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ENVELOPING ALGEBRAS OF LIE GROUPS WITH DISCRETE SERIES

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In this article we shall prove that the enveloping algebra of the Lie algebra of a class of unimodular Lie groups having discrete series, when localized at some element of the center, is isomorphic to the tensor product of a Weyl algebra over the ring of Laurent polynomials of one variable and the enveloping aglebra of some reductive algebra. In particular, it will be proved that the Lie algebra of a unimodular solvable Lie group having discrete series satisfies the Gelfand-Kirillov conjecture.

1. Introduction. Let G be a real connected Lie group with center Z, $\mathscr{G}$ and $\mathscr{Z}$ the Lie algebras of G and Z respectively. Let $\mathscr{G}^*$ be the linear dual of $\mathscr{G}$. Then G is said to be an *H-group* if there exists a linear functional $l \in \mathscr{G}^*$ such that the co-adjoint orbit of l in $\mathscr{G}^*$ is the hyperplane $l + \mathscr{Z}^{\perp}$ where $\mathscr{Z}^{\perp} = \{f \in \mathscr{G}^*; f(\mathscr{Z}) = 0\}$ (see Definition 2.1 of [**2**]).

In [**2**] it was proved that a connected Lie group G with center Z is an H-group if and only if G is unimodular and there exists $l \in \mathscr{G}^*$ such that $B_l(\cdot, \cdot) = l([\cdot, \cdot])$ is a non-degenerate skew-symmetric bilinear form on $\mathscr{G}/\mathscr{Z}$.

The class of H-groups plays the key role in the problem of classifying unimodular Lie groups with discrete series. Let us recall that a Lie algebra $\mathscr{H}$ is called an *H-algebra* if it is the Lie algebra of an H-group. The main results of [**1**] and [**2**] may be stated in another form as follows:

A Lie algebra $\mathscr{G}$ is the Lie algebra of some connected unimodular Lie group with discrete series iff $\mathscr{G}$ may be written as the semi direct product of an H-algebra $\mathscr{H}$ with center $\mathscr{Z}$ and a reductive Lie algebra S acting trivially on $\mathscr{Z}$ such that:

- *the maximal semisimple subalgebra of $\mathscr{S}$ has a compact Cartan subalgebra.*
- *the center of $\operatorname{ad}_{\mathscr{H}}(\mathscr{S})$ is the Lie subalgebra of $\operatorname{gl}(\mathscr{H})$ corresponding to a compact torus in $\operatorname{GL}(\mathscr{H})$*

Such an $\mathscr{S}$ clearly acts in a completely reducible manner on $\mathscr{H}$. In the following we shall consider a slightly more general situation:

namely $\mathscr{G}$ is the semidirect product of an H-algebra $\mathscr{H}$ with center $\mathscr{Z}$ and a subalgebra $\mathscr{S}$ acting trivially on $\mathscr{Z}$ such that $\mathscr{H}$ contains an $\mathscr{S}$-invariant subspace $\overline{\mathscr{H}}$ complementing $\mathscr{Z}$. Our aim is to determine the enveloping algebra of such a semidirect product and apply this result to compute the characters of discrete series representations later. In the present article we treat only the case $\dim(\mathscr{Z}) = 1$. Although the case $\dim(\mathscr{Z}) > 1$ is not much different from this, its proof requires one to extend the ground field to an arbitrary field of characteristic 0 and will be treated in another paper.

The main result many be stated as follows:

THEOREM 1. *Let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ and $\mathscr{Z}$ be as above.*[1] *Then for any $\zeta \neq 0$ in $\mathscr{Z}$, the localized ring $\mathbf{A} = \mathbf{U}(\mathscr{H})_\zeta$ is isomorphic to a Weyl algebra $A_n \otimes k[\zeta, \zeta^{-1}]$, where $n = \frac{1}{2}\dim(\mathscr{H}/\mathscr{Z})$. Moreover there exists a Lie algebra homomorphism $X \mapsto a_X$ from $\mathscr{S}$ into $\mathbf{A}$ such that $[X, u] = [a_X, u]$, $\forall u \in \mathbf{A}$. In particular $\mathbf{U}(\mathscr{G})_\zeta$ is isomorphic to $A_n \otimes k[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathscr{S})$.*

In fact, the above isomorphism will be described in more detail for later applications (see Theorem 4.3).

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2. Notation. $\mathbf{N}, \mathbf{R}, \mathbf{C}$ always stand for the natural integers, the real and complex numbers. Recall that if $\mathscr{G}$ is a Lie algebra with one-dimensional center $\mathscr{Z} = \mathbf{R}\zeta$, then the localized enveloping algebra $\mathbf{U}(\mathscr{G})_\zeta$ is defined to be the set of all elements of the form $\zeta^{-n}u$, $n \in \mathbf{N}$, $u \in \mathbf{U}(\mathscr{G})$ with the multiplication: $(\zeta^{-n}u)(\zeta^{-m}v) = \zeta^{-(n+m)}uv$. Let τ be the principal anti-automorphism of $\mathbf{U}(\mathscr{G})$ so that:

$$\tau(X_1X_2\cdots X_n) = (-1)^n X_nX_{n-1}\cdots X_1, \quad \forall X_1, \dots, X_n \in \mathscr{G}.$$

Then it is clear that τ may be extended to an anti-automorphism of $\mathbf{U}(\mathscr{G})_\zeta$ by defining: $\tau(\zeta^{-n}u) = (-1)^n\zeta^{-n}\tau(u)$. An element $u \in \mathbf{U}(\mathscr{G})_\zeta$ is said to be symmetric (resp. skew-symmetric) if $\tau(u) = u$ (resp. $\tau(u) = -u$).

