). *Let* F, Δ, m, n *etc. be given. Let* $\mathfrak{b} \neq R$ *be a proper integral ideal and let* $\psi \in \mathscr{X}(\mathfrak{b})$. *Then*

$$E^*(x, s; \psi, \mathfrak{b}) = \sum_{\alpha \in U} \varepsilon_{\psi, \mathfrak{b}}(\alpha x, s), \qquad \text{for } x \in P_{\mathbb{A}}^-. \tag{2.19}$$

Proof. Choose $\wp \in \mathfrak{f}$ such that $\wp | \mathfrak{b}$, and suppose that $\alpha \in G$ such that $\varepsilon(\alpha x, s) \neq 0$ (for $\varepsilon = \varepsilon_{\psi, \mathfrak{b}}$). Express $\alpha x = p\omega$ for $p \in P_{\mathbb{A}}$ and $\omega \in U_0(\mathfrak{b})$. Then $\alpha = p\omega x^{-1}$ where $(\omega x^{-1})_\wp = \left[\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right]$ for $d \in \mathrm{GL}_n(\Delta_\wp)$. Consequently,

$$\alpha_\wp = \begin{bmatrix} * & * \\ * & e \end{bmatrix} \qquad \text{for } e \in \mathrm{GL}_n(\Delta_\wp). \tag{2.20}$$

Simple matrix manipulation shows that $\alpha \in PU$. It is easy to check that U is a complete and irredundant list of representatives for $P \backslash PU$. □

3. Enter the Bessel functions. Our conventions here are different from those of other sections. Let $\Delta = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If T is a square matrix in Δ, let $|T|$ be the standard real absolute value of the reduced norm of T. (In other sections, we use the square of this norm for $\Delta = \mathbb{C}$ or $\mathbb{H}$.) Let ρ denote $1_{\mathbb{R}}$, conjugation or the main involution of $\mathbb{H}$ on $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively. For T a matrix in Δ, put $T^* = {}^t(T\rho) = ({}^tT)^\rho$; if T is invertible, use the notation $T^{-*} = (T^{-1})^* = (T^*)^{-1}$.

Fix $\imath = [\Delta : F]$. Suppose $m, n \in \mathbb{N}$ and T is an $n \times n$ matrix. The function $U \mapsto UT$ on $A = M_{m,n}(\Delta)$ changes the Haar measure of A by a factor of $|T|^{\imath n}$; there is an analogous factor for maps of the form $U \mapsto TU$. For $m \in \mathbb{N}$, put

$$U(m) = \{T \in M_m(\Delta) \colon TT^* = 1_m\}. \tag{3.1}$$

We remark

LEMMA 3.1. *Let* $m, n \in \mathbb{N}$ *and* $h \in M_{m,n}(\Delta) - \{0\}$. *Then there exists* $a \in U(m)$ *and* $b \in U(n)$ *so* $ahb = \left[\begin{smallmatrix} h_0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, *where* h_0 *is a non-degenerate square matrix.*

Proof. Trivial. □

Let $m, n \in \mathbb{N}$. Let $G_{m,n}$, $P_{m,n}$, N, etc. be as defined in §2 with respect to the "local" algebra Δ. Define $\varepsilon = \varepsilon_{m,n}$ on $\mathrm{GL}_{n+m}(\Delta)$ by the condition

$$(3.2) \qquad \varepsilon(y\omega) = |Nd| \quad \text{for } y = \begin{bmatrix} * & * \\ 0 & d \end{bmatrix} \quad \text{and} \quad \omega \in U(n+m).$$

Then

$$(3.3) \quad \varepsilon\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right) = |1 + xx^*|^{1/2} = |1 + x^*x|^{1/2} \quad \text{for } x \in M_{m,n}(\Delta).$$

For $s \in \mathbb{C}$, $A \in \mathrm{GL}_m(\Delta)$, $B \in \mathrm{GL}_n(\Delta)$ and $h \in M_{m,n}(\Delta)$, define a formal integral

$$(3.4) \quad k(h, m, n, A, B, s) = \int_{x \in M_{m,n}(\Delta)} \varepsilon\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)^{-s})e^{2\pi i \operatorname{tr}(hx)}\, dx,$$

and put $k(h, m, n, s) = k(h, m, n, 1_m, 1_n, s)$. Now

$$(3.5) \qquad \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 1 & 0 \\ B^{-1}xA & 1 \end{bmatrix}.$$

