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ON SIX-CONNECTED FINITE H-SPACES

JAMES P. LIN AND FRANK WILLIAMS

In this note we shall prove the following theorem.

MAIN THEOREM. ***Let*** X ***be a*** 6***-connected finite*** H***-space with associative*** mod 2 ***homology. Further, suppose that*** $\mathrm{Sq}^4 H^7(X; Z_2) = 0$ ***and*** $\mathrm{Sq}^{15} H^{15}(X; Z_2) = 0$. ***Then*** X ***is either contractible or has the homotopy type of a product of seven-spheres.***

0. Introduction. It should be noted that there are several results related to this theorem. Lin showed that any finite H-space with associative mod 2 homology has its first nonvanishing homotopy in degrees 1, 3, 7, or 15 (or is contractible). A seven-sphere is an H-space, but not a mod 2 homotopy-associative one [**4**, **10**]. Further work of Hubbuck [**5**], Sigrist and Suter [**12**], and others has shown that spaces whose mod 2 cohomology has the form

$$\Lambda(x_7, x_{11}) \quad \text{or} \quad \Lambda(x_7, x_{11}, x_{13})$$

are not realizable as H-spaces. (Here x_i denotes an element of degree i.) One is led to conjecture that

Conjecture 1. Every two-torsion-free 6-connected finite H-space is homotopy equivalent to a product of seven-spheres (or is acyclic).

Conjecture 2. Every two-torsion-free homotopy-associative 6-connected finite H-space is acyclic.

Conjecture 1 implies Conjecture 2 by [**4**, **11**].

Henceforth, X will denote an H-space that satisfies the hypotheses of the Main Theorem, and $H^*(X)$ will denote $H^*(X; Z_2)$. The proof of the Main Theorem will be accomplished in a series of steps, which we record here. Our goal is to show that under the hypotheses, X has mod 2 cohomology an exterior algebra on 7-dimensional generators. This relies heavily on the following theorem.

Steenrod Connections [**8**]. Let X be a finite simply-connected H-space with associative mod 2 homology. Then for $r \geq 0$, $k > 0$,

$$QH^{2^r+2^{r+1}k-1}(X; Z_2) = \mathrm{Sq}^{2^r k} QH^{2^r+2^r k-1}(X; Z_2), \text{ and}$$
$$\mathrm{Sq}^{2^r} QH^{2^r+2^{r+1}k-1}(X; Z_2) = 0.$$

(Here QH^* denotes the indecomposable quotient.)

In §1 we shall use a relation in the Steenrod algebra and the methods of [1] to produce a new factorization of Sq^{16}. We then apply this factorization to show that $H^{23}(X) = 0$. This implies that $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$, with trivial action of the Steenrod algebra. In §2 we use the Cartan formula for secondary operations, [7], and a particular factorization of the cube of a certain 8-dimensional cohomology class, [10], to show that $H^{15}(X) = 0$. In §3 we turn to the c-invariant, [14], to complete our calculations by showing that no algebra generators for $H^*(X)$ exist in degrees greater than seven. Once it is shown that the mod 2 cohomology is exterior on 7-dimensional generators, it follows by the Bockstein spectral sequence that the rational cohomology has the same form. But since the rational cohomology is isomorphic to the E_∞ term of the mod p Bockstein spectral for any prime p, it follows by [2] that $H^*(X; Z)$ has no odd torsion. Thus $H^*(X; Z)$ is torsion-free, and we may use the Hurewicz map together with the multiplication in X to obtain a homotopy equivalence

$$S^7 \times \cdots \times S^7 \to X.$$

1. $H^{23}(X)$**.** In this section we prove that there are no 23-dimensional generators in $H^*(X)$. We will also show that $H^*(X)$ is an exterior algebra with trivial action of the Steenrod algebra. We shall use the notation $\mathrm{Sq}^{i,j}$ to denote $\mathrm{Sq}^i\mathrm{Sq}^j$.

THEOREM 1.1. *Let Y be a space and $x \in H^k(Y)$ be the reduction of an integral class. If x is in the intersection of the kernels of Sq^2, Sq^7, Sq^8, and $\mathrm{Sq}^{8,4}$, then there exist classes $v_i \in H^{k+i}(Y)$, $i = 3, 7, 8, 10, 12, 13, 14, 15$, such that*

$$\begin{aligned}(1.1)\qquad \mathrm{Sq}^{16}x = {} & \mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 \\ & + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})v_8 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ & + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15}.\end{aligned}$$

Proof. Consider the following matrix of relations:

$$
(1.2)\qquad
\begin{matrix} v_3 \\ v_7 \\ v_8 \\ v_{10} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{matrix}
\begin{pmatrix}
\mathrm{Sq}^2 & 0 & 0 & 0 \\
0 & \mathrm{Sq}^1 & 0 & 0 \\
0 & \mathrm{Sq}^2 & \mathrm{Sq}^1 & 0 \\
\mathrm{Sq}^9 & \mathrm{Sq}^4 & \mathrm{Sq}^3 & 0 \\
\mathrm{Sq}^{8,2,1} & 0 & 0 & \mathrm{Sq}^1 \\
\mathrm{Sq}^{12} & 0 & \mathrm{Sq}^{4,2} & 0 \\
\mathrm{Sq}^{13}+\mathrm{Sq}^{12,1} & \mathrm{Sq}^8 & 0 & \mathrm{Sq}^3 \\
\mathrm{Sq}^{14} & 0 & \mathrm{Sq}^8 & \mathrm{Sq}^4
\end{pmatrix}
\begin{pmatrix}
\mathrm{Sq}^2 \\ \mathrm{Sq}^7 \\ \mathrm{Sq}^8 \\ \mathrm{Sq}^{8,4}
\end{pmatrix} = 0.
$$

