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INVARIANT SUBSPACES AND HARMONIC CONJUGATION ON COMPACT ABELIAN GROUPS

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Let Γ be a dense subgroup of the real line $\mathbb{R}$. Endow Γ with the discrete topology and the order it inherits from $\mathbb{R}$, and let K be the dual group of Γ. Helson's classic theory of generalized analyticity uses the spectral decomposability of unitary groups to establish a one-to-one correspondence between the cocycles on K and the normalized simply invariant subspaces of $L^2(K)$. This theory has been extended to the invariant subspaces of $L^p(K)$, $1 < p < \infty$, by using recent results concerning the spectral decomposability of uniformly bounded one-parameter groups acting on UMD spaces. We show here that each cocycle A on K can be used to transfer the classical Hilbert transform from $L^1(\mathbb{R})$ to $L^1(K)$ in terms of almost everywhere convergence on K so that in the interesting case (i.e., when A is not a coboundary) the corresponding invariant subspace of $L^p(K)$ is a generalized ergodic Hardy space. This description of the invariant subspaces explicitly identifies the role of the Hilbert transform in generalized analyticity on K. The formulation in terms of almost everywhere convergence on K provides an intrinsic viewpoint which extends to the case $p = 1$.

1. Cocycles and invariant subspaces. Throughout what follows K will be a compact abelian group other than $\{0\}$ or the unit circle $\mathbb{T}$ such that the dual group of K is archimedean ordered. Equivalently, we shall require K to be the dual group of Γ, where Γ arises as a dense subgroup of the additive real line $\mathbb{R}$, and Γ is then endowed with the natural order of $\mathbb{R}$ and the discrete topology. For each $\lambda \in \Gamma$ we denote by χ_λ the corresponding character on K (evaluation at λ), and for each $t \in \mathbb{R}$ we let e_t be the element of K defined by $e_t(\lambda) = e^{it\lambda}$ for all $\lambda \in \Gamma$. As is well known, $t \to e_t$ is a continuous isomorphism of $\mathbb{R}$ onto a dense subgroup of K. For $1 \le p < \infty$ we follow Helson [12] in defining a *simply invariant subspace* of $L^p(K)$ to be a closed subspace M of $L^p(K)$ such that $\chi_\lambda M \subseteq M$ for all $\lambda > 0$, but for some $\alpha < 0$, $\chi_\alpha M$ is not a subset of M. A simply invariant subspace M of $L^p(K)$ is said to be *normalized* provided $M = \bigcap\{\chi_\lambda M : \lambda \in \Gamma,\ \lambda < 0\}$. The set of all normalized simply invariant subspaces of $L^p(K)$ will be denoted by $\mathcal{S}_p$. A *cocycle* on

K is a Borel measurable function $A: \mathbb{R} \times K \to \mathbb{T}$ such that

$$A(t+u, x) = A(t, x)A(u, x+e_t), \quad \text{for } t \in \mathbb{R},\ u \in \mathbb{R},\ x \in K.$$

After identifying cocycles which are equal almost everywhere (with respect to the Haar measure of $\mathbb{R} \times K$), we denote by $\mathscr{C}$ the collection of all cocycles on K. It is convenient to note that for $A \in \mathscr{C}$, [**10**, Lemma VII.12.1] shows that $A(t, \cdot)$ moves continuously in $L^p(K)$, $1 \le p < \infty$, as t runs through $\mathbb{R}$. A cocycle having the form $A(t, x) = \phi(x)/\phi(x+e_t)$ for some Borel measurable function $\phi: K \to \mathbb{T}$ is called a *coboundary*. As will be observed in Corollary (3.4) below, the coboundaries determine the Beurling-type elements of $\mathscr{S}_p$.

For $1 \le p < \infty$, and $A \in \mathscr{C}$, we define the strongly continuous one-parameter group $\{U_t^{(A,p)}\}$, $t \in \mathbb{R}$, of isometries of $L^p(K)$ by setting

$$(1.1) \quad (U_t^{(A,p)} f)(x) = A(t, x) f(x+e_t), \qquad \text{for } t \in \mathbb{R},\ f \in L^p(K),\ x \in K.$$

As will be described in detail in §2 below, if $1 < p < \infty$, then the one-parameter group $\{U_t^{(A,p)}\}$ is decomposable by a one-parameter "spectral family" $\mathscr{E}^{(A,p)}$ of projections acting in $L^p(K)$ [**4**]. With $\mathscr{Q}^{(A,p)}(0)$ denoting the strong limit as $t \to 0^-$ of $(I - \mathscr{E}^{(A,p)}(t))$, it was shown in [**5**, Theorem (3.3)] that the mapping

$$\Psi_p : A \to (\mathscr{Q}^{(A,p)}(0)) L^p(K)$$

is a bijection of $\mathscr{C}$ onto $\mathscr{S}_p$. This result extends Helson's classic development of Ψ_2 in [**11**, §3], and can be used ([**5**, Theorem (5.7)]) to generalize Helson's characterization of $\Psi_2(A)$ in terms of analyticity in the upper half-plane ([**11**, §6]). However, the foregoing characterization of $\mathscr{S}_p$ by spectral decomposability can no longer be used when $p = 1$.