Simple coordinate change implies that

$$(3.6) \quad \begin{aligned} &k(h, m, n, A, B, s) = |B|^{-s+m\iota}|A|^{-n\iota}k(A^{-1}hB, m, n, s), \\ &k(\alpha h \beta, m, n, s) = k(h, m, n, s) \\ &\qquad\qquad\qquad \text{for } \alpha \in U(m) \text{ and } \beta \in U(n) \end{aligned}$$

in the sense that if the integral on either side exists, then both sides are defined and equal.

LEMMA 3.2. *Let $r, m, n \in \mathbb{N}$, $h_0 \in M_r(\Delta)$ and $s \in \mathbb{C}$. Assume that $r \leq m, n$, and let*

$$(3.7) \qquad h = \begin{bmatrix} h_0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{m,n}(\Delta).$$

Then

$$(3.8) \quad \begin{aligned} k(h, m, n, s) = {} & k(0, m, n-r, s) \\ & \cdot k(0, m-r, r, s-(n-r)\iota) \\ & \cdot k(h_0, r, r, s-(m+n-2r)\iota), \end{aligned}$$

in the sense that if the integrals on either side exist, then both sides are defined and equal.

Proof. We play the change of variables game. For $x \in M_{n,m}(\Delta)$, express $x = \left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right]$ where $y \in M_{r,m}(\Delta)$ and $z \in M_{n-r,m}(\Delta)$. Then

$$(3.9) \quad k(h, m, n, s) = \int_y \int_z |1 + y^*y + z^*z|^{-s/2} e^{2\pi i \operatorname{tr}(\left[\begin{smallmatrix} h_0 \\ 0 \end{smallmatrix}\right] y)}\, dz\, dy.$$

For each y, let $t \in M_m(\Delta)$ such that $t^*t = 1 + y^*y$, and substitute variables $z = z_0 t$ to get

$$\begin{aligned}(3.10)\quad & k(h, m, n, s)\\ &= k(0, m, n-r, s) \int_y |1 + y^*y|^{(\imath(n-r)-s)/2} e^{2\pi i \operatorname{tr}([{}^{h_0}_{0}]y)}\, dy\\ &= k(0, m, n-r, s) k\left(\begin{bmatrix} h_0 \\ 0 \end{bmatrix}, m, r, s - \imath(n-r)\right).\end{aligned}$$

Now decompose $y = (bc)$ where $b \in M_r(\Delta)$ and $c \in M_{r,m-r}(\Delta)$.

$$\begin{aligned}(3.11)\quad & K\left(\begin{bmatrix} h_0 \\ 0 \end{bmatrix}, m, r, s - \imath(n-r)\right)\\ &= \int_b \int_c |1 + bb^* + cc^*|^{(\imath(n-r)-s)/2} e^{2\pi i \operatorname{tr}(h_0 b)}\, dc\, db.\end{aligned}$$

For each b, choose $u \in M_r(\Delta)$ so that $uu^* = 1 + bb^*$ and replace $c = uc_0$ to get

$$\begin{aligned}(3.12)\quad & k\left(\begin{bmatrix} h_0 \\ 0 \end{bmatrix}, m, r, s - \imath(n-r)\right)\\ &= \int_b \int_c |1 + cc^*|^{(\imath(n-r)-s)/2} |1 + bb^*|^{(\imath(m+n-2r)-s)/2}\\ &\qquad\qquad \cdot e^{2\pi i \operatorname{tr}(h_0 b)}\, dc\, db\\ &= k(0, m-r, r, s - \imath(n-r))\\ &\quad \cdot k(h_0, r, r,)s - \imath(m+n-2r)).\end{aligned}$$

□

For $m \in \mathbb{N}$, define a meromorphic function on $s \in \mathbb{C}$ by

$$(3.13)\qquad \Gamma_m(s) = \prod_{j=0}^{m-1} \Gamma\left(s - \frac{j\imath}{2}\right)$$

where Γ without subscript is the usual gamma function. Adopt the convention that $\Gamma_0(s) = 1$. From the literature,