Let

$$w: K(Z, n) \to K(Z_2; n+2, n+7, n+8, n+12) = K_0$$

be defined by

$$w^*(\iota_{n+2}) = \mathrm{Sq}^2\iota_n;\ w^*(\iota_{n+7}) = \mathrm{Sq}^7\iota_n;$$
$$w^*(\iota_{n+8}) = \mathrm{Sq}^8\iota_n;\ w^*(\iota_{n+12}) = \mathrm{Sq}^{8,4}\iota_n.$$

If E is the fiber of w, we have the following diagram

$$
(1.3)\qquad
\begin{array}{ccc}
\Omega K_0 & & \\
\big\downarrow{\scriptstyle j} & & \\
E & & \\
\big\downarrow & & \\
K(Z, n) & \xrightarrow{\ w\ } & K_0
\end{array}
$$

and there exist elements $v_j \in PH^{n+j}(E; Z_2)$ defined by the relations (1.2).

A calculation shows that the element

$$z = \mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})v_8$$
$$+ \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} + Sq^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15}$$

lies in $PH^{16+n}(E) \cap \ker(j^*) = p^*PH^{16+n}(K(Z_2, n))$. It follows that $z = cp^*(\mathrm{Sq}^{16}\iota_n)$, where $c \in Z_2$. For $n = 16$ there is a commutative diagram

$$\begin{array}{ccccc} & & \Omega K_0 & & \\ & & \downarrow{\scriptstyle j} & & \\ & \overset{\tilde f}{\nearrow} & E & & \\ & & \downarrow & & \\ K(Z\,,2) & \xrightarrow{f} & K(Z\,,n) & \xrightarrow{w} & K_0 \end{array}$$

where $f^*(\iota_{16}) = \iota_2^8$. Now consider $\overline{w}: K(Z_2\,, 16) \to K_0 \times K(Z_2\,, 17)$ given by the same formulas as w on the fundamental classes in K_0 and such that $\overline{w}^*(\iota_{17}) = \mathrm{Sq}^1\iota_{16}$. Let $\overline{E}$ be the fiber of $\overline{w}$.

There exists a commutative diagram

(1.4)
$$\begin{array}{ccccccc} & & \Omega K_0 & \longrightarrow & \Omega K_0 \times K(Z_2\,, 16) \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \bar j} \\ & \overset{\tilde f}{\nearrow} & E & \xrightarrow{\bar h} & \overline{E} \\ & & \downarrow & & \downarrow \\ K(Z\,,2) & \xrightarrow{f} & K(Z\,,16) & \xrightarrow{h} & K(Z_2\,,16) \\ & & {\scriptstyle w}\downarrow & & {\scriptstyle \overline{w}}\downarrow \\ & & K_0 & \longrightarrow & K_0 \times K(Z_2\,, 17) \end{array}$$

Further, there is another lifting $\tilde h: K(Z\,, 2) \to \overline{E}$ of hf that has its H-deviation

$$D\tilde h: K(Z\,, 2) \times K(Z\,, 2) \to K(Z_2\,, 16)$$

given by $[D\tilde h] = \iota_2^4 \otimes \iota_2^4$. This holds because

$$B(hf)^* B\overline{w}^*(\iota_{18}) = \mathrm{Sq}^{9,4,2}\iota_3 = (\mathrm{Sq}^{4,2}\iota_3)^2$$

and because $B(hf)^* B\overline{w}^*$ is zero on the fundamental classes in K_0.

In $PH^*(\overline{E})$ there exist elements $\overline{v}_j$ such that $\bar h^*(\overline{v}_j) = v_j$. The components of $\overline{v}_j$ in $H^*(K(Z_2\,, 16))$ are:

(1.5) $\overline{v}_3$: $\mathrm{Sq}^3\iota_{16}$; $\overline{v}_7$: 0; $\overline{v}_8$: $\mathrm{Sq}^8\iota_{16}$; $\overline{v}_{10}$: 0;

$\overline{v}_{12}$: $\mathrm{Sq}^{8,4}\iota_{16}$; $\overline{v}_{13}$: $\mathrm{Sq}^{4,9}\iota_{16}$; $\overline{v}_{14}$: 0; $\overline{v}_{15}$: $\mathrm{Sq}^{15}\iota_{16}$.