The purpose of the present note is to develop, in terms more intrinsic to pointwise operations on K, a unified "Hilbert-transform" approach to the generalized analyticity and cocycle description of $\mathscr{S}_p$, $1 \le p < \infty$. More specifically, given $A \in \mathscr{C}$, we describe (in §2) a method for using the one-parameter group $\{U_t^{(A,1)}\}$ to transfer the Hilbert transform from $L^1(\mathbb{R})$ to an operator $\mathscr{H}^{(A)}$ on $L^1(K)$. In fact, denoting (now and henceforth) the normalized Haar measure of K by σ, we obtain $\mathscr{H}^{(A)} f$, for each $f \in L^1(K)$, as the σ-a.e. limit of the transferred truncated Hilbert transforms applied to f. For

$1 < p < \infty$, the restriction $\mathscr{H}^{(A,p)}$ of $\mathscr{H}^{(A)}$ to $L^p(K)$ becomes a bounded linear mapping of $L^p(K)$ into itself, and when $1 < p < \infty$ we use the above-described spectral decomposability considerations to show in Theorem (3.9) that if $\Psi_p(A)$ is not a Beurling-type subspace, then

$$\Psi_p(A) = \{f \in L^p(K) : f = i\mathscr{H}^{(A,p)} f \ \sigma\text{-a.e. on } K\}.$$

Thus for $1 < p < \infty$, the elements of $\mathscr{S}_p$ of interest can be thought of as generalized Hardy spaces. In terms of a suitably defined bijection Ψ_1 of $\mathscr{C}$ onto $\mathscr{S}_1$, this result extends to the case $p = 1$.

2. Background and preliminaries. In this section we blend some items from **[4]**, **[5]**, and **[3]** into a framework suitable for the formulation and treatment of generalized analyticity on K outlined in §1. The relevant facts concerning spectral decomposability will be phrased in terms of the following familiar notion.

DEFINITION. Let $\mathfrak{B}(X)$ be the Banach algebra of all bounded linear mappings of a Banach space X into itself, and denote the identity of $\mathfrak{B}(X)$ by I. A *spectral family of projections in* X is a projection-valued function $E : \mathbb{R} \to \mathfrak{B}(X)$ such that:

(i) $\sup\{\|E(t)\| : t \in \mathbb{R}\} < \infty$;
(ii) $E(s)E(t) = E(t)E(s) = E(s)$, for $-\infty < s \le t < +\infty$;
(iii) E is right-continuous on $\mathbb{R}$ in the strong operator topology of $\mathfrak{B}(X)$, and for each $t \in \mathbb{R}$, E has a strong left-hand limit $E(t^-)$ at t;
(iv) as $t \to +\infty$ (respectively, $t \to -\infty$), $E(t) \to I$ (respectively, $E(t) \to 0$) in the strong operator topology of $\mathfrak{B}(X)$.

We shall require some aspects of the integration theory of a spectral family E in X **[9**, Chapter 17]. For a compact interval $\Delta = [a, b]$ of $\mathbb{R}$, let $\mathbf{AC}(\Delta)$ be the Banach algebra of all complex-valued, absolutely continuous functions on Δ under the norm $\|\cdot\|_\Delta$ defined by

$$\|f\|_\Delta = |f(b)| + \operatorname{var}(f, \Delta),$$

where "var" denotes total variation. For each $f \in \mathbf{AC}(\Delta)$, the integral $\int_\Delta f(t)\,dE(t)$ exists as a strong limit of Riemann-Stieltjes sums, and we define $\int_\Delta^\oplus f(t)\,dE(t)$ by putting

$$\int_\Delta^\oplus f(t)\,dE(t) = f(a)E(a) + \int_\Delta f(t)\,dE(t).$$

The mapping $f \to \int_\Delta^\oplus f(t)\,dE(t)$ is an algebra homomorphism of $\mathbf{AC}(\Delta)$ into $\mathfrak{B}(X)$ such that

$$\left\| \int_\Delta^\oplus f(t)\,dE(t) \right\| \le \|f\|_\Delta \sup\{\|E(t)\| : t \in \mathbb{R}\}.$$

The Banach spaces X possessing the unconditionality property for martingale differences (written $X \in \mathrm{UMD}$) have been characterized in [7] and [8] as those Banach spaces for which the Hilbert kernel of $\mathbb{R}$ defines a bounded convolution operator on $L^p(\mathbb{R}, X)$ for some, and hence all, p in the range $1 < p < \infty$. When $X \in \mathrm{UMD}$, spectral families of projections in X occur naturally in accordance with the following theorem.