Theorem 3.3. *Let $m, n \in \mathbb{N}$, $s \in \mathbb{C}$, $A \in \mathrm{GL}_m(\Delta)$, $B \in \mathrm{GL}_n(\Delta)$ and $h \in M_{m,n}(\Delta)$. Then there is a bound b such that the integral for $k(h, m, n, A, B, s)$ converges for all s with $\mathrm{Re}(s) \geq b$, and the function has a meromorphic continuation to all $\mathbb{C}$. We let the notations $k(h, m, n, A, B, s)$ and $k(h, m, n, s)$ denote the continuations, and then (3.6) and (3.8) are true in the sense of equality of meromorphic functions. Moreover, for $r = \mathrm{rank}_\Delta(h)$, the product*

$\gamma(s)k(h, m, n, A, B, s)$ is entire for

$$\gamma(s)=\frac{\Gamma_m\left(\frac{s}{2}\right)}{\Gamma_{m-r}\left(\frac{s-n\iota}{2}\right)}=\frac{\Gamma_n\left(\frac{s}{2}\right)}{\Gamma_{n-r}\left(\frac{s-m\iota}{2}\right)}, \tag{3.14}$$

and $\gamma(s)k(0, m, n, s)$ is a non-zero constant.

Proof. In the case $\Delta=\mathbb{R}$, the functions $k(h, m, n, s)$ are special examples of the "K-Bessel" functions studied by Bengtson [**2**] and Terras [**14**]. When $h=0$ or h is invertible, the lemma can be verified using [**14**; §4.2.2, Theorem 2]. The previous lemmas show how the calculation can be reduced to these two extreme cases. The proofs in [**2**] and [**14**] easily generalize to the cases $\Delta=\mathbb{C}$ and $\mathbb{H}$, although some of the relevant constants must be changed. □

4. Integration over the "Big Cell". We begin by fixing some additive characters. For $\nu \in \infty \cup \mathfrak{f}$ and $x \in F_\nu$, define

$$\begin{aligned} (4.1)\quad \chi_\nu(x) &= e^{2\pi i\, \mathrm{tr}_{F_\nu/\mathbb{R}}(x)} && \text{if } \nu \in \infty, \\ \chi_\nu(x) &= e^{2\pi i r} && \text{if } \nu \text{ divides the rational prime } p \text{ and} \\ & && r \in \mathbb{Z}[1/p] \text{such that } r+\mathrm{tr}_{F_\nu/\mathbb{Q}_p}(x) \in \mathbb{Z}_p. \end{aligned}$$

Define χ on $F_{\mathbb{A}}$ to be the product of the local characters of (4.1). For $m \in \mathbb{N}$, and B a finite dimensional central simple F-algebra, extend χ to $M_m(B_{\mathbb{A}})$ (respectively, χ_ν to $M_m(B_\nu)$ for ν a prime) by composing the character above with the reduced trace function. We freely denote any and all of these characters by χ.

Let $m, n, \in \mathbb{N}$. Set

$$Q(m, n)=M_{m,n}(\Delta_{\mathbb{A}})/M_{m,n}(\Delta), \tag{4.2}$$

where the quotient is taken with respect to addition. For $\widehat{\Delta}$ equal either $\Delta_{\mathbb{A}}$ or Δ_ν for a prime ν, the pairing

$$\begin{aligned} M_{m,n}(\widehat{\Delta}) \times M_{n,m}(\widehat{\Delta}) &\longrightarrow T\ (\subseteq \mathbb{C}^*) \\ (U, T) &\longmapsto \chi(UT) \end{aligned} \tag{4.3}$$

induces a canonical identification of $M_{n,m}(\widehat{\Delta})$ with the character group of $M_{m,n}(\widehat{\Delta})$. Also, for $\widehat{\Delta}=\Delta_{\mathbb{A}}$ the pairing also induces an identification of $M_{n,m}(\widehat{\Delta})$ with the character group of $M_{m,n}(\widehat{\Delta})$. Also, for $\widehat{\Delta}=\Delta_{\mathbb{A}}$ the pairing also induces an identification of $M_{n,m}(\Delta) \subseteq M_{n,m}(\Delta_{\mathbb{A}})$ with the character group of $Q(m, n)$.

