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ASYMPTOTIC BEHAVIOR OF EIGENVALUES FOR A CLASS OF PSEUDODIFFERENTIAL OPERATORS ON $\mathbf{R}^n$

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We consider a pseudodifferential operator P whose symbol has an asymptotic expansion by quasi homogeneous symbols and the principal symbol is degenerate on a submanifold. Under appropriate conditions, P has the discrete spectrum. Then we can get the asymptotic behavior of the counting function of eigenvalues of P with remainder estimate according to various cases.

0. Introduction. We consider the asymptotic behavior of eigenvalues for a class of pseudodifferential operators on $\mathbf{R}^n$ containing the Schrödinger operator with magnetic field:

$$(0.1)\quad \begin{aligned} p^w(x, D) &= H(a) + V(x) \\ &= \sum_{j=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x)\right)^2 + V(x) \qquad (i=\sqrt{-1}). \end{aligned}$$

Throughout this paper we assume that the magnetic potential $a(x)$ satisfies:

$$a(x) = (a_1(x), a_2(x), \dots, a_n(x)) \in C^\infty(\mathbf{R}^n; \mathbf{R}^n)$$

and the scalar potential $V(x)$ satisfies $V(x) \in C^\infty(\mathbf{R}^n; \mathbf{R})$. We regard $p^w(x, D)$ as a linear operator in $L^2(\mathbf{R}^n)$ with domain $C_0^\infty(\mathbf{R}^n)$. Under appropriate conditions, we shall see that $p^w(x, D)$ is essentially self-adjoint in $L^2(\mathbf{R}^n)$ and its self-adjoint extension P is semibounded from below and has a compact resolvent in $L^2(\mathbf{R}^n)$. Therefore the spectrum $\sigma(P)$ of P is discrete, that is, $\sigma(P)$ consists only of eigenvalues of finite multiplicity. Thus we can denote the eigenvalues with repetition according to multiplicity by: $\lambda_1 \le \lambda_2 \le \cdots$, $\lim_{k\to\infty}\lambda_k = \infty$. We consult the asymptotic behavior of the counting function $N_P(\lambda)$ of eigenvalues:

$$(0.2)\qquad N_P(\lambda) = \#\{j; \lambda_j \le \lambda\}.$$

In the special case $a(x) = 0$, i.e., $p^w(x, D)$ is of the form:

$$(0.3)\qquad p^w(x, D) = -\Delta + V(x),$$

if $V(x)$ satisfies $\lim_{|x|\to\infty} V(x) = \infty$, then it is well known that

$$N_P(\lambda) = (2\pi)^{-n}\,\mathrm{Vol}[(x, \xi);\, |\xi|^2 + V(x) < \lambda](1 + o(1)) \tag{0.4}$$

as $\lambda \to \infty$. In particular, Helffer and Robert **[8]** have obtained the asymptotic formula of $N_P(\lambda)$ for a class of quasi elliptic pseudodifferential operators containing the anharmonic oscillator: $V(x) = a|x|^{2k}$ in (0.3) (a real > 0, k integer ≥ 2). They have found not only the first term but also the following several terms of $N_P(\lambda)$. Aramaki **[3]** extended the result to the case containing the operator of the form, for example, $V(x) = x_1^2 + x_2^4 + ax_2^3$ (a real > 0) in $\mathbf{R}^2$.

For general $a(x)$ and $n = 3$, under the condition in (0.3), Combes-Schrader-Seiler **[5]** had the result

$$N_P(\lambda) = M(\lambda)(1 + o(1)) \quad \text{as } \lambda \to \infty \tag{0.5}$$

where

$$M(\lambda) = (2\pi)^{-3}\,\mathrm{Vol}\left[\left\{(x, \xi);\, \sum_{j=1}^{3} (\xi_j - a_j(x))^2 + V(x) < \lambda\right\}\right].$$

In this paper we shall consider a class of pseudodifferential operator $p^w(x, D)$ of the form (0.1) containing the case, for example,

$$a(x) = (bx_3^{k+1}, 0, 0), \qquad V(x) = (x_1^2 + x_2^2)^l + ax_3^{k+1} \tag{0.6}$$

(a real > 0, b real, l positive integer and k odd integer). For such an operator, we seek the asymptotic behavior of $N_P(\lambda)$ of more precise form than (0.5):

$$N_P(\lambda) = M(\lambda)(1 + O(\lambda^{-\delta})) \tag{0.7}$$

as $\lambda \to \infty$ for some $\delta > 0$. Thus we consider a pseudodifferential operator $p^w(x, D)$ of order m with Weyl symbol $p(x, \xi)$ which has an asymptotic expansion by the quasi homogeneous functions:

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-jT/2}(x, \xi).$$

Such operators are treated by **[3]** in which he considered the case where $p^w(x, D)$ is quasi elliptic, i.e., $p_m(x, \xi) \neq 0$ for $(x, \xi) \neq 0$. In the present paper, we treat the case where $p_m(x, \xi)$ is degenerate on some closed submanifold in $\mathbf{R}^{2n}$. Under a suitable hypoelliptic condition, we shall get the asymptotic formula similar to (0.7).

The plan of this paper is as follows. In §1, we give the precise definition of the operators mentioned as above and give some hypotheses.

In §2, we construct the parametrices of $P-\zeta I$ for some $\zeta \in \mathbf{C}$ where I denotes the identity operator in $L^2(\mathbf{R}^n)$. Section 3 is devoted to the construction of complex powers P^z $(z \in \mathbf{C})$ of P. If the real part $\operatorname{Re} z$ of z is negative and sufficiently small, P^z is of trace class and the trace $\operatorname{Tr}[P^z]$ has a meromorphic extension $Z_P(z)$ to $\mathbf{C}$. Thus §4 is devoted to the study of the singularity of $Z_P(z)$. In §5 we examine asymptotic behavior of eigenvalues with the remainder using the technique of Aramaki [4]. Finally §6 gives an example which illustrates our theory.

1. Definitions of operators and some hypotheses. In this section we introduce some classes of pseudodifferential operators on $\mathbf{R}^n$ and give our hypotheses.

Throughout this paper, fix a multi-index $(h, k) = (h_1, h_2, \ldots, h_n, k_1, k_2, \ldots, k_n)$ such that $h_j, k_j \geq 1$, $h_j+k_j > T$ for $j = 1, 2, \ldots, n$ and put

$T =$ the least common multiple of $\{h_1, h_2, \ldots, h_n, k_1, k_2, \ldots, k_n\}$,

$$r(x, \xi) = \left[\sum_{j=1}^{n}\{|x_j|^{2T/h_j} + |\xi_j|^{2T/k_j}\}\right]^{1/(2T)}$$

for $(x, \xi) = (x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^{2n}$. Then we consider a symbol $p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ satisfying:

(1.1) There exists a sequence of functions $\{p_{m-jT/2}(x, \xi)\}_{j=0,1,\ldots}$ where $p_{m-jT/2}(x, \xi)$ are C^∞ functions in $\mathbf{R}^{2n}\backslash 0$ and quasi homogeneous of degree $m - jT/2$ of type (h, k) such that

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-jT/2}(x, \xi).$$

Here the quasi homogeneity of $p_{m-jT/2}$ of degree $m - jT/2$ of type (h, k) means that:

$$p_{m-jT/2}(\lambda^h \cdot x, \lambda^k \cdot \xi) = \lambda^{j-mT/2} p_{m-jT/2}(x, \xi)$$

for all $\lambda > 0$ and $(x, \xi) \in \mathbf{R}^{2n}\backslash 0$ where

$$\lambda^h \cdot x = (\lambda^{h_1} x_1, \ldots, \lambda^{h_n} x_n) \quad \text{and} \quad \lambda^k \cdot \xi = (\lambda^{k_1}\xi_1, \ldots, \lambda^{k_n}\xi_n).$$

Then the meaning of the asymptotic sum in (1.1) is as follows: For any integer $N \geq 1$ and multi-indices α, β, there exists a constant

$C_{\alpha\beta N} > 0$ such that

$$\left| D_x^\alpha D_\xi^\beta \left[p(x,\xi) - \sum_{j=0}^{N-1} p_{m-jT/2}(x,\xi) \right] \right| \le C_{\alpha\beta N} r(x,\xi)^{m-(NT/2)-\langle\alpha,h\rangle-\langle\beta,k\rangle}$$

for all $(x,\xi) \in \mathbf{R}^{2n}$ such that $r(x,\xi) \ge 1$ where $\langle\alpha,h\rangle = \sum_{j=1}^n \alpha_j h_j$ for multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $h = (h_1, h_2, \dots, h_n)$ as above (cf. Robert [**9**]).

Next we define a pseudodifferential operator P with the Weyl symbol $p(x,\xi)$ as above:

(1.2) $$p^w(x,D)u(x) = (2\pi)^{-n} \int\int e^{i(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi,$$

for all $u \in S(\mathbf{R}^n)$ which denotes the totality of rapidly decreasing C^∞ functions and $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$.

Our first assumption is:

(H.1) $p(x,\xi)$ is a real valued function on $\mathbf{R}^{2n}$.

Then it is well known that the operator $p^w(x,D)$ defined by (1.2) is formally self-adjoint, i.e., for all $u, v \in S(\mathbf{R}^n)$,

$$(p^w(x,D)u, v) = (u, p^w(x,D)v)$$

where (u,v) denotes the usual inner product of u and v in $L^2(\mathbf{R}^n)$.