(2.1) Stone's Theorem for UMD Spaces ([4, §5]). *Let* $\{V_t\}$, $t \in \mathbb{R}$, *be a strongly continuous one-parameter group of operators on a UMD space* X *such that* $\sup\{\|V_t\| : t \in \mathbb{R}\} < \infty$. *Then*:

(i) *there is a unique spectral family* $\mathscr{E}$ *in* X *(called the Stone-type spectral family of* $\{V_t\}$*) such that*

$$V_t x = \lim_{u \to +\infty} \int_{-u}^{u} e^{its}\,d\mathscr{E}(s)x, \quad \text{for } t \in \mathbb{R},\ x \in X;$$

(ii) *for each* $s \in \mathbb{R}$,

$$(\pi i)^{-1} \int_{\delta \le |t| \le \delta^{-1}} t^{-1} e^{ist} V_{-t}\,dt$$

converges in the strong operator topology of $\mathfrak{B}(X)$, *as* $\delta \to 0^+$, *to an operator* $J_s \in \mathfrak{B}(X)$;

(iii) $J_s = \mathscr{E}(s) + \mathscr{E}(s^-) - I$, *for all* $s \in \mathbb{R}$;

(iv) $\mathscr{E}(s) = I + 2^{-1}(J_s - J_s^2)$, *for all* $s \in \mathbb{R}$.

If μ is an arbitrary measure and $1 < p < \infty$, then $L^p(\mu) \in \mathrm{UMD}$. In particular in the setting of §1, if $A \in \mathscr{C}$ and $1 < p < \infty$, then we shall denote the Stone-type spectral family of the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1) by $\mathscr{E}^{(A,p)}$. This implements a bijection of $\mathscr{C}$ onto $\mathscr{S}_p$, as described in the next theorem.

(2.2) Theorem ([5, §3]). *For* $1 < p < \infty$ *and* $A \in \mathscr{C}$, *let*

$$\Psi_p(A) = \{I - \mathscr{E}^{(A,p)}(0^-)\} L^p(K).$$

Then Ψ_p *is a one-to-one mapping of* $\mathscr{C}$ *onto* $\mathscr{S}_p$.

In the sequel we shall cast the relationship between invariant subspaces and generalized analyticity on K in a Hilbert transform setting. In order to do so, we now take up the necessary background material from **[3]** concerning distributional control, which will later be applied to the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1). Suppose that $1 \le p < \infty$, (Ω, μ) is an arbitrary measure space, and $u \to S_u$ is a strongly continuous representation of a locally compact abelian group G in $L^p(\mu)$ such that for some positive real number C_p,

$$\sup\{\|S_u\| : u \in G\} \le C_p .$$

Fix a Haar measure ν on G, and for $k \in L^1(G, d\nu)$, let $S_k \in \mathfrak{B}(L^p(\mu))$ denote the *transferred convolution operator* defined by $L^p(\mu)$-valued Bochner integration as follows:

$$S_k f = \int_G k(u) S_{-u} f \, d\nu(u), \quad \text{for all } f \in L^p(\mu). \tag{2.3}$$

Suppose further that the representation $u \to S_u$ is also separation-preserving in the sense that whenever $f \in L^p(\mu)$, $g \in L^p(\mu)$, and $fg = 0$ μ-a.e., then $(S_u f)(S_u g) = 0$ μ-a.e. for all $u \in G$. Following **[3]**, we say that $u \to S_u$ is a *distributionally controlled* representation of G in $L^p(\mu)$ provided that there is a positive real constant C_∞ such that

$$\|S_u f\|_\infty \le C_\infty \|f\|_\infty, \quad \text{for all } f \in L^p(\mu) \cap L^\infty(\mu), \ u \in G. \tag{2.4}$$