For $\nu \in \infty \cup \mathfrak{f}$, let $\mu=\mu_\nu$ be the Haar measure on Δ_ν which is self-dual with respect to the identification of Δ_ν with its character group. Then $\mu=\mu_{\mathbb{A}}=\prod_{\nu \in \infty \cup \mathfrak{f}} \mu_\nu$ is self-dual on $\Delta_{\mathbb{A}}$. Define μ on

$M_{m,n}(\widehat{\Delta})$ to be the product of the coordinate measures. Also let μ denote the measure induced on $Q(m, n)$ (this is the Haar measure such that $\mu(Q(m, n)) = 1$).

Suppose f is a function on $G_{m,n;\mathbb{A}}$ which is automorphic with respect to the trivial factor of automorphy. The function

$$(x, y) \mapsto f(\tau(x)\omega(y)) \tag{4.4}$$

on $Q(n, m) \times D_\infty$ uniquely determine the original f. Now (4.4) has a "Fourier expansion" with respect to the variable x where the coefficients are functions in y. We are interested in the case $f = E^*(g, s)$ (for s fixed) is an Eisenstein series of §2, and our objective is to find a meromorphic factor $\Lambda(s)$ such that $\Lambda(s)E^*(g, s)$ has only simple poles at known positions. It suffices to find a meromorphic continuation for each Fourier coefficient of $E^*(g, s)$ and to compute a $\Lambda(s)$ which controls the poles of each continuation. Since each coefficient is an integral over a compact space, each coefficient has an analytic continuation to $\mathbb{C}$ with isolated singularities. Conversely, if $s_0 \in \mathbb{C}$ and $n \in \mathbb{Z}$ so $(s - s_0)^n \cdot E^*(g, s)$ is finite and non-zero at s_0, then $(s - s_0)^n$ times each coefficient is finite and at least one such integral is non-zero. Thus, the continuation is meromorphic and the poles of the coefficients are the poles of the series.

Let $\mathfrak{b}$ be an integral R-ideal and let $\psi \in \mathscr{X}(\mathfrak{b})$. Put $\varepsilon = \varepsilon_{\psi,\mathfrak{b}}$, and

$$\sigma = [\Delta : F]^{1/2}. \tag{4.5}$$

For $\wp \in \infty \cup \mathfrak{f}$, express $\Delta_\wp = M_a(\Delta_0)$ where Δ_0 is a division $F_\wp$-algebra, and define $\sigma_\wp = [\Delta_0 : F_\wp]^{1/2}$. For the rest of the section, we assume Standing Hypothesis (2.4) *and* that $\mathfrak{b} \neq R$. We have the simplified formula

$$E(\tau(x)\omega(y), s; \psi, \mathfrak{b}) = \sum_{\alpha \in U} \varepsilon_{\psi,\mathfrak{b}}(\alpha\tau(x)\omega(y), s) \tag{4.6}$$

for $(x, y) \in Q(n, m) \times D_\infty$.

Let $h \in M_{m,n}(\Delta)$ and denote the hth Fourier coefficient of $E^*(z, s; \psi, \mathfrak{b})$ by $c(h, y, s)$ for $y \in D_{m,n}(\Delta_\infty)$. The function ε is unaffected if the argument is multiplied on the right by an adelic matrix $\tau(c) \in U(\mathfrak{b}) \cap G_\mathfrak{f}$; it follows that the integral vanishes unless h is contained in

$$\begin{aligned} &\widehat{L}(m, n; \mathfrak{b}) \\ &\quad = \left\{ h \in M_{m,n}(\Delta) \colon \chi\left(h \cdot \prod_{\wp \in \mathfrak{f}} \mathfrak{b} M_{n,m}(S_\wp) \right) = \{1\} \right\}. \end{aligned} \tag{4.7}$$

For $\wp \in \mathfrak{f}$, define

$$(4.8) \qquad \widehat{L}(m, n; \mathfrak{b}, \wp) = \{h \in M_{m,n}(\Delta) : \chi_{\wp}(h \cdot \mathfrak{b} M_{n,m}(S_{\mathfrak{p}})) = \{1\}\},$$

the local version of (4.7)