Now we shall consider the operator $p^w(x,D)$ whose principal symbol $p_m(x,\xi)$ is non-negative and degenerate on some submanifold in $\mathbf{R}^{2n}\backslash 0$. In order to do so, let Σ_1 and Σ_2 be smooth closed quasi conic submanifolds of codimension d_1 and d_2 in $\mathbf{R}^{2n}\backslash 0$ respectively such that $d_1 + d_2 < 2n$. Here quasi conicity of Σ_i means that $(x,\xi) \in \Sigma_i$ implies $(\lambda^h \cdot x, \lambda^k \cdot \xi) \in \Sigma_i$ for any $\lambda > 0$.

The second assumption is:

(H.2) Σ_1 and Σ_2 intersect transversally. That is to say, $\Sigma \equiv \Sigma_1 \cap \Sigma_2$ is a closed quasi conic submanifold such that for every $\rho \in \Sigma$, the tangent space $T_\rho\Sigma$ of Σ at ρ is the intersection of $T_\rho\Sigma_i$ $(i = 1, 2)$: $T_\rho\Sigma = T_\rho\Sigma_1 \cap T_\rho\Sigma_2$.

Then the normal space $N_\rho\Sigma$ of Σ at ρ is identified with the direct sum of $N_\rho\Sigma_i$ $(i = 1, 2)$: $N_\rho\Sigma \equiv T_\rho\mathbf{R}^{2n}/T_\rho\Sigma = N_\rho\Sigma_1 \oplus N_\rho\Sigma_2$ (direct sum).

DEFINITION 1.1. Let m be a positive number, l positive integer and M non-negative integer. Then the space $\widetilde{S}^{m,M}_{(h,k;l)}$ is the set of all symbols $p(x, \xi)$ having an asymptotic expansion of type (1.1) and satisfying the following (1.3) and (1.4):

$$\Sigma = \{(x, \xi) \in \mathbf{R}^{2n}\backslash 0 ; p_m(x, \xi) = 0\}. \tag{1.3}$$

There exists a constant $C > 0$ such that

$$\frac{|p_{m-jT/2}(x, \xi)|}{r(x, \xi)^{m-jT/2}} \le C d_\Sigma(x, \xi)^{M-j} \tag{1.4}$$

for $j = 0, 1, \dots, M$ where

$$d_{\Sigma_i}(x, \xi) = \inf\left\{\left[\sum_{j=1}^{n}\left(\left(\frac{x_j}{r(x, \xi)^{h_j}} - y_j\right)^2 + \left(\frac{\xi_j}{r(x, \xi)^{k_j}} - \eta_j\right)^2\right)\right]^{1/2} ; (y, \eta) \in \Sigma_i\right\},$$

$i = 1, 2$ and

$$d_\Sigma = \{d_{\Sigma_1}(x, \xi)^2 + d_{\Sigma_2}(x, \xi)^{2l}\}^{1/2}.$$

We assume the following regular degeneracy of the principal symbol:

(H.3) There exists a constant $C > 0$ such that

$$p_m(x, \xi) \ge C r(x, \xi)^m d_\Sigma(x, \xi)^M.$$

Now for every $\rho \in \Sigma$ and $j = 0, 1, \dots, M$, we can define multilinear forms $\check{p}_{m-jT/2}(\rho)$ on $N_\rho\Sigma$ which may be identified with $\mathbf{R}^{d_1} \times \mathbf{R}^{d_2}$: For $X_1, \dots, X_{M-j} \in N_\rho\Sigma$,

$$\check{p}_{m-jT/2}(\rho)(X_1, \dots, X_{M-j}) = \frac{1}{(M-j)!}(\widetilde{X}_1 \cdots \widetilde{X}_{M-j} p_{m-jT/2})(\rho)$$

where $\widetilde{X}_j$ is a vector field extending X_j to a neighborhood of ρ. Then it is clear from (1.4) that $\check{p}_{m-jT/2}(\rho)$ is independent of the choice of extension $\widetilde{X}_j$ of X_j. Furthermore we define

$$\tilde{p}_{m-jT/2}(\rho, X) = \check{p}_{m-jT/2}(\rho)(X, \dots, X).$$

If we write $X = (X_1, X_2) \in N_\rho\Sigma = N_\rho\Sigma_1 \oplus N_\rho\Sigma_2$, then it follows from (1.4) that

$$\tilde{p}_{m-jT/2}(\rho, X) = \sum_{|\alpha_1| + |\alpha_2|/l = M-j} \frac{1}{\alpha_1!\alpha_2!}(\widetilde{X}_1^{\alpha_1}\widetilde{X}_2^{\alpha_2} p_{m-jT/2})(\rho).$$

Thus we define a form $\tilde{p}(\rho, X)$ on $N_\rho\Sigma$ and the set Γ_ρ $(\rho \in \Sigma)$ as follows:

$$\tilde{p}(\rho, X) = \sum_{j=0}^{M} \tilde{p}_{m-jT/2}(\rho, X),$$
$$\Gamma_\rho = \{\tilde{p}(\rho, X); X \in N_\rho\Sigma\}.$$

If we note that $\tilde{p}(\lambda\rho, X) = \lambda^{m-MT/2}\tilde{p}(\rho, \lambda^{T/2}X_1, \lambda^{T/(2l)}X_2)$ for $\lambda > 0$, we see that $\Gamma_{\lambda\rho} = \lambda^{m-MT/2}\Gamma_\rho$ (cf. Helffer **[6]**).

Moreover we assume the following:

(H.4) For all $\rho \in \Sigma$, Γ_ρ does not meet the origin, i.e., $\Gamma_\rho \cap \{0\} = \varnothing$.

(H.5) $m > MT/2$.

Under the above hypotheses (H.1) $\sim$ (H.4), $p^w(x, D)$ is hypoelliptic with loss of $MT/2$ derivatives. Therefore if we define an operator P_0 on $L^2(\mathbf{R}^n)$ with definition domain $D(P_0) = S(\mathbf{R}^n)$ so that $P_0 u = p^w(x, D)u$ for $u \in D(P_0)$, then P_0 is essentially self-adjoint. If we also assume (H.5) in addition to (H.1) $\sim$ (H.4), then the closure P of P_0 has a compact resolvent and the spectrum consisting only of eigenvalues of finite multiplicity. Here we note that the definition domain of P is $D(P) = \{u \in L^2(\mathbf{R}^n); p^w(x, D)u \in L^2(\mathbf{R}^n)\}$. Moreover by (H.3), P is semi-bounded from below, i.e., there exists a real number C such that for all $u \in D(P)$, $((P + C)u, u) \geq 0$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$, be the sequence of eigenvalues with repetition according to multiplicity and $N_P(\lambda)$ be the counting function of eigenvalues as in the introduction.

Finally, in our arguments, we may assume:

(H.6) P is positively definite, i.e., $\lambda_1 > 0$.

Now let $\rho \in \Sigma$. Then we can choose a local coordinate system $w = (u_1, u_2, v, r)$ in a quasi conic neighborhood W of ρ where $u_1 = (u_{11}, \dots, u_{1d_1})$, $u_2 = (u_{21}, \dots, u_{2d_2})$, $v = (v_1, \dots, v_{2n-d_1-d_2-1})$ such that u_{ij} $(i = 1, \dots, d_i, i = 1, 2)$ and v_k $(k = 1, \dots, 2n - d_1 - d_2 - 1)$ are quasi homogeneous functions of degree 0 with du_{ij}, dv_k being linearly independent and $\Sigma_i = \{u_{i1} = \cdots = u_{id_i} = 0\}$ $(i = 1, 2)$.

Then we must define a micro-local symbol class containing $\widetilde{S}^{m,M}_{(h,k;l)}$.

Definition 1.2. Let m, $M \in \mathbf{R}$, W, w be as above. Then the space $S^{m,M}_{(h,k;l)}(W, \Sigma)$ is the set of all $a(w) \in C^\infty(W)$ satisfying: For any integer $p \geq 0$ and multi-indices $(\alpha_1, \alpha_2, \beta)$, there exists a constant $C > 0$ such that

$$\left| \left(\frac{\partial}{\partial u_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial u_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial v}\right)^{\beta} \left(\frac{\partial}{\partial r}\right)^{p} a(w) \right| \leq C r^{m-p} \rho_\Sigma^{M-|\alpha_1|-|\alpha_2|/l}$$

where $\rho_\Sigma = (d_\Sigma^2 + r^{-T})^{1/2}$. Note that the symbol class is the Fréchet space with the usual semi-norms.

The following five propositions follow from routine considerations and so we omit the proofs (cf. Aramaki **[2]**, **[3]** and Helffer-Nourrigat **[7]**).