It follows that $\tilde{h}^*(\overline{v}_j)$ is primitive except for $\tilde{h}^*(\overline{v}_8)$ which has

$$\overline{\Delta}\tilde{h}^*(\overline{v}_8) = \iota_2^8 \otimes \iota_2^4 + \iota_2^4 \otimes \iota_2^8 = \overline{\Delta}(\iota_2^{12}).$$

Because $H^*(K(Z, 2))$ is trivial in odd degrees and is Z_2 in even degrees, we conclude

$$\tilde{h}^*(\overline{v}_j) = 0 \quad \text{if } j \neq 8, \quad \text{and}$$
$$\tilde{h}^*(\overline{v}_8) = \iota_2^{12}.$$

Therefore if

$$\overline{z} = \mathrm{Sq}^{11,2}\overline{v}_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})\overline{v}_7 + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})\overline{v}_8 + \mathrm{Sq}^{4,2}\overline{v}_{10} + \mathrm{Sq}^4\overline{v}_{12} + \mathrm{Sq}^3\overline{v}_{13} + \mathrm{Sq}^2\overline{v}_{14} + \mathrm{Sq}^1\overline{v}_{15},$$

it follows that

$$\tilde{h}^*(\overline{z}) = \mathrm{Sq}^8\tilde{h}^*(\overline{v}_8) = \mathrm{Sq}^8(\iota_2^{12}) = \iota_2^{16} = \mathrm{Sq}^{16}(\iota_2^8). \tag{1.6}$$

Now $\tilde{h}$ and $\overline{h}\tilde{f}$ both lift hf, so

$$\overline{h}\tilde{f} = \tilde{h} + \overline{j}F,$$

for some

$$F = (F_1, F_2): K(Z, 2) \to \Omega K_0 \times K(Z_2, 16).$$

But ΩK_0 is odd-dimensional with the exception of $K(Z_2, 22)$. If $\tilde{f}$ is altered by F_1, then

$$\overline{h}\tilde{f} = \tilde{h} + \overline{j}F_2,$$

and $[F_2] = d\iota_2^8$, $d \in Z_2$.

Using (1.5) we calculate that for all j

$$F_2^*\overline{j}^*(\overline{v}_j) = 0, \quad \text{and hence} \quad \tilde{h}^*(\overline{v}_j) = \tilde{f}^*(v_j). \tag{1.7}$$

Therefore

$$\tilde{f}^*(z) = \tilde{h}^*(\overline{z}) = \mathrm{Sq}^{16}(\iota_2^8).$$

It follows that $c = 1$. □

THEOREM 1.2. $QH^{23}(X) = 0$.

Proof. By the restrictions on the degrees of generators of $H^*(X)$, $H^i(X \wedge X) = 0$ for $i = 7, 15$, and 31. So by the Steenrod connections, all generators in degrees less than 63 may be chosen to be primitive. Further, in degrees < 40, $H^*(X)$ is an exterior algebra in which

$$QH^k(X) = 0, \quad k \neq 7, 15, 23, 27, 29, 31, 39. \tag{1.8}$$

The lowest-dimensional possible non-trivial Steenrod operation is Sq^8 acting on $H^{15}(X)$. So let $x_{23} = \mathrm{Sq}^8 x_{15} \neq 0$. By (1.8), Sq^2, Sq^8, and $\mathrm{Sq}^{8,4}$ are all zero on x_{23}, and $\mathrm{Sq}^7 x_{23} = \mathrm{Sq}^{15} x_{15}$, which is zero by hypothesis. Thus the factorization (1.1) applies to $\mathrm{Sq}^{16} x_{23}$. We now construct the universal example.

Let $p_0\colon E_0 \to K(Z, 23)$ be the fiber of the map

$$g\colon K(Z, 23) \to K(Z_2; 25, 30, 31, 35)$$

given by

$$\begin{aligned} g^*(\iota_{25}) &= \mathrm{Sq}^2(\iota_{23}), \\ g^*(\iota_{30}) &= \mathrm{Sq}^7(\iota_{23}), \\ g^*(\iota_{31}) &= \mathrm{Sq}^8(\iota_{23}), \quad \text{and} \\ g^*(\iota_{35}) &= \mathrm{Sq}^{8,4}(\iota_{23}). \end{aligned}$$

Next, define $p_1\colon E_1 \to E_0$ to be the fiber of the map

$$g_0\colon E_0 \to K_0 = K(Z_2; 26, 30, 33, 35, 36, 37, 38; \overline{32}, \overline{33}, \overline{35})$$

given by

$$g_0^*(\iota_{23+m}) = v_m \qquad (m \neq 8),$$

and

$$g_0^*(\bar\iota_{31+k}) = \mathrm{Sq}^k v_8 \qquad (k = 1, 2, 4).$$

Consider the element in $H^{47}(K_0)$:

$$\begin{aligned} \chi = {} & \mathrm{Sq}^8[\mathrm{Sq}^{11,2}\iota_{26} + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})\iota_{30} + \mathrm{Sq}^{4,2}\iota_{33} + \mathrm{Sq}^4\iota_{35} \\ & \qquad + \mathrm{Sq}^3\iota_{36} + \mathrm{Sq}^2\iota_{37} + \mathrm{Sq}^1\iota_{38}] \\ & + \mathrm{Sq}^{15}\bar\iota_{32} + (\mathrm{Sq}^{14} + \mathrm{Sq}^{10,4})\bar\iota_{33} + \mathrm{Sq}^{12}\bar\iota_{37}. \end{aligned}$$