For $h \in \widehat{L}(m, n; \mathfrak{b})$,

$$(4.9) \quad \begin{aligned} c(h, y, s) &= \int_{x \in Q(n,m)} \sum_{\alpha \in U} \varepsilon(\alpha\tau(x)\omega(y), s)\chi(-hx)\, d\mu(x) \\ &= \int_{x \in Q(n,m)} \sum_{u \in M_{n,m}(\Delta)} \varepsilon(\tau(x+u)\omega(y), s)\chi(-hx)\, d\mu(x) \\ &= \int_{x \in M_{n,m}(\Delta_{\mathbb{A}})} \varepsilon(\tau(x)\omega(y), s)\chi(-hx)\, d\mu(x) \\ &= \left\{ \prod_{\nu \in \infty} \int_{x \in M_{n,m}(\Delta_\nu)} \varepsilon_\nu(\tau(x)\omega(y_\nu), s)\chi_\nu(-h_\nu x)\, d\mu_\nu(x) \right\} \\ &\quad \times \left\{ \prod_{\wp \in \mathfrak{f}} \int_{x \in M_{n,m}(\Delta_\wp)} \varepsilon_\wp(\tau(x), s)\chi_\wp(-h_\wp x)\, d\mu_\wp(x) \right\}, \end{aligned}$$

where $h_\wp$ denotes h regarded as a matrix over $F_\wp$ for $\wp$ a prime.

The local integrals are of three types:

Infinite Primes: The integral at $\nu \in \infty$ is a K-Bessel function of the type discussed in §3. It has a meromorphic continuation which is an entire function times

$$(4.10) \qquad \frac{\Gamma_{(v-r)a,\Delta_\nu}\left(\frac{\kappa s - ua\iota}{2}\right)}{\Gamma_{va,\Delta_\nu}\left(\frac{\kappa s}{2}\right)},$$

where $a = \sigma/\sigma_\nu$, $v = \min\{m, n\}$, $u = \max\{m, n\}$, and $\kappa = 1$ if $\Delta_\nu \approx M_\sigma(\mathbb{R})$ and $\kappa = 2$ otherwise .

Finite Prime Divisors of $\mathfrak{b}$. Let $\wp$ be a finite prime which divides $\mathfrak{b}$. Then $\varepsilon(\tau(x), s) = 0$ unless $x \in \mathfrak{b}M_{n,m}(S_\wp)$. If $x \in \mathfrak{b}M_{n,m}(S_\wp)$, then $\varepsilon(\tau(x), s) = 1$. Thus, the integral is $\mu_\wp(\mathfrak{b}M_{n,m}(S_\wp))$.

Finite Primes which do not divide $\mathfrak{b}$: We refer to previous articles.

Let $\wp$ be a finite prime which does not divide $\mathfrak{b}$. Let $q = |R/\wp|$ and $a = \sigma/\sigma_\wp$. Define l on $M_{n,m}(\Delta_\wp)$ by $q^{-l(x)s} = \varepsilon_\wp(\tau(x), s)$, and then l factors $\mod(M_{n,m}(S_\wp))$. The local integral becomes

$$(4.11) \qquad \mu(S_\wp)^{m,n} \sum_{x \in M_{n,m}(\Delta_\wp)/M_{n,m}(S_\wp)} \psi^{l(x)} q^{-l(x)s}\chi(-hx),$$

where ψ in (4.11) is the value of the character $\psi_\wp$ on any generator of $\wp$. The expression in (4.11) is a non-zero constant times $\alpha(h, \psi q^{-s})$ where

(4.12) $$\alpha(h, t) = \sum_{x \in M_{n,m}(\Delta_\wp)/M_{n,m}(S_\wp)} t^{l(x)} \chi(-hx),$$

is a power series in an indeterminate t.

The series (4.12) were evaluated in [7; Theorem 2.1]. Identify $\Delta_\wp \approx M_a(\Delta_0)$ for Δ_0 a local division algebra in such a manner that $S_\wp$ is identified with $M_a(S_0)$ for S_0 the maximal order of Δ_0. Then $l(x)\sigma_\wp = j[\Delta_0, S_0](x)$ for $j[\Delta_0, S_0]$ defined in [7; (1.5, 6, 7)].