PROPOSITION 1.3. *Let X be a vector field with C^∞ coefficients which are quasi homogeneous of degree 0 on $T^*\mathbf{R}^n$. Then we have:*

(i) *X is a continuous linear mapping from $S^{m,M}_{(h,k;l)}(W, \Sigma)$ to $S^{m,M-1}_{(h,k;l)}(W, \Sigma)$.*

(ii) *If X is tangent to Σ_1, then X is a continuous linear mapping from $S^{m,M}_{(h,k;l)}(W, \Sigma)$ to $S^{m,M-1/l}_{(h,k;l)}(W, \Sigma)$.*

(iii) *If X is tangent to Σ_1 and Σ_2, then X is a continuous linear mapping from $S^{m,M}_{(h,k;l)}(W, \Sigma)$ to $S^{m,M}_{(h,k;l)}(W, \Sigma)$.*

PROPOSITION 1.4. *We have an inclusion: For any $q \geq 0$,*

$$S^{m,M}_{(h,k;l)}(W, \Sigma) \subset S^{m+q/2, M+q/T}_{(h,k;l)}(W, \Sigma).$$

PROPOSITION 1.5. *If M is a non-negative integer, then we have*

$$\widetilde{S}^{m,M}_{(h,k;l)} \subset S^{m,M}_{(h,k;l)}(\mathbf{R}^{2n}, \Sigma).$$

PROPOSITION 1.6. *If*

$$p_i \in S^{m_i, M_i}_{(h,k;l)}(W, \Sigma)$$

for $i = 1, 2$, then we have

$$p_1 \# p_2 \in S^{m_1+m_2, M_1+M_2}_{(h,k;l)}(W, \Sigma)$$

where

$$(1.5) \qquad p_1 \# p_2 \sim \sum_{k=0}^{\infty} 2^{-k} \sum_{|\alpha+\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial_\xi^\alpha D_x^\beta p_1 \partial_\xi^\beta D_x^\alpha p_2.$$

PROPOSITION 1.7. *Assume that* $p \in S^{m,M}_{(h,k;l)}(W, \Sigma)$ *satisfies* $|p| \geq Cr^m \rho_\Sigma^M$ *in* W *for a constant* $C > 0$. *Then we have*

$$p^{-1} \in S^{-m,-M}_{(h,k;l)}(W, \Sigma).$$

2. Construction of parametrices. In this section we shall construct the parametrices of $p^w(x, D) - \zeta I$ for some $\zeta \in \mathbf{C}$. For this purpose, let $\rho \in \Sigma$. As in §1, we can choose a local coordinate system $w = (u_1, u_2, v, r)$ in a quasi conic neighborhood W of ρ where $u_1 = (u_{11}, \dots, u_{1d_1})$, $u_2 = (u_{21}, \dots, u_{2d_2})$, $v = (v_1, \dots, v_{2n-d_1-d_2-1})$ such that u_{ij} $(i = 1, \dots, d_i, i = 1, 2)$ and v_k $(k = 1, 2, \dots, 2n - d_1 - d_2 - 1)$ are quasi homogeneous functions of degree 0 with du_{ij}, dv_k being linearly independent and $\Sigma_i = \{u_{i1} = \cdots = u_{id_i} = 0\}$, $(i = 1, 2)$. In order to construct parametrices for $p^w(x, D) - \zeta I$, we must also define a symbol class with a parameter ζ.

DEFINITION 2.1. Let $\rho \in \Sigma$, W be a quasi conic neighborhood of ρ having a local coordinate system (u_1, u_2, v, r) as above and Λ an open set in the complex plane $\mathbf{C}$ and $s, t \in \mathbf{R}$. Then the class $S^{s,t}_{(h,k;l)}(W, \Sigma, \Lambda)$ is the set of all C^∞ functions $a(w, \zeta)$ on $W \times \Lambda$ satisfying the following (i), (ii) and (iii):

(i) For any $\zeta \in \Lambda$,

$$a(w, \zeta) \in S^{s,t}_{(h,k;l)}(W, \Sigma).$$

(ii) For any $w \in W$, $a(w, \zeta)$ is holomorphic in Λ.

(iii) For any $(\alpha_1, \alpha_2, \beta, p)$, there exists a constant $C = C(\alpha_1, \alpha_2, \beta, p) > 0$ (independent of $\zeta \in \Lambda$) such that

$$|\zeta| \left| \left(\frac{\partial}{\partial u_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial u_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial v}\right)^{\beta} \left(\frac{\partial}{\partial r}\right)^{p} a(w, \zeta) \right| \leq C r^{m+s-p} \rho_\Sigma^{M+t-|\alpha_1|-|\alpha_2|/l}$$

for all $(w, \zeta) \in W \times \Lambda$.

Since $(h, k; l)$ is fixed throughout this paper, we omit the subscript of symbol classes $S^{m,M}_{(h,k;l)}(W, \Sigma)$ and $S^{s,t}_{(h,k;l)}(W, \Sigma, \Lambda)$ and we denote the class of pseudodifferential operators defined by (1.2) with the Weyl symbols with support contained in W in $S^{m,M}(W, \Sigma, \Lambda)$ by $OPS^{m,M}(W, \Sigma, \Lambda)$.

By the Taylor theorem we can write, for $j \leq M$

$$p_{m-jT/2} = \sum_{|\alpha_1|+|\alpha_2|/l = M-j} a_{\alpha_1 \alpha_{2j}}(u_1, u_2, v, r) u_1^{\alpha_1} u_2^{\alpha_2}$$

in W and we note that $\rho_\Sigma(x, \xi)$ is equivalent to

$$\left\{ \sum_{j=1}^{d_1} |u_{1j}|^2 + \sum_{j=1}^{d_2} |u_{2j}|^{2l} + r^{-T} \right\}^{1/2}.$$

If we identify $X = (X_1, X_2) \in N_\rho\Sigma = N_\rho\Sigma_1 \oplus N_\rho\Sigma_2$ with (u_1, u_2) and $\rho \in \Sigma$ with $(0, 0, v, r)$, we can write

$$\tilde{p}(\rho, u) = \sum_{j=0}^{M} \sum_{|\alpha_1|+|\alpha_2|/l=M-j} a_{\alpha_1\alpha_2 j}(0, 0, v, r) u_1^{\alpha_1} u_2^{\alpha_2}.$$

Proposition 2.2. *For every $\rho \in \Sigma$, there exists a quasi conic neighborhood W of ρ having a local coordinate system (u_1, u_2, v, r) as above and $q_i(\zeta) = q_i(\zeta; x, \xi) \in S^{-m,-M}(W, \Sigma, \Lambda)$, $i = 1, 2$, where Λ is the union of an open cone in $\mathbf{C}$ having the vertex with the origin containing the negative real line and a set $\{\zeta \in \mathbf{C}; |\zeta| < \varepsilon\}$ for some $\varepsilon > 0$ such that*

$$(p - \zeta)\#q_i(\zeta) = 1 + r_i(\zeta) \tag{2.1}$$

where

$$r_1(\zeta) = r_{11}(\zeta) + r_{12}(\zeta) \quad \textit{and} \quad r_2(\zeta) = r_{21}(\zeta) + r_{22}(\zeta),$$
$$r_{11}(\zeta) \in S^{0,1}(W, \Sigma, \Lambda), \quad r_{21}(\zeta) \in S^{-T/2,-1}(W, \Sigma, \Lambda)$$

and

$$r_{12}(\zeta), r_{22}(\zeta) \in S^{-T_0/2,0}(W, \Sigma, \Lambda)$$

where $T_0 = \mathrm{Min}\{T_1, T\}$, $T_1 = \mathrm{Min}\{h_j + k_j; j = 1, \dots, n\} - T$.

Proof. Choose a function $\chi \in C^\infty(\mathbf{R}^{2n})$ such that $\chi(x, \xi) = 1$ for $r(x, \xi) \geq 1$ and $\chi(x, \xi) = 0$ for $r(x, \xi) \leq 1/2$. First we construct $q_1(\zeta; x, \xi)$. In a quasi conic neighborhood W of $\rho \in \Sigma$, put

$$q_1(\zeta; u_1, u_2, v, r) = \chi(u_1, u_2, v, r)(\tilde{p}(\rho, u) - \zeta)^{-1}.$$

Then we have

$$\begin{aligned}
&(p - \zeta)\#q_1(\zeta) \\
&\quad = \chi\left\{(\tilde{p} - \zeta)\#(\tilde{p} - \zeta)^{-1} + \left(p - \sum_{j=0}^{M} p_{m-jT/2}\right)\#(\tilde{p} - \zeta)^{-1}\right. \\
&\qquad\qquad \left. + \sum_{j=0}^{M}(p_{m-jT/2} - \tilde{p}_{m-jT/2})\#(\tilde{p} - \zeta)^{-1}\right\} \\
&\qquad + [p, \chi](\tilde{p} - \zeta)^{-1}
\end{aligned}$$

where $[p, \chi] = p\#\chi - \chi\#p$. Since $(\tilde{p}-\zeta)^{-1} \in S^{-m,-M}(W, \Sigma, \Lambda)$ we can write

$$\partial_{x_j} = C_{1j}\cdot\partial_{u_1} + C_{2j}\cdot\partial_{u_2} + C_{3j}\cdot\partial_v + C_{4j}\partial_r,$$
$$\partial_{\xi_j} = D_{1j}\cdot\partial_{u_1} + D_{2j}\cdot\partial_{u_2} + D_{3j}\cdot\partial_v + D_{4j}\partial_r$$

where C_{ij}, D_{ij} $(i = 1, 2, 3)$ are quasi homogeneous of degree $-h_j$, $-k_j$, C_{4j}, D_{4j} are of degree $1-h_j$, $1-k_j$ respectively, the formula (1.5) leads to

$$(\tilde{p}-\zeta)\#(\tilde{p}-\zeta)^{-1} - 1 \in S^{-T_0,0}(W, \Sigma, \Lambda).$$

Since

$$p - \sum_{j=0}^{M} p_{m-jT/2} \in S^{m-(M+1)T/2,0}(W, \Sigma, \Lambda),$$

we have

$$\left(p - \sum_{j=0}^{M} p_{m-jT/2}\right)\#(\tilde{p}-\zeta)^{-1} \in S^{-(M+1)T/2,-M}(W, \Sigma, \Lambda)$$
$$\subset S^{-T_0/2,0}(W, \Sigma, \Lambda).$$

It is easy to see that $[p, \chi](p-\zeta)^{-1} \in S^{-\infty}(W, \Sigma, \Lambda)$. Since for $j = 0, 1, \dots, M$,

$$p_{m-jT/2} - \tilde{p}_{m-jT/2} \in S^{m-jT/2,M-j+1}(W, \Sigma, \Lambda),$$

we have

$$\sum_{j=0}^{M}(p_{m-jT/2} - \tilde{p}_{m-jT/2})\#(\tilde{p}-\zeta)^{-1} = r_{11}(\zeta) + r_{12}(\zeta)$$

where

$$r_{11}(\zeta) = (p_m - \tilde{p}_m)(\tilde{p}-\zeta)^{-1} \in S^{0,1}(W, \Sigma, \Lambda) \tag{2.2}$$

and it follows from the formula (1.5) that $r_{12} \in S^{-T_0/2,0}(W, \Sigma, \Lambda)$.