Applying g_0^* to χ, we get

$$\begin{aligned} g_0^*(\chi) &= \mathrm{Sq}^8[\mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ &\qquad + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15}] \\ &\quad + (\mathrm{Sq}^{15,1} + \mathrm{Sq}^{14,2} + \mathrm{Sq}^{12,4} + \mathrm{Sq}^{10,4,2})v_8 \\ &= \mathrm{Sq}^8[\mathrm{Sq}^{11,2}v_3 + (\mathrm{Sq}^{7,2} + \mathrm{Sq}^{6,3})v_7 + \mathrm{Sq}^{4,2}v_{10} + \mathrm{Sq}^4 v_{12} \\ &\qquad + \mathrm{Sq}^3 v_{13} + \mathrm{Sq}^2 v_{14} + \mathrm{Sq}^1 v_{15} + (\mathrm{Sq}^8 + \mathrm{Sq}^{6,2})v_8] \\ &= \mathrm{Sq}^8\mathrm{Sq}^{16} p_0^*(\iota_{23}) \\ &= (\mathrm{Sq}^{24} + \mathrm{Sq}^{23,1} + \mathrm{Sq}^{22,2} + \mathrm{Sq}^{20,4})p_0^*(\iota_{23}). \end{aligned}$$

The values of the last three operations on ι_{23} are in the kernel of p_0^*. So

$$g_0^*(\chi) = \mathrm{Sq}^{24} p_0^*(\iota_{23}).$$

Hence there exists an element $v \in H^{46}(E_1)$ such that $\overline{\Delta}(v) = p_1^* p_0^*(\iota_{23}) \otimes p_1^* p_0^*(\iota_{23})$ and $j_1^*(v) = \sigma^*(\chi)$, where j_1 is the fiber of p_1.

We now need to map X into E_1. Let $f\colon X \to K(Z, 23)$ be such that $f^*(\iota_{23}) = x_{23}$. We remark that f can be chosen to be an H-map, since $H^{23}(X \wedge X; Z) = 0$. Since the composition $g \circ f$ is nullhomotopic, there exists a lifting $f_0\colon X \to E_0$ of f. The H-deviation of f_0 factors through j_0, the fiber of p_0, say $Df_0 = j_0 \circ \widetilde{D}_0$. The map $\widetilde{D}_0$ corresponds to a set of classes in $H^k(X \wedge X)$, $k = 24, 29, 30$, and 34.

We shall work in P_2X, the projection plane of X. Recall that there is an exact triangle [3]

$$(1.9)\qquad \begin{array}{ccc} H^*(P_2X) & \longrightarrow & IH^*(X) \\ & \nwarrow \quad \swarrow & \\ & IH^*(X) \otimes IH^*(X) & \end{array}$$

that relates P_2X to X. This implies that

$$(1.10)\qquad H^k(P_2X) = 0 \qquad (17 \leq k \leq 22).$$

Let $u_{16} \in H^{16}(P_2X)$ correspond to x_{15} and set $u_{24} = \mathrm{Sq}^8 u_{16}$. By (1.10) and the Adem relations, Sq^2, Sq^8, and $\mathrm{Sq}^{8,4}$ are all zero on u_{24}. So by [3], the components of $\widetilde{D}_0$ in degrees 24, 30, and 34 are all zero. Thus $\widetilde{D}_0 \in H^{29}(X \wedge X)$, so it is a sum of terms of the form $x_7 \otimes x_7' x_{15}$, $x_7 x_7' \otimes x_{15}$, and twists of these terms. Consider the elements $f_0^* \circ g_0^*(\iota)$, where ι is one of the fundamental classes of K_0. We have

$$\overline{\Delta}(f_0^* \circ g_0^*(\iota)) = (Df_0)^* g_0^*(\iota) = \widetilde{D}_0^* \circ j_0^* \circ g_0^*(\iota).$$

Referring to the matrix relation (1.2), we see that the only possible non-zero values can be when $\iota = \iota_{37}$, when

$$\widetilde{D}_0^* \circ j_0^* \circ g_0^*(\iota) = \mathrm{Sq}^8 \widetilde{D}_0^*(\iota_{29}).$$

Hence the images under $f_0^* \circ g_0^*$ of all the fundamental classes of K_0, with the possible exception of ι_{37}, are primitive, so for degree reasons they must be zero. We might possibly have

$$f_0^* \circ g_0^*(\iota_{37}) = \sum x_{i,7} x'_{i,7} x_{i,23}.$$

But since $\mathrm{Sq}^8\colon H^{15}(X) \to H^{23}(X)$ is onto, we may alter the lift f_0 by the action of the fiber on the map $\tilde{f}\colon X \to K(Z_2, 29)$ given by

$$\tilde{f}^*(\iota_{29}) = \sum x_{i,7} x'_{i,7} x_{i,15}$$

so as to make, for the altered f_0, $f_0^* \circ g_0^*(\iota_{37}) = 0$. Thus there exists a lifting $f_1\colon X \to E_1$.