Suppose $r \in \mathbb{N}$. Put $U_r = \mathrm{GL}_r(S_0)$ and $\Phi_r = \mathrm{GL}_r(\Delta_0) \cap M_r(S_0)$. For $T \in M_r(S_0)$, define $\nu(T)$ by

(4.13) $$q^{\nu(T)} = [S_0^r : T \cdot S_0^r] = [S_0^r : S_0^r \cdot T].$$

Essentially, $q^{\nu(T)}$ is the reduced norm raised to the power $\sigma_\wp$. In particular, $\nu(T)$ is an integer divisible by $\sigma_\wp$. There is $\delta \in S_0 \cap \Delta_0^*$ such that

(1.14) $$\widehat{L}(1, 1; R, \wp) = \delta^{-1} \cdot S_0.$$

For $E \in \Phi_r$, define $p(E, t)$ a polynomial in $\mathbb{Z}[t]$ by

(4.15) $$p(E, t) = \sum_{\{D \in U_r \backslash \Phi_r \,:\, ED^{-1} \in \Phi_r\}} t^{\nu(D)/\sigma_\wp},$$

where the indexing set is finite. If $E \in U_r$, the $p(E, t) = 1$.

Now suppose $h \in \widehat{L}(m, n; R, \wp)$. There is a unique polynomial $A(\wp, h, t) \in \mathbb{Z}[t]$ with the property that

(4.16.a) $A(\wp, h, t) = 1$ if $h = 0$,

(4.16.b) if $\beta \in \mathrm{GL}_n(S_0)$, $\alpha \in \mathrm{GL}_m(S_0)$, $r \in \mathbb{N}$ and$E \in \mathrm{GL}_r(\Delta_0)$ so

$$\alpha h \beta = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix},$$

then $\delta E \in \Phi_r$ and $A(\wp, h, t) = p(\delta E, t)$.

Note that the set of primes $\wp \nmid \mathfrak{b}$ at which $A(\wp, h, t) \neq 1$ is finite.

Theorem 4.1 (*Corollary of* [7; Theorem 2.1]). *Let $\mathfrak{b}$ be an ideal of R, $\psi \in \mathscr{X}(\mathfrak{b})$ and $h \in \widehat{L}(m, n; R, \wp)$. Let $\wp$ be a finite prime which does not divide $\mathfrak{b}$. Let τ be the additive Haar measure on $M_{n,m}(\Delta_\wp)$ for which $\tau(M_{n,m}(S_\wp)) = 1$. Put $\varepsilon = \varepsilon_{\psi, \mathfrak{b}, \wp}$ and let $r = \mathrm{rank}_\Delta(h)$.*

Express $\Delta_\wp = M_a(\Delta_0)$ for Δ_0 a division algebra, and let $\sigma' = \sigma/a$. Then

$$(4.17)\quad \int_{x\in M_{n,m}(\Delta_\wp)} \varepsilon(\tau(x), s)\chi(-hx)d\tau(x) = \frac{\prod_{j=0}^{va-1}\left(1-\psi q^{\sigma'(j-s)}\right)}{\prod_{j=0}^{(v-r)a-1}\left(1-\psi q^{\sigma'(ua+j-s)}\right)} A(\wp, h, \psi q^{-s}),$$

where $\psi = \psi(\pi)$ for π any local generator of $\wp$, $v = \min\{m, n\}$, $u = \max\{m, n\}$ and $q = N\wp = |R/\wp|$. □

Next, we measure the global effect. Let $\mathfrak{A}$ be an integral ideal of R and let ϕ be a Hecke character of F such that $\phi_\nu = 1$ if $F_\nu \approx \mathbb{C}$. Let c be the conductor of ϕ and let ϕ^* denote the corresponding function on ideals prime to c. We *do not* assume that c divides $\mathfrak{A}$. For $\nu \in \infty$, define $\delta(\nu) = 0$ if $\phi_\nu(-1) = 1$ and $\delta(\nu) = 1$ otherwise. Define

$$(4.18)\quad \begin{aligned} L_{\mathfrak{A}}(s, \phi) &= \prod_{\wp\nmid\mathfrak{A}c}(1-\phi^*(\wp)N\wp^{-s}),\\ \mathscr{L}_{\mathfrak{A}}(s, \phi) &= L_{\mathfrak{A}}(s, \phi)\prod_{\nu\in\infty}\Gamma\left(\frac{[F_\nu : \mathbb{R}]s+\delta(\nu)}{2}\right), \end{aligned}$$

in the sense that the first product converges for $\mathrm{Re}(s) > 1$ and admits a meromorphic continuation. We prefer to work with $\mathscr{L}$-functions, as these have no zeros outside the critical strip. The function $L_{\mathfrak{b}}(s, \phi)$ has no poles unless $\phi = 1$. If $\phi = 1$, then it has a simple pole at $s = 1$ and no others unless $\mathfrak{b} = R$, in which case it has a simple pole at $s = 0$ as well.