This terminology is motivated by **[3**, §2**]**, where it is shown that under the above assumptions on the representation $u \to S_u$, (2.4) is equivalent to the existence of positive real constants c and α such that S interacts with distribution functions in the following manner:

$$\mu\{\omega \in \Omega : |(S_u f)(\omega)| > y\} \le c\mu\{\omega \in \Omega : |f(\omega)| > \alpha y\},$$

for all $u \in G$, $f \in L^p(\mu)$, and $y > 0$. The latter characterization permits distributionally controlled representations to transfer weak type bounds of maximal convolution operators, while, apart from (2.4), the separation-preserving requirement permits distributionally controlled representations to transfer strong type maximal bounds (for the case of strong bounds, see **[1]**, **[2]**). To be more precise, suppose that $\{k_n\}_{n=1}^\infty \subseteq L^1(G, d\nu)$. With $1 \le p < \infty$, let $N_p(\{k_n\})$ and $N_p^{(w)}(\{k_n\})$ denote the (possibly infinite) strong type (p, p) and weak type (p, p) norms, respectively, of the maximal convolution operator on $L^p(G, d\nu)$ defined by the sequence $\{k_n\}_{n=1}^\infty$. Let $u \to S_u$ be a distributionally controlled representation of G in $L^p(\mu)$ having constants C_p and C_∞ as described above. Denote by $\mathscr{M}$ the maximal operator on $L^p(\mu)$ defined by the sequence $\{S_{k_n}\}_{n=1}^\infty$. Under these circumstances we have, in particular, the following two results concerning transference of maximal bounds and transference of almost everywhere convergence ([**3**, Theorem (2.14), Corollary (2.22)], [**1**, Théorème 1]).

(2.5) Proposition. *With the above notation and assumptions we have:*

(i) *for all* $f \in L^p(\mu)$,

$$\|\mathscr{M} f\|_p \le C_p^2 N_p(\{k_n\})\|f\|_p;$$

(ii) *for all* $f \in L^p(\mu)$ *and all* $y > 0$,

$$\mu\{\omega \in \Omega: |(\mathscr{M} f)(\omega)| > y\} \le (C_p C_\infty)^{2p}[N_p^{(w)}(\{k_n\})\|f\|_p y^{-1}]^p.$$

(2.6) Proposition. *In the foregoing assumptions let* $p = 1$, *and suppose that the sequence* $\{k_n\}_{n=1}^\infty \subseteq L^1(G, d\nu)$ *has the following additional properties:*

(i) $N_1^{(w)}(\{k_n\}) < \infty$ *and* $N_r^{(w)}(\{k_n\}) < \infty$ *for some* $r \in (1, +\infty)$;

(ii) *the corresponding sequence of Fourier transforms* $\{\widehat{k_n}\}_{n=1}^\infty$ *is uniformly bounded and converges pointwise on the dual group* $\widehat{G}$;

(iii) *for each* $s \in G$ *and each* $\varepsilon > 0$, *there is a corresponding relatively compact open neighborhood* W *of the identity element of* G *such that*

$$\int_{G\backslash W} |k_n(u) - k_n(u+s)|\, d\nu(u) < \varepsilon,$$

for all sufficiently large n.

Then for each $\varphi \in L^1(G, d\nu)$, *the sequence of convolutions* $\{k_n * \varphi\}_{n=1}^\infty$ *converges* ν*-a.e. on* G, *and for each* $f \in L^1(\mu)$, *the sequence* $\{S_{k_n} f\}_{n=1}^\infty$ *converges* μ*-a.e. on* Ω.

If κ is a Calderón-Zygmund singular integral kernel of the kind treated in [**15**, §II.4], then the sequence of truncates of κ, $\{\kappa_n\}_{n=1}^\infty \subseteq L^1(\mathbb{R}^N)$, satisfies the requirements (2.6)–(i), (ii), (iii). Moreover, $N_r(\{\kappa_n\}) < \infty$ for all $r \in (1, +\infty)$. In particular, these comments hold for the sequence $\{h_n\}_{n=1}^\infty$ of truncates of the Hilbert kernel h on $\mathbb{R}$:

$$h(t) = (\pi t)^{-1}, \quad \text{for } t \in \mathbb{R}\backslash\{0\}; \tag{2.7}$$