For the case $i = 2$, we put

$$q_2(\zeta; x, \xi) = (p_m(x, \xi) + r^{m-MT/2} - \zeta)^{-1}.$$

Then by the same arguments as the case $i = 1$, we also see that (2.1) also holds for $i = 2$.

Now we shall construct global parametrices of $p^w(x, D) - \zeta I$ ($\zeta \in \Lambda$). In order to do so, let $\rho \in \Sigma$ and W be a quasi conic neighborhood of ρ as in Proposition 2.6. Then choose a function $\varphi(x, \xi) \in$

$C^\infty(\mathbf{R}^{2n})$ which is quasi homogeneous of degree 0 and $\operatorname{supp}\varphi \subset W$ and define

$$(2.3)\quad q_{10}^w(\zeta; x, D) = \varphi^w(x, D)\{q_1^w(\zeta; x, D) - q_2^w(\zeta; x, D)r_1^w(\zeta; x, D)\},$$

$$(2.4)\quad q_{20}^w(\zeta; x, D) = \varphi^w(x, D)\{q_2^w(\zeta; x, D) - q_1^w(\zeta; x, D)r_2^w(\zeta)(x, D)\}.$$

Then we have

$$(p^w(x, D) - \zeta I)q_{j0}^w(\zeta; x, D) = \varphi^w(x, D) + d_j^w(\zeta; x, D) \qquad (j = 1, 2)$$

where

$$d_j^w(\zeta; x, D) \in OPS^{-T_0/2, 0}(W, \Sigma, \Lambda).$$

Moreover, if we define for every $j = 1, 2$,

$$q_{jl}^w(\zeta; x, D) = q_{j0}^w(\zeta; x, D)(-d_j^w(\zeta; x, D))^l, \qquad l = 1, 2, \dots,$$

we can find

$$q_j^w(\zeta; x, D) \in OPS^{-m, -M}(W, \Sigma, \Lambda)$$

such that

$$q_j^w(\zeta; x, D) - \sum_{l=0}^{N-1} q_{jl}^w(\zeta; x, D) \in OPS^{-m-NT_0/2, -M}(W, \Sigma, \Lambda).$$

Thus we see

$$(p^w(x, D) - \zeta I)q_j^w(\zeta; x, D) \equiv \varphi^w(x, D)$$

modulo $OPS^{-\infty}(W, \Sigma, \Lambda) = \bigcap_m OPS^{-m, -M}(W, \Sigma, \Lambda)$. Of course, since $p^w(x, D)$ is elliptic outside Σ, we construct a usual parametrix there and by a partition of unity, we can construct the global parametrix for $p^w(x, D) - \zeta I$.

3. Construction of complex powers. In this section we construct complex powers for $p^w(x, D)$. For this purpose, define an operator P_0 on $L^2(\mathbf{R}^n)$ so that

$$P_0 u = p^w(x, D)u, \qquad u \in D(P_0),$$

where $D(P_0) = S(\mathbf{R}^n)$. Under the hypotheses (H.1) $\sim$ (H.5), P_0 has the closure P whose spectrum is discrete. Moreover, P is bounded from below, so by (H.6) we may assume that there exists a positive number $\gamma > 0$ such that

$$(Pu, u) \geq \gamma \|u\|^2_{L^2(\mathbf{R}^n)}$$

for all $u \in D(P)$. Then we can define complex powers P^z of P as follows.

$$P^z = \frac{-1}{2\pi i} \int_\Gamma \zeta^z (P - \zeta I)^{-1} \, d\zeta \tag{3.1}$$

for $\operatorname{Re} z < 0$. For $\operatorname{Re} z \geq 0$, choose a positive integer k such that $\operatorname{Re} z < k$ and define $P^z = P^k P^{z-k}$. Here Γ is a curve beginning at infinity, passing along the negative real line to a circle $|\zeta| = \varepsilon_0$ $(0 < \varepsilon_0 < \gamma)$, then clockwise about the circle, and back to the infinity along the negative real line. Note that the definition of P^z $(z \in \mathbf{C})$ is well defined (cf. Shubin **[11]** and Seeley **[10]**).

We set Λ as the union of a small open convex cone containing the negative real line and $\{\zeta \in \mathbf{C}; |\zeta| < (\varepsilon_0 + \gamma)/2\}$. Then we define the symbol, for $\operatorname{Re} z < 0$,

$$p_{i,z}(x, \xi) = \frac{-1}{2\pi i} \int_\Gamma \zeta^z q_i(\zeta; x, \xi) \, d\zeta \qquad (i = 1, 2) \tag{3.2}$$

and denote the pseudodifferential operator with the Weyl symbol $p_{i,z}(x, \xi)$ by $p^w_{i,z}(x, D)$. If $k - 1 \leq \operatorname{Re} z < k$ for some positive integer k, we define $p^w_{i,z}(x, D) = p^w(x, D)^k p^w_{i,z-k}(x, D)$. Then we have

THEOREM 3.1. *Assume that* $p(x, \xi) \in \widetilde{S}^{m,M}_{(h,k;l)}$ *satisfies* (H.1) $\sim$ (H.6). *Then we have*

(i) $P^z \in OPS^{m \operatorname{Re} z, M \operatorname{Re} z}_{(h,k;l)}$ *and has the Weyl symbol* $p_{i,z}(x, \xi)$ $(i = 1, 2)$.

(ii) *For any* $a < 0$ *and* $m', M' \in \mathbf{R}$ *such that* $ma < m'$,

$$(m - MT/2)a < m' - M'T/2,$$

$p_{i,z}(x, \xi)$ *are holomorphic on any compact set in* $\Pi_a = \{z; \operatorname{Re} z < \alpha\}$ *with values in* $S^{m',M'}_{(h,k;l)}$. *More precisely, for any compact set* K *in* Π_a *and* $\alpha_1, \alpha_2, \beta, p$, *there exists a constant* $C = C_{K,\alpha_1,\alpha_2,\beta,p}$ *independent of* $z \in K$ *such that*

$$|\partial^{\alpha_1}_{u_1} \partial^{\alpha_2}_{u_2} \partial^\beta_v \partial^p_r p_{i,z}| \leq C r^{m'-p} \rho_\Sigma^{M'-|\alpha_1|-|\alpha_2|/l}. \tag{3.3}$$

Later, we denote the class satisfying (i), (ii) and (iii) of Theorem 3.1 by $HS^{m \operatorname{Re} z, M \operatorname{Re} z}$.

For the proof we need the following lemma.

LEMMA 3.2. *Let* $a(\zeta)(x, \xi) \in S^{s,t}(W, \Sigma, \Lambda)$ *and define*

$$a_z(x, \xi) = \frac{-1}{2\pi i} \int_\Gamma \zeta^z a(\zeta)(x, \xi)\, d\zeta.$$

Then $a_z \in HS^{m \operatorname{Re} z + m + s, M \operatorname{Re} z + M + t}$.

Proof. Since $a(\zeta)(x, \xi)$ is holomorphic in

$$\Gamma_{\rho(x,\xi)} = \{\zeta\,;\, \operatorname{Im}\zeta = 0,\, \operatorname{Re}\zeta \geq 0\} \cup \{\zeta\,;\, |\zeta| \leq 2\delta\rho(x, \xi)\}$$

with values in $S^{s,t}_{(h,k;l)}(W, \Sigma, \Lambda)$ where $\rho(x, \xi) = r^m \rho_\Sigma^M$, by the Cauchy theorem we may replace the contour Γ in the integral with $\Gamma_{\rho(x,\xi)}$. Moreover for any $\alpha_1, \alpha_2, \beta, p$, there exists a constant $C = C_{\alpha_1, \alpha_2, \beta, p}$ such that

$$|\partial_{u_1}^{\alpha_1} \partial_{u_2}^{\alpha_2} \partial_v^\beta \partial_r^p a(\zeta)(x, \xi)| \leq C|\zeta|^{-1} \rho(x, \xi) r^{s-p} \rho_\Sigma^{t-|\alpha_1|-|\alpha_2|/l}.$$

Now we decompose $\Gamma_{\rho(x,\xi)}$ in (3.2) into $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ as follows:

$$\begin{aligned} &\Gamma_1\,;\, \zeta = -s, && -\delta\rho(x, \xi) \leq s < \infty, \\ &\Gamma_2\,;\, \zeta = \rho(x, \xi) e^{-i\theta}, && -\pi \leq \theta \leq \pi, \\ &\Gamma_3\,;\, \zeta = s, && \delta\rho(x, \xi) \leq s < \infty. \end{aligned}$$

For $i = 1, 3$, we have, for some constant C and C_z

$$\begin{aligned} &\int_{\Gamma_i} |\zeta^z \partial_{u_1}^{\alpha_1} \partial_{u_2}^{\alpha_2} \partial_v^\beta \partial_r^p a(\zeta)|\,|d\zeta| \\ &\quad \leq C\rho(x, \xi) r^{s-p} \rho_\sigma^{t-|\alpha_1|-|\alpha_2|/l} \int_{\delta\rho(x,\xi)}^\infty s^{\operatorname{Re} z - 1}\, ds \\ &\quad \leq C_z (r^m \rho_\sigma^M)^{\operatorname{Re} z + 1} r^{s-p} \rho_\Sigma^{t-|\alpha_1|-|\alpha_2|/l}. \end{aligned}$$

For $i = 2$, we have

$$\begin{aligned} &\int_{\Gamma_2} |\zeta^z \partial_{u_1}^{\alpha_1} \partial_{u_2}^{\alpha_2} \partial_v^\beta \partial_r^p a(\zeta)|\,|d\zeta| \\ &\quad \leq C\rho(x, \xi) r^{s-p} \rho_\Sigma^{t-|\alpha_1|-|\alpha_2|/l} \int_{|\zeta|=\delta\rho(x,\xi)} |\zeta|^{\operatorname{Re} z - 1} |\,d\zeta| \\ &\quad = C(r^m \rho_\Sigma^M)^{\operatorname{Re} z + 1} r^{s-p} \rho_\Sigma^{t-|\alpha_1|-|\alpha_2|/l}. \end{aligned}$$

This completes the proof.