We now consider the element $f_1^*(v) \in H^{46}(X)$. We have

$$\overline{\Delta}(f_1^*(v)) = (f_1^* \otimes f_1^*)(\overline{\Delta}v) + (Df_1)^*(v) = x_{23} \otimes x_{23} + (Df_1)^*(v).$$

There is no term in $H^*(X)$ whose coproduct has $x_{23} \otimes x_{23}$ as a summand. Now

$$Df_1 = \theta + j_1 \circ \widetilde{D}_1,$$

where $\theta\colon X \wedge X \to E_1$ is a map given by applying the Cartan formula, Theorem 3.1 of [7], to Df_0. The map θ factors through cohomology classes in $H^*(X \wedge X)$ of which one factor is a primary or secondary operation applied to a decomposable element, and, by the Cartan formulae for primary and secondary operations, such operations cannot hit x_{23}. Also, $(j_1 \circ \widetilde{D}_1)^*(v)$ lies in the image of Steenrod operations applied to elements of degrees $\neq 30$ or 38, so $x_{23} \otimes x_{23}$ cannot be in this image. Thus Sq^8 is identically zero on $H^{15}(X)$ and hence $QH^{23}(X) = 0$. □

COROLLARY 1.3. *$H^*(X)$ is an exterior algebra on generators concentrated in degrees of the form $2^d - 1$ for $d \geq 3$. Further, the action of the Steenrod algebra on $H^*(X)$ is trivial.*

Proof. By the Steenrod connections, any element of $QH^*(X)$ not in a degree of the form $2^d - 1$ lies in the image of Steenrod operations applied to generators in degrees of the form $2^d - 1$. By Theorem 1.2 and the Steenrod Connections, it follows that

$$\mathrm{Sq}^{2^i} QH^{2^d-1}(X) = 0 \quad \text{for } i = 0, 1, 2, 3.$$

By [**1**], Sq^{2^i} factors through secondary operations for $i \geq 4$ if x_{2^d-1} lies in the kernel of Sq^{2^j} for $0 \leq j \leq i - 1$.

So consider the first nontrivial Steenrod operation, say $\mathrm{Sq}^{2^i} x_{2^d-1}$. By the Cartan formula, $\mathrm{Sq}^{2^i} x_{2^d-1}$ is primitive, so it must be a generator. By the Steenrod connections we must have $i = d-1$. By Theorem 1.2 we must have $d \geq 5$, so $i \geq 4$. But this implies $\mathrm{Sq}^{2^{d-1}} x_{2^d-1}$ is in the image of Steenrod operations of lower degree, which cannot happen. Thus the action of the Steenrod algebra on $H^*(X)$ is trivial. Hence $H^*(X)$ is an exterior algebra on generators in degrees of the form $2^d - 1$, $d \geq 3$. □

2. $H^{15}(X)$.

THEOREM 2.1. $H^{15}(X) = 0$.

Proof. Let x_{15} be a nonzero element of $H^{15}(X)$. We define a cohomology operation as follows. Consider the diagram:

$$\begin{array}{ccccc} & & E_2 & & \\ & \overset{f_2}{\nearrow} & \downarrow{\scriptstyle p_2} & & \\ & & E_1 & \xrightarrow{g_2} & K(Z_2;\,18,\,19,\,22,\,23,\,24,\,30) \\ & \overset{f_1}{\nearrow} & \downarrow{\scriptstyle p_1} & & \\ X & \xrightarrow{f} & K(Z,\,15) & \xrightarrow{g_1} & K(Z_2;\,17,\,19,\,23) \end{array} \tag{2.1}$$

which is associated with a factorization of Sq^{16} as

$$\mathrm{Sq}^{16} = \sum \alpha_{ij}\varphi_{ij}$$

in which the α_{ij} are Steenrod operations and the φ_{ij} are the secondary operations of Adams, **[1]**, and is constructed as follows.

The map g_1 is given by the formulas

$$g_1^*(\iota_{15+2^k}) = \mathrm{Sq}^{2^k}(\iota_{15}), \qquad k = 1, 2, 3.$$

The map g_2 is given by the formulas

$$g_2^*(\iota_{14+2^i+2^j}) = v_{ij},$$

where v_{ij} is an element in $H^*(E_1)$ that represents the secondary operation φ_{ij}.

The map f represents the element x_{15}. The lift f_1 exists since all Steenrod operations are zero on x_{15}, by Corollary 1.3. Now the H-deviation of f_1 factors through the fiber of p_1, namely $K(Z_2;\,16,\,18,\,22)$. Hence the reduced coproducts of the $f_1^*(v_{ij})$ are in the image of Steenrod operations, which are all zero. Hence the $f_1^*(v_{ij})$ are primitive, so they are all zero. Therefore the lift f_2 exists.

In $H^{30}(E_2)$ there is an element v whose reduced coproduct is $p_2^*p_1^*(\iota_{15}) \otimes p_2^*p_1^*(\iota_{15})$. We shall show that the reduced coproduct of $f_2^*(v)$ contains a term $x_{15} \otimes x_{15}$, which will be a contradiction. Let us write the factorization of the H-deviation Df_1 of f_1 as

$$Df_1 = \widetilde{D} \circ j_1.$$

The map $\widetilde{D}$ determines elements in degrees 16, 18, and 22 of $X \wedge X$. Checking possibilities, we see that the components in degrees 16 and