We can now extract L-factors from (4.9). Fix $v = \min\{m, n\}$ and $u = \max\{m, n\}$. Let $h \in \widehat{L}(m, n; \mathfrak{b}, \wp)$, and set $r = \mathrm{rank}_\Delta(h)$. Outside a finite set of primes, the local factor at a finite prime $\wp$ contains a local factor for

$$(4.19)\quad \frac{\prod_{j=0}^{\sigma(v-r)-1} L_*(s-u\sigma-j, \psi)}{\prod_{j=0}^{\sigma v-1} L_*(s-j, \psi)},$$

where the relevant ideals can be determined later. The numerator terms in (4.19) will ultimately indicate exceptional poles. The poles of the integrals at the infinite primes are controlled by Γ-factors. In fact, the Γ-factors from the $\mathscr{L}$-functions will cancel out the Γ-factors of the infinite integrals.

Put

$$\begin{aligned}(4.20)\qquad &\infty_{\mathbb{C}} = \{\nu \in \infty \colon F_\nu \approx \mathbb{C}\},\\ &\infty_{\mathbb{R}} = \{\nu \in \infty - \infty_{\mathbb{C}} \colon \Delta \text{ splits at } \nu\},\\ &\infty_0 = \{\nu \in \infty - \infty_{\mathbb{C}} - \infty_{\mathbb{R}} \colon \psi_\nu \text{ is even}\},\\ &\infty_1 = \infty - \infty_{\mathbb{C}} - \infty_{\mathbb{R}} - \infty_0.\end{aligned}$$

We may express $c(h, y, s)$ times a finite product of polynomials in powers q^{-s} (from Theorem 4.1), a ratio of $\mathscr{L}$-functions similar to (4.19), a holomorphic function derived from the Bessel functions of Theorem 3.3, and the following ratio of Γ-factors:

$$(4.21)\qquad \prod_{\delta=0}^{1} \prod_{\nu\in\infty_\delta} \frac{\prod_{j=0}^{\sigma(v-r)/2-1} \Gamma(s-u\sigma-2j) \prod_{\beta=0}^{\sigma v-1} \Gamma\left(\frac{s-\beta+\delta}{2}\right)}{\prod_{j=0}^{\sigma v/2-1} \Gamma(s-2j) \prod_{\beta=0}^{\sigma(v-r)-1} \Gamma\left(\frac{s-\beta-u\sigma+\delta}{2}\right)}.$$

where $\infty_0 = \infty_1 = \varnothing$ unless $2|\sigma$. Define

$$(4.22)\qquad \forall c \in \mathbb{Z},\quad \forall s \in \mathbb{C},$$

$$f(c, s) = \frac{\Gamma(s+c)}{\Gamma(s)} = \begin{cases} \prod_{j=0}^{c-1}(s+j) & \text{if } c \geq 0, \\ \prod_{j=1}^{|c|}(s-j) & \text{if } c < 0. \end{cases}$$

Recall that

$$(4.23)\qquad \Gamma(s/2)\Gamma((s+1)/2) = \pi^{1/2}2^{1-s}\Gamma(s).$$

We use (4.23) to combine terms of the form $\Gamma((s+\kappa)/2)$ in (4.21) and then use (4.22) to replace ratios of Γ-functions with ratios of polynomials and exponential terms. The result is a non-vanishing entire function times

$$(4.24)\qquad \prod_{\nu\in\infty_0} \frac{\prod_{j=0}^{\sigma(v-r)/2-1}(s-u\sigma-2j-1)}{\prod_{j=0}^{\sigma v/2-1}(s-2j-1)} = \prod_{\nu\in\infty_0} 2^{-\sigma r/2} \frac{\prod_{j=0}^{\sigma(v-r)/2-1} f\left(1, \frac{s-u\sigma-(2j+1)}{2}\right)}{\prod_{j=0}^{\sigma v/2-1} f\left(1, \frac{s-(2j+1)}{2}\right)}.$$

The product of the last expression with the Γ-factors of the $\mathscr{L}$-functions can be expressed as a product of Γ-functions with shifted arguments.