$$h_n(t) = \begin{cases} h(t) & \text{if } n^{-1} \le |t| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, if $A \in \mathscr{C}$ and $1 \leq p < \infty$, then the one-parameter group $\{U_t^{(A,p)}\}$ in (1.1) is a distributionally controlled representation of $\mathbb{R}$ in $L^p(K, \sigma)$, and we shall now specialize the discussion surrounding Propositions (2.5) and (2.6) to $S = \{U_t^{(A,p)}\}$ and to the sequence of truncated Hilbert kernels $\{h_n\}_{n=1}^{\infty}$ in (2.7). The Haar measure ν of $G = \mathbb{R}$ will be ordinary Lebesgue measure. Given $k \in L^1(\mathbb{R})$, the corresponding transferred convolution operator $S_k \in \mathfrak{B}(L^p(K, \sigma))$ defined in (2.3) by $L^p(\sigma)$-valued Bochner integration will be denoted by $U_k^{(A,p)}$ in order to signify the present context explicitly. Applying Fubini's theorem for locally compact spaces ([**13**, §13]) to (2.3) and (1.1), we easily see that for each $f \in L^p(\sigma)$,

$$(2.8)\quad (U_k^{(A,p)}f)(x) = \int_{\mathbb{R}} k(t)A(-t, x)f(x - e_t)\,dt, \qquad \text{for } \sigma\text{-almost all } x \in K.$$

In particular, $U_k^{(A,p)}$ is the restriction to $L^p(\sigma)$ of $U_k^{(A,1)}$. Hence we can conveniently economize on notation by writing $U_k^{(A)}$ to denote the operator $U_k^{(A,1)}$ on $L^1(\sigma)$ when there is no danger of confusion. For each positive integer n, let $\mathscr{H}_n^{(A)}$ be the operator $U_{h_n}^{(A)}$. Thus, applying Proposition (2.6) to the one-parameter group $\{U_t^{(A,1)}\}$ and the sequence $\{h_n\}$, we obtain the *transferred Hilbert transform* $\mathscr{H}^{(A)}$ on $L^1(\sigma)$, which is defined for each $f \in L^1(\sigma)$ by taking $\mathscr{H}^{(A)}f$ to be the limit σ-a.e. on K of the sequence $\{\mathscr{H}_n^{(A)}f\}_{n=1}^{\infty}$. In view of (2.8), we have for each $f \in L^1(\sigma)$,

$$(2.9)\quad (\mathscr{H}^{(A)}f)(x) = \lim_{n\to+\infty} \int_{n^{-1}\leq|t|\leq n} (\pi t)^{-1}A(-t, x)f(x - e_t)\,dt, \qquad \text{for } \sigma\text{-almost all } x \in K.$$

By either Proposition (2.5)–(i) or by taking $s = 0$ in Theorem (2.1)–(ii), we see that for $1 < p < \infty$, the restriction of $\mathscr{H}^{(A)}$ to $L^p(\sigma)$ (denoted $\mathscr{H}^{(A,p)}$) is a bounded linear mapping of $L^p(\sigma)$ into itself, and that the sequence $\{U_{h_n}^{(A,p)}\}_{n=1}^{\infty}$ converges in the strong operator topology of $\mathfrak{B}(L^p(\sigma))$ to $\mathscr{H}^{(A,p)}$. Moreover, from the definition of $\mathscr{H}^{(A,p)}$, together with Theorem (2.1)–(ii), (iii), (iv), we obtain the following lemma for use in §3 (in conjunction with Theorem (2.2)).

(2.10) LEMMA. *If $A \in \mathscr{C}$ and $1 < p < \infty$, then the Stone-type spectral family $\mathscr{E}^{(A,p)}$ of $\{U_t^{(A,p)}\}$ satisfies*:

(i) $-i\mathscr{H}^{(A,p)} = \mathscr{E}^{(A,p)}(0) + \mathscr{E}^{(A,p)}(0^-) - I$;

(ii) $\mathscr{E}^{(A,p)}(0) = I + 2^{-1}(\{\mathscr{H}^{(A,p)}\}^2 - i\mathscr{H}^{(A,p)})$.

Hence

(iii) $I - \mathscr{E}^{(A,p)}(0^-) = I + 2^{-1}(\{\mathscr{H}^{(A,p)}\}^2 + i\mathscr{H}^{(A,p)})$.

We close this section by listing in the next proposition some applications of [**3**, §3] to $\mathscr{H}^{(A)}$ which will be needed for our treatment of generalized analyticity when $p = 1$.

(2.11) PROPOSITION. *Given* $A \in \mathscr{C}$, *the following assertions hold.*

(i) $\mathscr{H}^{(A)}$ *is of weak type* (1.1) *on* $L^1(\sigma)$.