18 are zero, while we may express the component in degree 22 as

$$\widetilde{D}^*(\iota_{22}) = \sum x_{7,i} \otimes x_{15,i}$$

for elements $x_{7,i}$ and $x_{15,i}$ in degrees 7 and 15 respectively. Use of the Cartan formula, [7], now enables us to express the H-deviation of f_2 as the sum of terms in the image of Steenrod operations (which are all zero) together with terms of the form

$$\psi_i(x_{7,i}) \otimes x_{15,i},$$

where the ψ_i are secondary operations. We need to check that it cannot happen for $x_{15,i}$ and $\psi_i(x_{7,i})$ both to be x_{15}. To determine the secondary operations involved here, we may consider the diagram

$$\begin{array}{ccccc} & & G & & \\ & \overset{h_1}{\nearrow} & \downarrow \pi & & \\ X & \xrightarrow{h} & K(Z, 7) & \xrightarrow{\mathrm{Sq}^2, \mathrm{Sq}^4} & K(Z_2; 9, 11). \end{array}$$

Using either the Serre or the Eilenberg-Moore spectral sequence we see that a basis for $H^{15}(G)$ is given by elements in the image of Steenrod operations together with an element $\tilde{w}_{0,3}$ that restricts to the fiber of π to be $(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1})\iota_{10}$. So we need to determine whether $h_1^*(\tilde{w}_{0,3})$ can be x_{15}.

For dimensional reasons, h_1 is an H-map. Hence it determines a map $\hat{h}_1$: $P_2X \to BG$, where BG denotes the classifying space of G. If $h_1^*(\tilde{w}_{0,3}) = x_{15}$, then $y_{16} = \hat{h}_1^*(B\tilde{w}_{0,3})$ is a representative in $H^*(P_2X)$ of the primitive class x_{15}. In **[10]**, Corollary 1.3, we derived the formula (in the cohomology of BG)

$$(B\pi^*(\iota_8))^3 \equiv \mathrm{Sq}^8(B\tilde{w}_{0,3}), \quad \text{modulo}\, \mathrm{Im}(\mathrm{Sq}^{12}, \mathrm{Sq}^{6,3}, \mathrm{Sq}^{4,2,1}).$$

In general, three-fold cup products in $H^*(P_2X)$ are all zero. By the hypotheses on X and (1.9), $H^*(P_2X) = 0$ in degrees 12, 15, and 17. So $\mathrm{Sq}^8\hat{h}_1^*(B\tilde{w}_{0,3}) = 0$. By **[13]**, $\mathrm{Sq}^8(y_{16}) = \sum y_{8,i}y_{16,i}$, where the $y_{8,i}$ and $y_{16,i}$ correspond to $x_{7,i}$ and $x_{15,i}$, respectively. So we obtain that $\psi_i(x_{7,i})$ cannot contain x_{15} as a summand; hence the reduced coproduct

$$\overline{\Delta}h_2^*(v) = x_{15} \otimes x_{15},$$

which, as stated above, is a contradiction. □

3. $QH^{2^k-1}(X)$**.** By Corollary 1.3 and Theorem 2.1, $H^*(X)$ is an exterior algebra on generators in degrees 7 and $2^d - 1$, for $d \geq 5$, and has trivial action of the Steenrod algebra.

THEOREM 3.1. *$QH^*(X)$ is concentrated in degree* 7.

Proof. Let $x = x_{2^k-1}$, $k \geq 5$, be a generator of lowest degree greater than seven. Let $\xi H^*(X)$ denote the image of the cup-squaring map $\xi(x) = x^2$. Since $H_*(X)$ is associative, we may assume by **[8]** that $\bar{\Delta} x \in \xi H^*(X) \otimes H^*(X)$, which is trivial since $\xi H^*(X) = 0$. Hence x may be chosen to be primitive. We shall construct an operation similar to that in the proof of Theorem 1.4. Consider the following diagram:

$$(3.1)\qquad \begin{array}{ccccc} & & E_2 & & \\ & {\scriptstyle f_2}\nearrow & \downarrow{\scriptstyle p_2} & & \\ & & E_1 & \xrightarrow{g_2} & K_2 \\ & {\scriptstyle f_1}\nearrow & \downarrow{\scriptstyle p_1} & & \\ X & \xrightarrow{f} & K(Z, 2^k_1) & \xrightarrow{g_1} & K_1, \end{array}$$

in which $K_1 = \prod_i K(Z_2; 2^k - 1 + 2^n)$, $1 \leq n \leq k-1$, and

$$g_1^*(\iota_{2^k-1+2^n}) = \mathrm{Sq}^{2^n} \iota_{2^k-1},$$

and in which $K_2 = \prod_{i,j} K(Z_2; 2^k - 2 + 2^i + 2^j)$, and g_2 represents the secondary operations φ_{ij} associated with a factorization of Sq^{2^k}.

By Corollary 1.3, all Steenrod equations vanish on x, so $g_1 f \simeq *$ and the lift f_1 exists.

We note that in degrees below $2^k - 1$, $H^*(X)$ is concentrated in degrees divisible by seven. Since x is primitive, f is an H-map. Therefore $D_{g_2 f_1}$ factors through the fiber of p_1. Hence the formula for the H-deviation of a composition yields that $D_{g_2 f_1}$ is in the image of primary operations in $H^*(X \wedge X)$, so it is zero by Corollary 1.3. Hence $g_2 f_1$ is represented by primitive elements of $H^*(X)$ in degrees not of the form $2^d - 1$. Since all primitives are concentrated in degrees of the form $2^d - 1$, $g_2 f_1$ is nullhomotopic, and the lift f_2 exists.