After tedious calculation, with special attention to the constant term $h = 0$, we can summarize with:

DEFINITION 4.1. Let Δ be the central simple division F-algebra. Put $\sigma = [\Delta : F]^{1/2}$. For $\wp \in \mathfrak{f}$, express $\Delta_\wp \approx M_a(\Delta_0)$ where Δ_0 is a division $F_\wp$-algebra, and put $\sigma_\wp = [\Delta_0 : F_\wp]^{1/2}$. Let $\mathfrak{A}$ be the product of finite prime ideals at which Δ does not split. Let $\mathfrak{b}$ be an integral ideal of R and let $\psi \in \mathscr{X}(\mathfrak{b})$ where $\mathscr{X}(\mathfrak{b})$ is given in (2.10). Put

$$\begin{aligned} \infty_{\mathbb{C}} &= \{\nu \in \infty\colon F_\nu \approx \mathbb{C}\}, \\ \infty_{\mathbb{R}} &= \{\nu \in \infty - \infty_{\mathbb{C}}\colon \Delta \text{ splits at } \nu\}, \\ \infty_{\mathbb{H}} &= \infty - \infty_{\mathbb{C}} - \infty_{\mathbb{R}}. \end{aligned} \tag{4.25}$$

For $m, n \in \mathbb{N}$, put

$$\begin{aligned} (4.26)\quad &\Lambda_{m,n}(s; \psi, \mathfrak{b}) \\ &= \prod_{k=0}^{\sigma m-1} L_{\mathfrak{b}\mathfrak{A}_k}(s-k, \psi) \times \prod_{\nu\in\infty_{\mathbb{R}}} \left\{ \prod_{j=0}^{\sigma m-1} \Gamma\left(\frac{s-j}{2}\right) \right\} \\ &\times \prod_{\nu\in\infty_{\mathbb{C}}} \left\{ \prod_{j=0}^{\sigma m-1} \Gamma(s-j) \right\} \times \prod_{\nu\in\infty_{\mathbb{H}}} \left\{ \prod_{j=0}^{\sigma m/2-1} \Gamma(s-2j) \right\} \\ &\times \prod_{\wp|\mathfrak{A},\, \wp\nmid\mathfrak{b}} \left\{ \prod_{j=0}^{\sigma m/\sigma_\wp - 1} (1 - \psi^*(\wp) N\wp^{n\sigma+\sigma_\wp(j-s)}) \right\}. \end{aligned}$$

THEOREM 4.2 (*Context of* §1). *Let* $\mathfrak{b} \neq R$ *be a proper integral ideal and* $\psi \in \mathscr{X}(\mathfrak{b})$. *Let* $v = \min\{m, n\}$ *and* $u = \max\{m, n\}$.

(A) *If* $\psi \neq 1$, *then* $\Lambda_{v,u}(s; \psi, \mathfrak{b}) E_{m,n}(z, s; \psi, \mathfrak{b})$ *is entire.*

(B) *Suppose* $\psi = 1$. *If there is an infinite prime at which* Δ *does not split, then* $\Lambda_{v,u}(s; \psi, \mathfrak{b}) E_{m,n}(z, s; \psi, \mathfrak{b})$ *is analytic on* $s \in \mathbb{C}$ *except for (possible) simple poles at* $s = \sigma u + 2j$ *for* $j = 1, \ldots, \sigma v/2$. *If* Δ *splits at every infinite prime, then* $\Lambda_{m,n}(s; \psi, \mathfrak{b}) \cdot E_{m,n}(z, s; \psi, \mathfrak{b})$ *is analytic on* $s \in \mathbb{C}$ *except for (possible) simple poles at* $s = \sigma u + j$ *for* $j = 1, \ldots, \sigma v$. *In either case, the residue at* $s = \sigma(m+n) = \sigma(u+v)$ *is a non-zero constant.*

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