(ii) *Suppose* $f \in L^1(\sigma)$, $\mathscr{H}^{(A)} f \in L^1(\sigma)$, *and* $g \in L^1(\mathbb{R})$. *Then*

$$\mathscr{H}^{(A)}(U_g^{(A)} f) = U_g^{(A)}(\mathscr{H}^{(A)} f) \quad \sigma\text{-a.e.}$$

(iii) *For* $1 \le p < \infty$, *let*

$$Y^{(A,p)} = \{f \in L^p(\sigma) \colon U_t^{(A,p)} f = f \text{ for all } t \in \mathbb{R}\},$$

and let $Z^{(A,p)}$ *be the* $L^p(\sigma)$*-closure of the linear manifold in* $L^p(\sigma)$ *spanned by the ranges of the operators* $U_g^{(A,p)}$ *for all* $g \in L^1(\mathbb{R})$ *such that* $\mathbb{R}\backslash\{0\}$ *contains the support of* $\hat{g}$. *Then*

$$L^p(\sigma) = Y^{(A,p)} \oplus Z^{(A,p)}.$$

(iv) *If* $f \in L^1(\sigma)$, *and* $f = i\mathscr{H}^{(A)} f$ σ*-a.e., then there is a sequence* $\{g_n\}_{n=1}^{\infty} \subseteq L^\infty(\sigma)$ *such that* $g_n = i\mathscr{H}^{(A)} g_n$ σ*-a.e. for each* n, *and* $\|g_n - f\|_1 \to 0$, *as* $n \to \infty$.

3. Invariant subspaces as Hardy spaces. In this section we shall describe the elements of S_p, $1 \le p < \infty$, by means of Hardy spaces defined from their corresponding cocycles. In order to include the case $p = 1$ in our considerations, we begin by defining a bijection Ψ_1 of $\mathscr{C}$ onto $\mathscr{S}_1$. This is accomplished by recourse to the following scholium, which is readily deduced from [**10**, Theorem V.6.1] with the aid of the first lemma in [**12**, §1.6].

(3.1) SCHOLIUM. *For each* $M \in \mathscr{S}_2$, *let* $\Theta(M)$ *be the* $L^1(\sigma)$*-closure of* M. *Then* Θ *is a one-to-one mapping of* $\mathscr{S}_2$ *onto* $\mathscr{S}_1$, *with inverse mapping given by*

$$\Theta^{-1}(N) = N \cap L^2(\sigma), \quad \text{for all } N \in \mathscr{S}_1.$$

(3.2) DEFINITION. We denote by Ψ_1 the one-to-one mapping of $\mathscr{C}$ onto $\mathscr{S}_1$ given by

$$\Psi_1(A) = \Theta(\Psi_2(A)), \quad \text{for all } A \in \mathscr{C}.$$

The simplest and most obvious elements of $\mathscr{S}_p$, are the Beurling type subspaces, which are described as follows. For $1 \leq p < \infty$ let $H^p(K)$ be the standard Hardy space defined by $H^p(K) = \{f \in L^p(\sigma) \colon \hat{f}(\gamma) = 0$ whenever $\gamma \in \Gamma$ and $\gamma < 0\}$. A *Beurling type subspace* of $L^p(\sigma)$ is a subspace of the form $\phi H^p(K)$ for some Borel measurable function $\phi \colon K \to \mathbb{T}$. Since the multiplication operator on $L^p(K)$ defined by such a function ϕ is a surjective linear isometry which preserves σ-a.e. convergence, the structural features of the Beurling type subspaces mirror those of $H^p(K)$. Consequently, we shall focus on $H^p(K)$ initially in order to elucidate and dispense with the rather special properties of the Beurling type subspaces. This will smooth the way for the subsequent treatment of the main result (Theorem (3.9)).

If A_0 denotes the trivial cocycle, $A_0(x, t) \equiv 1$, then for $1 \leq p < \infty$ and $t \in \mathbb{R}$, the operator $U_t^{(A_0, p)}$ in (1.1) is translation by e_t on $L^p(\sigma)$. Application of (2.9) shows that $\mathscr{H}^{(A_0)}\chi_\lambda = -i\,\mathrm{sgn}(\lambda)\chi_\lambda$ for each $\lambda \in \Gamma$, where, as usual, the function sgn is defined on Γ by putting $\mathrm{sgn}(\gamma)$ equal to 1, 0, or -1, according as $\gamma > 0$, $\gamma = 0$, or $\gamma < 0$. It follows that for $1 < p < \infty$, $\mathscr{H}^{(A_0, p)}$ is the translation-invariant operator on $L^p(K)$ corresponding to the $L^p(K)$-Fourier multiplier $(-i\,\mathrm{sgn})$ (that is, Bochner's abstract harmonic conjugation operator [**6**, Theorem 16]), and hence by (2.10)–(iii), $(I - \mathscr{E}^{(A_0, p)}(0^-))$ is the translation-invariant operator on $L^p(K)$ corresponding to the $L^p(K)$-Fourier multiplier given by the characteristic function on Γ of $\{\lambda \in \Gamma \colon \lambda \geq 0\}$. This observation leads us to the following scholium (which was observed for the case $p = 2$ in [**12**, 2.3.(13)]).