End of proof of Theorem 3.1.
Since $q_i(\zeta) \in S^{-m,-M}_{(h,k;l)}(W, \Sigma, \Lambda)$ and

$$(P - \zeta I)^{-1} - q_i^w(\zeta)(x, D) \in OPS^{-\infty}(W, \Sigma, \Lambda),$$

(i) follows from Lemma 3.2, (ii) follows from the same arguments of the proof of Aramaki [1; Proposition 3.1].

Next we clarify the symbol of P^z.

PROPOSITION 3.3. *Let W be a small quasi conic neighborhood of $\rho \in \Sigma$. Then we have in W*

(i) $\sigma(P^z) = \tilde{p}(\rho, u)^z + d_{1,z}$.

(ii) $\sigma(P^z) = (p_m + r^{m-MT/2})^z + d_{2,z}$ *where*

$$d_{1,z} = d_{11,z} + d_{12,z} \quad \text{and} \quad d_{2,z} = d_{21,z} + d_{22,z},$$
$$d_{11,z} \in HS^{m \operatorname{Re} z, M \operatorname{Re} z+1}, \; d_{21,z} \in HS^{m \operatorname{Re} z - T/2, M \operatorname{Re} z-1}$$

and

$$d_{12,z}, d_{22,z} \in HS^{m \operatorname{Re} z - T_0/2, M \operatorname{Re} z}.$$

Proof. (i) First, we consider the symbol $q_1(\zeta)(x, \xi)$ in (2.3). By the Cauchy theorem, we have

$$\frac{-1}{2\pi i} \int_\Gamma \zeta^z q_1(\zeta)(x, \xi)\, d\zeta = \chi \tilde{p}(\rho, u)^z.$$

Since $q_2(\zeta)\#r_1(\zeta) - q_2(\zeta)r_{11}(\zeta) \in S^{-m-MT/2,-M}(W, \Lambda)$, it suffices to consider

$$d_{11,z} = \frac{-1}{2\pi i} \int_\Gamma \zeta^z q_2(\zeta) r_{11}(\zeta)\, d\zeta. \tag{3.4}$$

Since $q_2(\zeta)r_{11}(\zeta) \in S^{-m,-M+1}(W, \Lambda)$, it follows from Lemma 3.2 that $d_{11,z} \in HS^{m \operatorname{Re} z, M \operatorname{Re} z+1}$. Thus (i) holds. Taking (2.3) into consideration, (ii) also follows.

4. The singularity of trace of P^z. In this section we consider the singularities of trace of P^z and determine the order of the poles and the coefficients of the Laurent expansions at the points. Let $p_z(x, \xi)$ be the Weyl symbol of P^z and holomorphic function of z. It is well known that if

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} |p_z(x, \xi)|\, dx\, d\xi \le C_z$$

for some constant C_z, then P^z is an operator of trace class and the trace is given by:

$$\mathrm{Tr}[P^z] = (2\pi)^{-n} \int_{\mathbf{R}^n \times \mathbf{R}^n} p_z(x, \xi)\, dx\, d\xi.$$

Since

$$\int_{r \leq 1} p_z(x, \xi)\, dx\, d\xi$$

is an entire function, we may consider:

$$\int_{r \geq 1} p_z(x, \xi)\, dx\, d\xi.$$

In order to do so, we need the following proposition.

PROPOSITION 4.1. *Let $f(z; x, \xi)$ be a C^∞ function on $\mathbf{C} \times \mathbf{R}^n \times \mathbf{R}^n$ satisfying*

(i) *For every $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$, $f(z; x, \xi)$ is a holomorphic function in $\mathbf{C}$.*

(ii) *For every compact set K in $\mathbf{C}$, $f(z; x, \xi)/r^{m \operatorname{Re} z - j} \rho_\Sigma^{M \operatorname{Re} z - i}$ is bounded uniformly in $z \in K$.*

Then the integrals

$$I_{ji}(z) = \int_{r \geq 1} f(z, x, \xi)\, dx\, d\xi$$

holomorphic in $\Pi_a = \{z; \operatorname{Re} z < a\}$ if a satisfies any one of the following (I) *and* (II):

(I) *$Ma - i + d_1 + d_2/l < 0$ and $(m - MT/2)a - j + Ti/2 + |h| + |k| - Td_1/2 - Td_2/(2l) < 0$,*

(II) *$Ma - i + d_1 + d_2/l \geq 0$ and $ma - j + |h| + |k| < 0$.*

Proof. Let K be any compact subset in Π_a. Then there exists a constant $C_K > 0$ (independent of $z \in K$) such that

$$|f(z, x, \xi)| \leq C_K r^{m \operatorname{Re} z - j} \rho_\Sigma^{M \operatorname{Re} z - i} \leq C_K (r^m \rho_\Sigma^M)^a r^{-j} \rho_\Sigma^{-i}$$

for all $z \in K$. In fact, we have from (H.5), $r^m \rho_\Sigma^M \geq r^{m - MT/2} \geq 1$. Let W be a quasi conic neighborhood of $\rho \in \Sigma$ and (u_1, u_2, v, r) a local coordinate system in W as in §2. Then

$$dx\, d\xi = J(u_1, u_2, v, r)\, du_1\, du_2\, dv\, dr$$

where $J(u_1, u_2, v, r)$ is quasi homogeneous of degree $|h|+|k|-1$. Since $\rho_\Sigma = \{|u_1|^2 + |u_2|^{2l} + r^{-T}\}^{1/2}$ in W, we have, for $z \in K$,

$$\begin{aligned} |I|_{ji}(z) &\equiv \int_{W\cap\{r\geq 1\}} |f(z, x, \xi)|\, dx\, d\xi \\ &\leq C \int_{W\cap\{r\geq 1\}} (r^m \rho_\Sigma^M)^a r^{-j} \rho_\Sigma^{-i} r^{|h|+|k|-1}\, du_1\, du_2\, dv\, dr \\ &= C \int_1^\infty r^{ma-j+|h|+|k|-1}\, dr \\ &\quad \times \int_{W\cup\{r\geq 1\}} (|u_1|^2 + |u_2|^{2l} + r^{-T})^{(Ma-i)/2}\, du_1\, du_2\, dv. \end{aligned}$$

Since $|u_i|$, $|v|$ are bounded in W, we may assume that $|u_i|$, $|v| \leq 1$. By the change of variable $(u_1, u_2) \to (r^{-T/2}u_1, r^{-T/(2l)}u_2)$, we have with another constant C,

$$|I|_{ji}(z) \leq C \int_1^\infty r^{ma-j+|h|+|k|-1-(l(Ma-i)+ld_1+d_2)T/(2l)} J_i(r)\, dr$$

where

$$\begin{aligned} J_i(r) &= \int_{|u_1|\leq r^{T/2},\, |u_2|\leq r^{T/(2l)}} (|u_1|^2 + |u_2|^{2l} + 1)^{(Ma-i)/2}\, du_1\, du_2 \\ &\leq C \int_0^{r^{T/(2l)}} \int_0^{r^{T/2}} (t^2 + s^{2l} + 1)^{(Ma-i)/2} t^{d_1-1} s^{d_2-1}\, dt\, ds. \end{aligned}$$

Moreover, we take the change of variable: $t = R\cos\theta$, $s = R^{1/l}\sin^{1/l}\theta$. Since the Jacobian is

$$\frac{D(t, s)}{D(R, \theta)} = \frac{1}{l} R^{1/l} \sin^{(1/l)-1}\theta,$$

we have

$$\begin{aligned} J_i(r) &\leq C \int_0^{r^{T/2}} (R^2+1)^{(Ma-i)/2} R^{\{(ld_1+d_2)/l\}-1}\, dR \\ &\quad \times \int_0^{\pi/2} \cos^{d_1-1}\theta \sin^{(d_2/l)-1}\theta\, d\theta. \end{aligned}$$

Here we note that

$$\int_0^{\pi/2} \cos^{d_1-1}\theta \sin^{(d_2/l)-1}\theta\, d\theta = \frac{1}{2} B\left(\frac{d_1}{2}, \frac{d_2}{2l}\right)$$

where $B(\cdot, \cdot)$ denotes the Beta function and

$$\begin{aligned} &\int_0^\infty (R^2+1)^{(Ma-i)/2} R^{\{(ld_1+d_2)/l\}-1}\, dR \\ &\quad = \frac{1}{2} \frac{\Gamma(\{ld_1+d_2\}/2l)\Gamma(\{l(i-Ma)-ld_1-d_2\}/(2l))}{\Gamma((i-Ma)/2)} \end{aligned}$$

if $Ma - i + d_1 + d_2/l < 0$. When $Ma - i + d_1 + d_2/l \geq 0$,

$$J_i(r) \leq C \int_0^{r^{T/2}} R^{\{Ma-i+(ld_1+d_2)/l\}-1}\, dR = O(r^{(l(Ma-i)+ld_1+d_2)/(2l)} \log r)$$

as $r \to \infty$. Thus we have, with an another constant $C > 0$,

$$|I|_{ji}(z) \leq C \int_1^{\infty} r^{ma-j+|h|+|k|-1-T(l(Ma-i)+ld_1+d_2)/(2l)}\, dr,$$

if $Ma - i + d_1 + d_2/2 < 0$ and

$$|I|_{ji}(z) \leq C \int_1^{\infty} r^{ma-j+|h|+|k|-1} \log r\, dr,$$

if $Ma - i + d_1 + d_2/2 \geq 0$. Therefore the integral $I_{ji}(z)$ is absolutely convergent for each case (I) or (II). Outside Σ, by the ellipticity of $p(x, \xi)$, (I) or (II) is clear. This completes the proof.