To simplify the situation, we loop the entire diagram to obtain

$$\begin{array}{ccccc} & & \Omega E_2 & & \\ & \nearrow^{\Omega f_1} & \downarrow{\scriptstyle \Omega p_2} & & \\ & & \Omega E_1 & \xrightarrow{\Omega g_2} & \Omega K_2 \\ & \nearrow^{\Omega f_2} & \downarrow{\scriptstyle \Omega p_1} & & \\ \Omega X & \xrightarrow{\Omega f} & K(Z\,,\,2^k-2) & \xrightarrow{\Omega g_1} & \Omega K_1. \end{array} \tag{3.2}$$

Note. The c-invariant was introduced in **[14]** as the obstruction to an H-map between two homotopy-commutative H-spaces preserving the homotopy-commutative structure. There are various choices for this invariant, which depend on the choice of homotopy realizing the H-map. It was observed that if Y and Z are H-spaces and h: $Y \to Z$ a map, then the composition

$$\sum \Omega Y \wedge \sum \Omega Y \xrightarrow{\varepsilon\wedge\varepsilon} Y \wedge Y \xrightarrow{Dh} Z \tag{3.3}$$

has as its double adjoint $\Omega Y \wedge \Omega Y \to \Omega Z$ a particular choice for the c-invariant $c(\Omega h)$. In the sequel we shall always make this choice for our c-invariants.

We have a suspension element v in $H^{2^{k+1}-1}(\Omega E_2)$ such that

$$c(v) = (\Omega(p_1p_2))^*\iota_{2^k-2} \otimes (\Omega(p_1p_2))^*\iota_{2^k-2}.$$

We shall consider the c-invariant of the element

$$(\Omega f_2)^*[v] \in H^{2^{k+1}-1}(\Omega X) = 0.$$

Let $u_{2^k-2} = \sigma^*(x_{2^k-1})$. Then, applying (3.3) to the formula for the H-deviation for a composition of maps, we obtain

$$0 = c((\Omega f_2)^*[v]) = u_{2^k-2} \otimes u_{2^k-2} + c(\Omega f_2)^*[v].$$

Since x_{2^k-1} is primitive, u_{2^k-2} is a c-class. Hence $c(\Omega f_1)$ factors as

$$\Omega X \wedge \Omega X \xrightarrow{\tilde{c}} \Omega^3 K_1 \to \Omega^2 E_1.$$

We have a commutative diagram

$$\begin{array}{ccc} & & \Omega^2 E_2 \\ & \nearrow^{c(\Omega f_2)} & \downarrow{\scriptstyle \Omega^2 p_2} \\ \Omega X \wedge \Omega X & \xrightarrow[c(\Omega f_1)]{} & \Omega^2 E_1 \end{array} \tag{3.4}$$

Now $c(\Omega f_1)$ is adjoint to

$$\begin{array}{ccccc} \sum \Omega X \wedge \sum \Omega x & \longrightarrow & \Omega K_1 & \longrightarrow & E_1, \\ & \searrow & \swarrow{\scriptstyle \widetilde{D}_1} & & \\ & X \wedge X & & & \end{array}$$

hence $[c(\Omega f_1)] \in (PH^*(\Omega X) \otimes PH^*(\Omega X))^{2^k+2^n-4}$.

According to [**6**], there is an isomorphism of coalgebras

$$\operatorname{Tor}_{H^*(X)}(Z_2, Z_2) \cong H^*(\Omega X).$$

It follows that $H^*(\Omega X)$ in degrees less than $2^k - 2$ is a divided polynomial coalgebra on primitive elements of degree 6. Therefore

(3.5) $[c\Omega f_1)] \in PH^6(\Omega X) \otimes PH^{2^k-2}(\Omega X) + PH^{2^k-2}(\Omega X) \otimes PH^6(\Omega X)$.

Further, the indecomposables of $H^*(\Omega X)$ in degrees less than $2^k - 2$ are concentrated in degrees of the form $3 \cdot 2^r$. But if $k > 4$, no Steenrod operation on an element in one of these degrees can hit an indecomposable in degree $2^k - 2$, so u_{2^k-2} is not in the image of the Steenrod algebra.

An analysis of the Cartan formula [7] for secondary operations applied to diagrams 3.4 and 3.5 yields that $u_{2^k-2} = \psi(u_6)$, where ψ is a secondary operation defined on 6-dimensional primitives in the kernel of all Steenrod operations. We proceed to study all such operations. Note that ψ has degree $2^k - 8$. The possibilities come from the suspension elements in $H^{2^k-2}(G)$, where G is the space defined as follows. Let G' be defined to be the fiber of the horizontal map g' in the diagram

$$\begin{array}{ccc} G' & & \\ \big\downarrow & & \\ K(Z, 2^k - 1) & \xrightarrow[g']{\mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^{k-1}}} & \Pi K(Z_2; 2^k - 1 + 2^n) \end{array}$$

Now set

$$G = \Omega^{2^k-7} G' \quad \text{and} \quad g = \Omega^{2^k-7} g'.$$

So G is fibered as $\pi \colon G \to K(Z, 6)$. We shall see that in $H^{2^k-2}(G)$, $\operatorname{im}(\sigma^*) \subset \overline{A(2)} \cdot H^*(G)$. For, if an element ψ of $H^{2^k-2}(G)$ is a stable operation, then by [1] ψ can be expressed as a sum

$$\psi = \sum \alpha_{ij} v_{ij},$$

in which the v_{ij} represent the operations ψ_{ij} applied to $\pi^*(\iota_6)$. We note that none of the v_{ij} occurs in degree 2^k-2.