(3.3) SCHOLIUM. *Let $A_0 \in \mathscr{C}$ be the trivial cocycle, $A_0(t, x) \equiv 1$. Then for $1 \leq p < \infty$,*

$$\Psi_p(A_0) = H^p(K).$$

Proof. In view of the foregoing and the definition of Ψ_p, for $1 < p < \infty$, in Theorem (2.2), only the case $p = 1$ remains to be considered. By (3.2) and Scholium (3.1), $\Psi_1(A_0)$ is the $L^1(\sigma)$-closure of $H^2(K)$. Let $\{\tau_\delta\}$ be an approximate identity for $L^1(\sigma)$ consisting of trigonometric polynomials ([**14**, Theorem (33.12)]). If $f \in H^1(K)$, then the convolution $\tau_\delta * f \in H^2(K)$, for each δ, and we have $\|\tau_\delta * f - f\|_1 \to 0$. Hence $H^1(K) \subseteq \Psi_1(A_0)$. The reverse inclusion is obvious. □

(3.4) COROLLARY. *Let* $A \in \mathscr{C}$ *be a coboundary—that is,*

(3.5) $A(t, x) = \phi(x)/\phi(x + e_t)$

for some Borel measurable function $\phi: K \to \mathbb{T}$.

Then for $1 \le p < \infty$, $\Psi_p(A) = \phi H^p(K)$.

Proof. For each $t \in \mathbb{R}$, $U_t^{(A,p)} = \phi U_t^{(A_0,p)}\overline{\phi}$. Hence when $1 < p < \infty$, $\mathscr{E}^{(A,p)}(\cdot) = \phi\mathscr{E}^{(A_0,p)}(\cdot)\overline{\phi}$, and the desired conclusion follows in this case from Theorem (2.2), and Scholium (3.3). The desired conclusion for $p = 1$ is immediate from the case $p = 2$, together with (3.2) and Scholium (3.3). □

Thus for $1 \le p < \infty$, the Beurling type subspaces are the elements of $\mathscr{S}_p$ corresponding to the coboundaries under Ψ_p, and the study of the Beurling type subspaces from a Hardy space standpoint can be reduced to the cocycle $A_0 \equiv 1$ and the standard Hardy spaces $H^p(K)$. In particular, if A is a coboundary as in Corollary (3.4), then it follows from (2.9) that $\mathscr{H}^{(A)} = \phi\mathscr{H}^{(A_0)}\overline{\phi}$, which renders $\mathscr{H}^{(A,p)}$ isometrically equivalent to Bochner's abstract harmonic conjugation operator when A is a coboundary and $1 < p < \infty$. Since there is no novelty to be gained by studying the Beurling type subspaces from the Hardy space standpoint, we shall be concerned with their complement in $\mathscr{S}_p$, or, what comes to the same thing, the cocycles which are not coboundaries. In order to expedite these considerations, it will be convenient to characterize the coboundaries A in terms of the existence of nonzero fixed points for the corresponding group $\{U_t^{(A,p)}\}$. This matter is attended to in the following proposition.

(3.6) PROPOSITION. *Let* $A \in \mathscr{C}$, *and for* $1 \le p < \infty$, *let* $Y^{(A,p)}$ *be as described in Proposition* (2.11)–(iii). *The following assertions are valid.*

(i) *For* $1 < p < \infty$, $Y^{(A,p)} = \{\mathscr{E}^{(A,p)}(0) - \mathscr{E}^{(A,p)}(0^-)\}L^p(\sigma)$.

(ii) *If* $1 \le p < \infty$, *then* $Y^{(A,p)} \ne \{0\}$ *if and only if* A *is a coboundary. If* A *satisfies* (3.5), *then* $Y^{(A,p)}$ *is the one-dimensional subspace spanned by* ϕ.