For brevity of notations, we put

$$N_1 = \frac{ld_1 + d_2}{Ml}, \quad N_2 = \frac{|h| + |k|}{m} \quad \text{and}$$
$$N_3 = \frac{2(|h| + |k|) - T(ld_1 + d_2)/l}{2m - MT}.$$

COROLLARY 4.2. *Let* $d_{ij,z}$ $(i, j = 1, 2)$ *be as in Proposition* 3.3. *Then we have the following three cases.*

(i) *When* $N_1 > N_2$, $\mathrm{Tr}[d^w_{2,z}]$ *is holomorphic for* $\mathrm{Re}\, z < -N_2 + \delta_1$ *where* $\delta_1 = \mathrm{Min}\{1/(2m), N_2 - N_3\}$.

(ii) *When* $N_1 = N_2$, $\mathrm{Tr}[d^w_{1,z}]$ *is holomorphic for* $\mathrm{Re}\, z < -N_2 + \delta_2$ *for some* $\delta_2 > 0$ *except* $z = -N_2$ *which is at most a simple pole.*

(iii) *When* $N_1 < N_2$, $\mathrm{Tr}[d^w_{1,z}]$ *is holomorphic for* $\mathrm{Re}\, z < -N_3 + \delta_3$ *where* $\delta_3 = \mathrm{Min}\{1/T(2m - MT), N_3 - N_2\}$.

Proof. First we consider the case (i). In this case, we have $-N_1 < -N_2 < -N_3$. Since $d_{21,z} \in HS^{m\,\mathrm{Re}\, z - T/2, M\,\mathrm{Re}\, z - 1}$ and $d_{22,z} \in HS^{m\,\mathrm{Re}\, z - T_0/2, M\,\mathrm{Re}\, z}$, it follows from Proposition 4.1 that $\mathrm{Tr}[d^w_{2,z}]$ is holomorphic for $\mathrm{Re}\, z < -N_2 + \delta_1$.

In the case (iii), note that $-N_3 < -N_2 < -N_1$. If we consider the trace of $d_{11,z} \in HS^{m\,\mathrm{Re}\, z, M\,\mathrm{Re}\, z+1}$ and apply Proposition 4.1, it is easy to see that $\mathrm{Tr}[d^w_{1,z}]$ is holomorphic for $\mathrm{Re}\, z < -N_3 + \delta_3$.

The case (ii) is more delicate. In this case, we have $-N_1 = -N_2 = -N_3$. Since it easily follows from Proposition 4.1 that $\mathrm{Tr}[d_{12,z}]$ is holomorphic for $\mathrm{Re}\, z < -N_2 + 1/(2m)$ and $\mathrm{Tr}[d^w_{11,z}]$ is also holomorphic for $\mathrm{Re}\, z < -N_2$. Therefore it suffices to show that $\mathrm{Tr}[d^w_{11,z}]$

is holomorphic for $\operatorname{Re} z < -N_2+\delta_2$ except $z = -N_2$ which is at most a simple pole. By (3.4),

$$d_{11,z} = \frac{-1}{2\pi i}(p_m - \tilde{p}_m)\int_\Gamma \zeta^z (p_m + r^{m-MT/2} - \zeta)^{-1}(\tilde{p}_m - \zeta)^{-1}\,d\zeta.$$

However by Proposition 4.1, we can replace $(\tilde{p} - \zeta)^{-1}$ with $(\tilde{p}_m + r^{m-MT/2} - \zeta)^{-1}$. Thus the Cauchy theorem leads to

$$d_{11,z} = (\tilde{p}_m + r^{m-MT/2})^z - (p_m + r^{m-MT/2})^z.$$

Since by the Taylor theorem,

$$\begin{aligned} d_{11,z} &= z(\tilde{p}_m - p_m)\int_0^1 \{\tilde{p}_m + r^{m-MT/2} + \theta(p_m - \tilde{p}_m)\}^{z-1}\,d\theta \\ &= z d'_{11,z} + z(z-1)d''_{11,z} \end{aligned}$$

where

$$d'_{11,z} = (\tilde{p}_m - p_m)\int_0^1 \{\tilde{p}_m + \theta(p_m - \tilde{p}_m)\}^{z-1}\,d\theta$$

and

$$\begin{aligned} d''_{11,z} &= (\tilde{p}_m - p_m)r^{m-MT/2} \\ &\quad\times \int_0^1\int_0^1 \{\tilde{p}_m + \theta(p_m - \tilde{p}_m) + \chi r^{m-MT/2}\}^{z/2}\,d\chi\,d\theta. \end{aligned}$$

At first we consider $d'_{11,z}$. Let $a \le \operatorname{Re} z \le b$ where $b < -N_2$. Then

$$\begin{aligned} (2\pi)^{-n}&\int_{r\ge 1} d'_{11,z}(x,\xi)\,dx\,d\xi \\ &= \frac{-1}{(2\pi)^n(mz + |h| + |k|)} \\ &\quad\times \int_{S*\mathbf{R}^n\cap W} (\tilde{p}_m(\omega) - p_m(\omega)) \\ &\quad\times \int_0^1 \{\tilde{p}_m(\omega) + \theta(p_m(\omega) - \tilde{p}_m(\omega))\}^{z-1}\,d\theta\,d\omega. \end{aligned}$$

Since $\tilde{p}_m(\omega) + \theta(p_m(\omega) - \tilde{p}_m(\omega))$ is equivalent to $(|u_1|^2 + |u_2|^{2l})^{M/2}$, we may assume that the integral is equivalent to

$$\begin{aligned} \int_{|u_i|\le 1} &(|u_1|^2 + |u_2|^{2l})^{(M\operatorname{Re} z+1)/2}\,du_1\,du_2 \\ &= C\int_0^1\int_0^1 (t^2 + s^{2l})^{(M\operatorname{Re} z+1)/2} t^{d_1-1}s^{d_2-1}\,dt\,ds \\ &\le B\left(\frac{d_1}{2}, \frac{d_2}{2l}\right)\int_0^1 R^{M(a+1)+d_1+d_2/l-1}\,dR. \end{aligned}$$

If $a \le \operatorname{Re} z$ where $a > -N_2 - 1/M$, the integral is convergent. Thus $\operatorname{Tr}[d'^{w}_{11,z}]$ is holomorphic for $\operatorname{Re} z < -N_2+\delta_2$ for some $\delta_2 > 0$ except $z = -N_2$ which is at most a simple pole. Next we shall show that $\operatorname{Tr}[d''^{w}_{11,z}]$ is holomorphic for $-N_2 - \delta_2 < \operatorname{Re} z < -N_2 + \delta_2$ for some $\delta_2 > 0$. Since $\tilde{p}_m + \theta(p_m - \tilde{p}_m)$ is equivalent to $r^m(|u_1|^2 + |u_2|^{2l})^{M/2}$, we may consider the integral

$$I = \int_{r\ge 1,\, |u_i|\le 1} r^m(|u_1|^2 + |u_2|^{2l})^{(M+1)/2} r^{m-MT/2}\, dr \times \int_0^1 \{r^m(|u_1|^2 + |u_2|^{2l})^{M/2} + \chi r^{m-MT/2}\}^{z-2}\, d\chi\, du_1\, du_2.$$

Choose $0 < \varepsilon < 1/2$, a and b such that $a < -N_2 + M\varepsilon/2$, $b > -N_2 + \varepsilon - 1/2$ and let $a \le \operatorname{Re} z \le b$. Then

$$\begin{aligned}\int_0^1 &\{(r^m(|u_1|^2 + |u_2|^{2l}))^{M \operatorname{Re} z/2} + \chi r^{m-MT/2}\}^{\operatorname{Re} z-2}\, d\chi \\ &\le \{(r^m(|u_1|^2 + |u_2|^{2l})^{M/2}\}^{\operatorname{Re} z-1-\varepsilon} \int_0^1 (\chi r^{m-MT/2})^{\varepsilon-1}\, d\chi \\ &\le \frac{1}{\varepsilon}\{(r^m(|u_1|^2 + |u_2|^{2l})^{M/2}\}^{\operatorname{Re} z-1-\varepsilon} r^{(m-MT/2)(\varepsilon-1)}.\end{aligned}$$

Therefore

$$I \le \int_1^\infty r^{mb-M\varepsilon/2+|h|+|k|-1}\, dr \times \int_{|u_i|\le 1} (|u_1|^2 + |u_2|^{2l})^{M(a-\varepsilon+1/M)/2}\, du_1\, du_2.$$

By the same change of variable as in Proposition 4.1, we see that the integral is covergent. Thus $\operatorname{Tr}[d''^{w}_{11,z}]$ is holomorphic for $-N_2 + \varepsilon - 1/M < \operatorname{Re} z < -N_2 + M\varepsilon/(2m)$. This completes the proof.