If $v \in H^{2^k-2}(G)$ represents an unstable operation, then it must be in the image of $(\sigma^*)^N$ but not in the image of $(\sigma^*)^{N+1}$, for some N. Write $v = (\sigma^*)^N[\hat{v}]$, $\hat{v} \in H^{2^k-2+N}(B^N G)$. Since $\hat{v}$ is not a suspension, its a_m-obstruction **[12]** must be non-zero for some m. Such an obstruction must arise from having

$$\iota_{N+7}^m \in \mathrm{Im}(B^{N+1}g)^*$$

for some m of the form $m = 2^r$.

If $r > 1$, then $\iota_{N+7}^m = \mathrm{Sq}^{2^{r-1}(N+7)}\gamma\iota_{N+7}$, where

$$\gamma = \mathrm{Sq}^{2^{r-2}(N+7)} \cdots \mathrm{Sq}^{N+7}.$$

If $r = 1$, then $N = 2^k - 14$, so that $\iota_{N+7}^2 = \mathrm{Sq}^1\gamma\iota_{N+7}$, where $\gamma = \mathrm{Sq}^{2^k-8}$. In either case there is a relation

$$\gamma = \sum \alpha_n \mathrm{Sq}^{2^n}, \quad \alpha_n \in A(2),$$

so there exists an element $w \in H^{2^k-3}(G)$ that restricts to the fiber to be $\sum \alpha_n \iota_{2^n+5}$. Hence a representative of v is given by $\mathrm{Sq}^1 w$ if $r = 1$ and by $\mathrm{Sq}^{2^{r-1}(N+7)}w$ if $r > 1$.

Thus $\psi(u_6)$ must be in the image of the Steenrod operations. This implies that u_{2^k-2} lies in the image of Steenrod operations which is a contradiction. Since

$$\sigma^*\colon QH^{2^k-1}(X) \to PH^{2^k-2}(\Omega X)$$

is monic, we conclude that $QH^{2^k-1}(X) = 0$. □

Proof of the Main Theorem. We now know that $H^*(X)$ is an exterior algebra on seven-dimensional generators. If $H^*(X; Z)$ has odd torsion, then for some odd prime p, there is an even generator of the form $\beta_1 P^n x_{2n+1}$ by **[9]**. Applying the Bockstein spectral sequence, this yields an odd generator in the rational cohomology of degree $(2np+2)p^d - 1$ for $d \geq 1$. But

$$(2np+2)p^d - 1 > 7$$

so $H^*(X; Z)$ has no odd torsion. Hence it is torsion-free. Therefore

$$H^*(X; Z) \cong \Lambda(x_1, \ldots, x_r)$$

where $\deg(x_i) = 7$.

We now use the Hurewicz isomorphism to obtain our desired homotopy equivalence

$$S^7 \times \cdots \times S^7 \xrightarrow{f} X. \qquad \square$$

References

[1] J. F. Adams, *On the nonexistence of elements of Hopf invariant one*, Ann. of Math., **72** (1960), 20–103.

[2] W. Browder, *On differential Hopf algebras*, Trans. Amer. Math. Soc., **107** (1963), 153–178.

[3] W. Browder and E. Thomas, *On the projective plane of an H-space*, Illinois J. Math., **7** (1963), 492–502.

[4] D. Goncalves, Mod 2 *homotopy-associative H-spaces*, Geometric Applications of Homotopy Theory I, Proceedings Evanston, Springer Lec. Notes **657** (1977), 198–216.

[5] J. Hubbuck, *Generalized cohomology operations and H-spaces of low rank*, Trans. Amer. Math. Soc., **141** (1969), 335–360.

[6] R. Kane, *On loop spaces without p-torsion*, Pacific J. Math., **60** (1975), 189–201.

[7] J. Lin, *Higher order operations in the* mod 2 *cohomology of a finite H-space*, Amer. J. Math., **105** (1983), 855–937.

[8] ——, *Steenrod connections and connectivity in H-spaces*, Mem. Amer. Math. Soc., **369** (1987), 1–87.

[9] ——, *Torsion in H-spaces* II, Ann. of Math., **107** (1978), 41–88.

[10] J. Lin and F. Williams, *Two torsion and homotopy-associative H-spaces*, J. Math. Kyoto Univ., **30** (1990), 523–541.

[11] F. Sigrist and U. Suter, *Eine Anwendung der K-theorie in der Theorie der H-Raume*, Comment. Math. Helv., **47** (1972), 36–52.

[12] J. D. Stasheff, *Homotopy associativity of H-spaces*, I, II, Trans. Amer. Math. Soc., **108** (1963), 275–312.

[13] F. Williams, *The H-deviation of a lift*, Proc. Amer. Math. Soc., **104** (1988), 1291–1295.

[14] A. Zabrodsky, *Cohomology operations and homotopy commutative H-spaces*, in *The Steenrod Algebra and its Applications*, Springer Lecture Notes in Math., **168** (1970), 308–317.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92043

AND

NEW MEXICO STATE UNIVERSITY
LAS CRUCES, NM 88003