Proof. (i) Since $\mathscr{E}^{(A,p)}$ is the Stone-type spectral family of $\{U_t^{(A,p)}\}$, direct calculation from (2.1)–(i) shows that if

$$f = \{\mathscr{E}^{(A,p)}(0) - \mathscr{E}^{(A,p)}(0^-)\}f,$$

then $f \in Y^{(A,p)}$. Conversely, if $U_t^{(A,p)}f = f$ for all $t \in \mathbb{R}$, then $U_{h_n}^{(A,p)}f = 0$ for all n, and hence $\mathscr{H}^{(A,p)}f = 0$. Using this in (2.10)–(i), (ii), we find that $\mathscr{E}^{(A,p)}(0)f = f$ and $\mathscr{E}^{(A,p)}(0^-)f = 0$. (ii) If $1 \le p < \infty$, and A has the form (3.5), then direct calculation with (1.1) shows that $\phi \in Y^{(A,p)}$. Conversely, suppose $Y^{(A,p)}$ contains a unit vector ψ. Applying (1.1) to ψ gives, for each $t \in \mathbb{R}$,

$$\psi(x) = A(t, x)\psi(x + e_t) \quad \text{for } \sigma\text{-almost all } x \in K. \tag{3.7}$$

Taking absolute values, we see that for each $t \in \mathbb{R}$, $\varrho \equiv |\psi|$ satisfies

$$\varrho(x + e_t) = \varrho(x) \quad \text{for } \sigma\text{-almost all } x \in K.$$

Taking Fourier transforms on K, we find that $\hat{\varrho}(\gamma) = 0$ for all $\gamma \in \Gamma\backslash\{0\}$. So $|\psi| = 1$ σ-a.e. on K. Without loss of generality, we can assume that ψ is Borel measurable and has modulus identically 1 on K. Dividing both sides of (3.7) by $\psi(x + e_t)$ completes the proof of the first assertion in (3.6)–(ii). An obvious variation on the foregoing reasoning with Fourier transforms shows that for a given coboundary A, a unimodular Borel function ϕ satisfying (3.5) for almost all $(t, x) \in \mathbb{R} \times K$ has its equivalence class modulo equality σ-a.e. determined to within a multiplicative unimodular constant. Using this observation together with the foregoing argument also establishes the second assertion in (3.6)–(ii). □

The stage is now set for our main result identifying the non-Beurling type subspaces in $\mathscr{S}_p$ by means of the following generalization of ergodic Hardy space.

(3.8) Definition. For $A \in \mathscr{C}$ and $1 \le p < \infty$, we define $\mathfrak{H}^p(A)$ by

$$\mathfrak{H}^p(A) = \{f \in L^p(\sigma) : f = i\mathscr{H}^{(A)}f \ \ \sigma\text{-a.e. on } K\}.$$

Notice that $\mathfrak{H}^p(A)$ is a closed subspace of $L^p(\sigma)$. When $1 < p < \infty$ this observation is an immediate consequence of the fact that $\mathscr{H}^{(A,p)} \in \mathfrak{B}(L^p(\sigma))$, whereas when $p = 1$, it follows from (2.11)–(i).

(3.9) Theorem. *If $A \in \mathscr{C}$, and A is not a coboundary, then for $1 \le p < \infty$,*

$$\Psi_p(A) = \mathfrak{H}^p(A).$$

Proof. Suppose first that $1 < p < \infty$. Since A is not a coboundary, it follows from Proposition (3.6) that $\mathscr{E}^{(A,p)}(0) = \mathscr{E}^{(A,p)}(0^-)$. Using this in (2.10)–(i), we obtain for all $f \in L^p(\sigma)$:

$$(3.10) \qquad (I - \mathscr{E}^{(A,p)}(0^-))f = 2^{-1}(I + i\mathscr{H}^{(A,p)})f.$$

If $f \in \Psi_p(A)$, then the right-hand side of (3.10) equals f, and so $f \in \mathfrak{H}^p(A)$. Similarly, if $f \in \mathfrak{H}^p(A)$, then the left-hand side of (3.10) equals f, and so $f \in \Psi_p(A)$. It remains only to prove the conclusion of the theorem in the case $p = 1$. Suppose first that $f \in \Psi_1(A)$. From the definition of Ψ_1 in (3.2), there is a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \Psi_2(A)$ such that $\|f_n - f\|_1 \to 0$. It follows with the aid of (2.11)–(i) that, with respect to σ, $\mathscr{H}^{(A)}f_n \to \mathscr{H}^{(A)}f$ in measure. Since $f_n = i\mathscr{H}^{(A)}f_n$ for all n, it is clear that $f \in \mathfrak{H}^1(A)$. Conversely if $f \in \mathfrak{H}^1(A)$, then by (2.11)–(iv) and the conclusion already established when $p = 2$, there is a sequence $\{g_n\}_{n=1}^{\infty} \subseteq \mathfrak{H}^2(A) = \Psi_2(A)$ such that $\|g_n - f\|_1 \to 0$. Hence $f \in \Psi_1(A)$. □

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