Now we consult

$$I_1(z) = (2\pi)^{-n} \int\int_{r\ge 1} \varphi \tilde{p}(\rho, u)^z\, dx\, d\xi$$

and

$$I_2(z) = (2\pi)^{-n} \int\int_{r\ge 1} \varphi (p_m + r^{m-MT/2})^z\, dx\, d\xi.$$

In order to do so, we define, for $\rho \in \Sigma$ and $X = (X_1, X_2) \in N_\rho\Sigma$,

$$\operatorname{Hess} \tilde{p}_m(\rho, X) = \sum_{|\alpha_1|+|\alpha_2|/l=M} \frac{1}{\alpha_1!\alpha_2!}(X_1^{\alpha_1} X_2^{\alpha_2} p_m)(\rho).$$

Note that it follows from (H.3) that Hess $\tilde{p}_m(\rho, X) > 0$, for all $X = (X_1, X_2) \in N_\rho\Sigma$ so that $X \neq 0$. Define a measure dX_ρ on $N_\rho\Sigma$ such that

$$\int_{\text{Hess}\,\tilde{p}_m(\rho, X)<1} dX_\rho = 1. \tag{4.1}$$

Then it is easily seen that $dX_{\lambda\cdot\rho} = \lambda^{m(d_1+d_2/l)/M}\,dX_\rho$ for $\lambda > 0$, i.e., dX_ρ is quasi homogeneous of degree $m(d_1 + d_2/l)/M = mN_1$. Next we define a positive C^∞ density on Σ as follows: Choose a local coordinate system (u, v') where $u = (u_1, u_2)$ is as in §2 so that $dx\,d\xi = du\,dv'$, so dv' is quasi homogeneous of degree $|h| + |k|$. Then we define $d\rho = f(\rho)\,dv'|_\rho$ where

$$f(\rho) = \int_{\sum_{|\alpha_1|+|\alpha_2|/l=M} a_{\alpha_1,\alpha_2,0}(0, v, r(\rho))u_1^{\alpha_1}u_2^{\alpha_2}<1} du_1\,du_2. \tag{4.2}$$

It follows that $d\rho$ is quasi homogeneous of degree $|h| + |k| - mN_1$.

Moreover, taking Proposition 4.1 into consideration, we note that $\varphi = \varphi|_\Sigma + r_1$ in the integral $I_1(z)$ where $r_1 \in S^{0,1}$ and $r_1\tilde{p}(\rho, u)^z \in S^{m\,\text{Re}\,z,\,M\,\text{Re}\,z+1}$, so we may put

$$I_1(z) = (2\pi)^{-n} \int\int_{r\geq 1} \tilde{p}(\rho, u)^z\,dX_\rho\,d\rho.$$

THEOREM 4.3. *Assume that* $p(x, \xi) \in \widetilde{S}^{m,M}_{(h,k;l)}$ *satisfies the hypotheses* (H.1) $\sim$ (H.6). *Then there are three cases for the singularities of* $Z_P(z) = \text{Tr}[P^z]$.

(I) *When* $N_1 > N_2$, $Z_P(z)$ *is holomorphic for* $\text{Re}\,z < -N_2 + \delta_1$ *where* δ_1 *is as in Proposition* 4.2 *except only one singularity at* $z = -N_2$ *which is a simple pole and the residue* $R_1(-N_2)$ *is given by*:

$$R_1(-N_2) = \frac{-1}{m}(2\pi)^{-n} \int_{S_q^*\mathbf{R}^n} p_m(\omega)^{-N_2}\,d\omega$$

where $S_q^*\mathbf{R}^n = \{(x, \xi) \in T^*\mathbf{R}^n;\ r(x, \xi) = 1\}$.

(II) *When* $N_1 = N_2$, $Z_P(z)$ *is holomorphic for* $\text{Re}\,z < -N_2 + \delta_0$ *for some* $\delta_0 > 0$ *except only one singularity at* $z = -N_2$ *which is a double pole and the coefficient* $R_2(-N_2)$ *of* $(z + N_2)^{-2}$ *of the Laurent expansion at* $z = -N_2$ *is given by*:

$$R_2(-N_2) = \frac{|h| + |k| - mN_1}{M(m - MT/2)}(2\pi)^{-n} \int_{S_q^*\Sigma}\int_{SN_\omega\Sigma} \tilde{p}_m(\omega, Y)^{-N_2}\,dY_\omega\,d\omega,$$

where $S_q^*\Sigma = S_q^*\mathbf{R}^n \cap \Sigma$ *and* $SN_\omega\Sigma = \{X \in N_\omega\Sigma;\ \text{Hess}\,\tilde{p}_m(\omega, X) = 1\}$.

(III) *When* $N_1 < N_2$, $Z_P(z)$ *is holomorphic for* $\mathrm{Re}\, z < -N_3 + \delta_3$ *where* δ_3 *is as in Proposition* 4.2 *except only one singularity at* $z = -N_3$ *which is a simple pole and the residue* $R_1(-N_3)$ *is given by*:

$$R_1(-N_3) = \frac{-(|h| + |k| - mN_1)}{m - MT/2} (2\pi)^{-n} \int_{S_q^*\Sigma} \int_{N_\omega\Sigma} \tilde{p}(\omega, X)^{-N_3}\, dX_\omega\, d\omega.$$

Proof. By Corollary 4.2, we may consider

$$I_1(z) = (2\pi)^{-n} \int \int_{r \geq 1} \tilde{p}(\rho, X)^z\, dX_\rho\, d\rho$$

for the case (II), (III) and

$$I_2(z) = (2\pi)^{-n} \int \int_{r \geq 1} (p_m + r^{m-MT/2})^z\, dx\, d\xi$$

for the case (I). First we consider $I_1(z)$. Note that by the quasi homogeneity of $d\rho$ we can write

$$d\rho = (|h| + |k| - mN_1) r^{|h|+|k|-mN_1-1} f(\omega)\, dr\, d\omega$$

where $d\omega$ is the measure on $S_q^*\Sigma$ and $f(\omega)$ is as in (4.2). Then

$$\begin{aligned} I_1(z) &= (2\pi)^{-n} \int_1^\infty \int_{S_q^*\Sigma} \int_{N_\omega\Sigma} \tilde{p}(\omega, r^{T/2} X_1, r^{T/(2l)} X_2)^z \\ &\qquad\qquad \times r^{(m-MT/2)z + |h| + |k| - 1}\, dX_\omega\, d\omega\, dr \\ &= (2\pi)^{-n} (|h| + |k| - mN_1) \\ &\quad \times \int_1^\infty r^{(m-MT/2)(z+N_3)-1}\, dr \int_{S_q^*\Sigma} \int_{N_\omega\Sigma} \tilde{p}(\omega, X)^z\, dX_\omega\, d\omega \\ &= \frac{-(|h| + |k| - mN_1)}{(m - MT/2)(z + N_3)} \int_{S_q^*\Sigma} \int_{N_\omega\Sigma} \tilde{p}(\omega, X)^z\, dX_\omega\, d\omega. \end{aligned}$$

Here we consider

$$J(z) = \int_{N_\omega\Sigma} \tilde{p}(\omega, X)^z\, dX_\omega.$$

For brevity of notations, put $|X|_\omega = \{\mathrm{Hess}\, \tilde{p}_m(\omega, X)\}^{1/M}$ for $X \in N_\omega\Sigma$ which is equivalent to $(|X_1|^1 + |X_2|^{2l})^{1/2}$. Then by (H.4), there exists a constant $C > 0$ such that $\tilde{p}(\omega, X) \geq C$. Therefore

$$\int_{|X|_\omega \leq 1} \tilde{p}(\omega, X)^z\, dX_\omega$$

is an entire function of z. Thus we may consider

$$J_1(z)=\int_{|X|_\omega\geq 1}\tilde{p}(\omega, X)^z\,dX_\omega.$$

Choose a real number a so that $a<-N_3$ and let $\operatorname{Re} z\leq a$. Then by similar arguments as in Proposition 4.1,

$$\int_{|X|_\omega\geq 1}\tilde{p}(\omega, X)^a\,dX_\omega\leq\int_1^\infty s^{Ma+d_1+d_2/l-1}\,ds\int_{SN_\omega\Sigma}\tilde{p}(\omega, Y)^a\,dY_\omega$$

where $SN_\omega\Sigma=\{Y\in N_\omega\Sigma;\ |Y|_\omega=1\}$. Therefore $J(z)$ is holomorphic for $\operatorname{Re} a<-N_1$. Thus the case (III) follows.

The case (II): Since $\tilde{p}(\omega, X)-\tilde{p}_m(\omega, X)=O(|X|_\omega^{M-1})$ as $|X|_\omega\to\infty$, we have $\tilde{p}(\omega, X)^z-\tilde{p}_m(\omega, X)^z=O(|X|_\omega^{M\operatorname{Re} z-1})$. Thus we may consider

$$J_2(z)=\int_{|X|_\omega\geq 1}\tilde{p}_m(\omega, X)^z\,dX_\omega.$$

Since $\tilde{p}_m(\omega, X)$ is quasi homogeneous of degree M in (X_1, X_2), we see that

$$\begin{aligned}J_2(z)&=\int_1^\infty s^{Mz+d_1+d_2/l-1}\,ds\int_{SN_\omega\Sigma}\tilde{p}_m(\omega, Y)^z\,dY_\omega\\&=\frac{-1}{Mz+d_1+d_2/l}\int_{SN_\omega\Sigma}\tilde{p}_m(\omega, Y)^z\,dY_\omega.\end{aligned}$$

Here the integral is an entire function of z. Thus the case (II) follows.

The case (I): In this case we note that the integral

$$\int_{S_q^*R^n}p_m(\omega)^{-N_2}d\omega=\lim_{\varepsilon\to 0}\int_{S_q^*\mathbf{R}^n\cap\{p_m\geq\varepsilon\}}p_m(\omega)^{-N_2}\,d\omega$$

exists. Now we must consider

$$I_2(z)=(2\pi)^{-n}\int\int(p_m+r^{m-MT/2})^z\,dx\,d\xi.$$

By the Taylor theorem and Proposition 4.1, we are reduced to studying

$$I_2'(z)=(2\pi)^{-n}\int_{S_q^*\mathbf{R}^n}(r^m p_m(\omega)+1)^z r^{|h|+|k|-1}\,dr\,d\omega.$$

The change of variable: $rp_m(\omega)^{1/m}=s$ leads to

$$\begin{aligned}I_2'(z)&=(2\pi)^{-n}\int_0^\infty(s^m+1)^z s^{|h|+|k|-1}\int_{S_q^*\mathbf{R}^n}p_m(\omega)^{-N_2}\,d\omega\\&=\frac{1}{m}\frac{\Gamma(N_2)\Gamma(-z-N_2)}{\Gamma(-z)}\int_{S_q^*\mathbf{R}^n}p_m(\omega)^{-N_2}\,d\omega\end{aligned}$$

if $\operatorname{Re} z<-N_2$. Thus the case (I) follows. This completes the proof.

5. Asymptotic behavior of eigenvalues. In this section we shall consider the asymptotic behavior of eigenvalues of P under the hypotheses (H.1) $\sim$ (H.6). Let the eigenvalues of P according to multiplicity be $\lambda_1 \leq \lambda_2 \leq \cdots$ and $N_P(\lambda)$ be the counting function of eigenvalues: $N_P(\lambda) = \#\{j\,;\, \lambda_j \leq \lambda\}$. The following theorem is useful in the sequel.

THEOREM 5.1 (cf. [4]). *Let P be a positively definite self-adjoint operator on a separable Hilbert space H with domain of definition K which is a dense subspace of H and the canonical injection from K to H is a compact operator. Here we regard K equipped with the graph norm as a Hilbert space. Assume that*

(i) *P^{-s} is of trace class for large* $\operatorname{Re} s > 0$ *and* $\operatorname{Tr}[P^{-s}]$ *has a meromorphic extension $Z_P(s)$ in $D_\delta = \{s \in \mathbf{C}\,;\, \operatorname{Re} s > a - \delta\}$ for some $a > 0$ and $\delta > 0$.*

(ii) *$Z_P(s)$ has the first singularity at $s = a$ (> 0) and*

$$Z_P(s) - \sum_{j=1}^{p} \frac{A_j}{(j-1)!}\left(-\frac{d}{ds}\right)^{j-1} \frac{1}{s-a}$$

is holomorphic in D_δ.

(iii) *$Z_P(z)$ is of at most polynomial order in* $\operatorname{Im} s$ *in all vertical strips in D_δ, excluding neighborhood of the pole $s = a$.*

Then we have:

$$N_P(\lambda) = \sum_{j=1}^{p} \frac{A_j}{(j-1)!}\left(\frac{d}{ds}\right)^{j-1} \left(\frac{\lambda^s}{s}\right)\bigg|_{s=a} + O(\lambda^{a-\delta})$$

as $\lambda \to \infty$.

The proof is essentially due to the inverse Mellin transformation and given by [4].

Now we return to our consideration. Here we note from the construction of the parametrix of $P - \zeta I$ and the same arguments as in [3] that the condition (iii) of Theorem 5.1 holds.

PROPOSITION 5.2. *Assume that $p(x, \xi) \in \widetilde{S}^{m,M}_{(h,k;l)}$ satisfies* (H.1) $\sim$ (H.6). *Then we have three cases according to Theorem* 4.3.

(I) *When $N_1 > N_2$, we have*

$$N_P(\lambda) = B_1 \lambda^{N_2} + O(\lambda^{N_2 - \delta_1})$$

as $\lambda \to \infty$ where

$$B_1 = \frac{1}{|h| + |k|}(2\pi)^{-n} \int_{S_q^* \mathbf{R}^n} p_m(\omega)^{-N_2}\, d\omega.$$

(II) *When* $N_1 = N_2$, *we have*

$$N_P(\lambda) = B_2\lambda^{N_2}\log\lambda + O(\lambda^{N_2-\delta_0})$$

as $\lambda \to \infty$ *where*

$$B_2 = \frac{|h|+|k|-mN_1}{MT(|h|+|k|-Td_1/2-Td_2/(2l))} \times (2\pi)^{-n}\int_{S_q^*\Sigma}\int_{SN_\omega\Sigma}\tilde{p}_m(\omega, Y)^{-N_2}\,dY_\omega\,d\omega.$$

(III) *When* $N_1 < N_2$, *we have*

$$N_P(\lambda) = B_3\lambda^{N_3} + O(\lambda^{N_3-\delta_3})$$

as $\lambda \to \infty$ *where*

$$B_3 = \frac{|h|+|k|-mN_1}{|h|+|k|-Td_1/2-Td_2/(2l)} \times (2\pi)^{-n}\int_{S_q^*\Sigma}\int_{N_\omega\Sigma}\tilde{p}(\omega, X)^{-N_3}\,dX_\omega\,d\omega.$$

6. Example. We consider the example (0.1):

$$p^w(x, D) = H_{(a,b)} + V(x) \quad \text{on } \mathbf{R}^3$$

where $a(x) = (bx_3^{k+1}, 0, 0)$ and $V(x) = (x_1^2+x_2^2)^l + ax_3^{k+1}$ (b real number, $a > 0$, $k > 0$ odd integer and l positive integer). Then we have $m = 2l(k+1)$, $M = 2$ and $T = l(k+1)$. If we put $\Sigma_1 = \{(x, \xi); \xi_1 = bx_3^{k+1}, \xi_2 = \xi_3 = 0\}$ and $\Sigma_2 = \{(x, \xi); x_1 = x_2 = 0\}$, it is easily seen that (H.2) holds and $d_1 = 3$ and $d_2 = 2$. Moreover we see $N_1 = 3/2 + 1/l < N_2 = 3/2 + 1/l + 1/(2(k+1))$ and $N_3 = 3/2 + 1/l + 1/(k+1)$. Thus by Proposition 5.2 (III), we have

$$N_P(\lambda) = B_4\lambda^{(3/2+1/(k+1)+1/l)} + O(\lambda^{(3/2+1/(k+1)+1/l)-\delta})$$

as $\lambda \to \infty$ where

$$B_4 = \frac{l}{k+1+l+3l(k+1)/2}(2\pi)^{-3} \times \int_{S_q^*\Sigma}\int_{N_\omega\Sigma}\tilde{p}(\omega, X)^{-(3/2+1/(k+1)+1/l)}\,dX_\omega\,d\omega.$$

Here by simple calculation, we see that

$$\tilde{p}(\omega, X) = |X_1|^2 + |X_2|^{2l} + a/\sqrt{b^2+1},$$

so we have

$$
\begin{aligned}
I &= \int_{S_q^*\Sigma}\int_{N_\omega\Sigma}(|X_1|^2+|X_2|^{2l}+a/\sqrt{b^2+1})^{-(3/2+1/(k+1)+1/l)}\,dX_\omega\,d\omega \\
&= \mathrm{Vol}[S_q^*\Sigma]\prod_{j=1}^{2} S_{(d_j-1)} \\
&\quad\times\int_0^\infty\int_0^\infty (s^2+t^{2l}+a/\sqrt{b^2+1})^{-(3/2+1/(k+1)+1/l)} s^{d_1-1}t^{d_2-1}\,ds\,dt \\
&= 2\pi^{5/2}a^{-1/(k+1)}\frac{\Gamma(1/l)\Gamma(1/(k+1))}{l\Gamma(3/2+1/l+1/(k+1))}
\end{aligned}
$$

where $S_{(d_j-1)}$ is the surface area of the unit sphere in $\mathbf{R}^{d_j}$ and we used $\mathrm{Vol}[S_q^*\Sigma]=2(b^2+1)^{-1/(2(k+1))}$. Thus we have

$$B_4=\frac{\Gamma(1/l)\Gamma(1/(k+1))}{2\pi^{1/2}\{3l(k+1)+2(k+1+l)\}a^{1/(k+1)}\Gamma(3/2+1/l+1/(k+1))}.$$

In the particular case $k=l=1$, we have

$$N_P(\lambda)=\frac{1}{48a^{1/2}}\lambda^3+O(\lambda^{3-\delta})$$

as $\lambda\to\infty$.

REMARK 6.1. When $b=0$, we can regard $H_{(a,b)}+V(x)$ as a quasi elliptic operator of order $2l(k+1)$ of type $(k+1, k+1, 2l, l(k+1), l(k+1), l(k+1))$. In this case the result also follows from [3].

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