[1] $\odot$ denotes the semidirect product.

Let R be an algebra over $\mathbf{R}$, $n \in \mathbf{N}$; then the *Weyl algebra* $A_n(R)$ is the algebra over R generated by the set $\mathscr{W} = \{\overline{p}_1, \dots, \overline{p}_n, \overline{q}_1, \dots, \overline{q}_n\}$ with relations:

$$\overline{p}_i \overline{q}_j - \overline{q}_j \overline{p}_i = \delta_{ij}, \qquad 1 \le i \le n$$

where δ_{ij} is the Kronecker symbol. We also say that $\mathscr{W}$ is a *Gelfand-Kirillov basis* of $A_n(R)$. More generally, let $\mathbf{A}(R)$ be any algebra over R; then a generating subset $\mathscr{W} = \{p_1, \dots p_n, q_1, \dots, q_n\}$ is said to be a *Gelfand-Kirillov basis* of $\mathbf{A}(R)$ if the mapping: $p_i \mapsto \overline{p}_i$, $q_i \mapsto \overline{q}_i$, $1 \le i \le n$ may be extended to an algebra isomorphism between $\mathbf{A}(R)$ and $A_n(R)$. We often identify $\mathbf{A}(R)$ with $A_n(R)$ and p_i with $\overline{p}_i$, q_i with $\overline{q}_i$, $1 \le i \le n$.

Let $\mathscr{H}_n$ be the $(2n+1)$-dimensional Heisenberg algebra with the standard basis ζ, ξ_i, η_i, $1 \le i \le n$ such that the only nonzero Lie brackets among the elements of this basis are:

$$[\xi_i, \eta_i] = \zeta, \qquad 1 \le i \le n.$$

It is clear that $\mathrm{U}(\mathscr{H}_n)_\zeta$ is a Weyl algebra over $\mathbf{R}[\zeta, \zeta^{-1}]$ with Gelfand-Kirillov basis $p_i = \xi_i$, $q_i = \zeta^{-1}\eta_i$, $1 \le i \le n$. Let τ be the principal anti-automorphism of $\mathrm{U}(\mathscr{H}_n)_\zeta$. Then we have:

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i, \qquad 1 \le i \le n$$

and

$$\tau(\zeta) = -\zeta, \qquad \tau(\zeta^{-1}) = -\zeta^{-1}$$

Such an anti-automorphism of the Weyl algebra $\mathbf{A}_n = A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}]$ is also called the *principal anti-automorphism* of $\mathbf{A}_n$.

3. The nilpotent case. Let $\mathscr{H} = \mathscr{H}_n$ be the Heisenberg algebra with standard basis ζ, ξ_i, η_i, $1 \le i \le n$ as above. Let $\overline{\mathscr{H}} = \sum_{i=1}^n (\mathbf{R}\xi_i + \mathbf{R}\eta_i)$. Then there is a natural symplectic form on $\overline{\mathscr{H}}$ with the canonical symplectic basis ξ_i, η_i, $1 \le i \le n$. The matrix of any $X \in \mathrm{sp}(\overline{\mathscr{H}})$ with respect to this basis has the form:

$$\begin{pmatrix} \mathbf{a}^X & \mathbf{b}^X \\ \mathbf{c}^X & -{}^t\mathbf{a}^X \end{pmatrix}$$

where $\mathbf{a}^X$, $\mathbf{b}^X$, $\mathbf{c}^X$ are $n \times n$-real matrices such that $\mathbf{b}^X$ and $\mathbf{c}^X$ are symmetric, and ${}^t\mathbf{a}^X$ is the transpose of $\mathbf{a}^X$. Put

$$\begin{aligned} S_X &= -\frac{1}{2}\zeta^{-1}\sum_{i,j=1}^{n}\mathbf{a}_{ij}^X(\xi_i\eta_j+\eta_j\xi_i)+\frac{1}{2}\zeta^{-1}\sum_{i,j=1}^{n}(\mathbf{b}_{ij}^X\xi_i\xi_j-\mathbf{c}_{ij}^X\eta_i\eta_j) \\ &= -\frac{1}{2}\sum_{i,j=1}^{n}\mathbf{a}_{ij}^X(p_iq_j+q_jp_i)+\frac{1}{2}\zeta^{-1}\sum_{i,j=1}^{n}\mathbf{b}_{ij}^X p_ip_j \\ &\quad -\frac{1}{2}\zeta\sum_{i,j=1}^{n}\mathbf{c}_{ij}^X q_iq_j. \end{aligned}$$

LEMMA 3.1. *$X \mapsto S_X$ is a Lie algebra homomorphism from* $\mathrm{sp}(\overline{\mathscr{H}})$ *into* $\mathbf{U}(\mathscr{H})_\zeta$ *such that*

$$[X, u] = [S_X, u], \quad \forall X \in \mathrm{sp}(\overline{\mathscr{H}}), \ \forall u \in \mathbf{U}(\mathscr{H})_\zeta.$$

Proof. For $1 \leq i_0 \leq n$ we have:

$$[S_X, \xi_{i_0}] = \sum_{i=1}^{n}\mathbf{a}_{ii_0}^X\xi_i + \sum_{i=1}^{n}\mathbf{c}_{ii_0}^X\eta_i = [X, \xi_{i_0}].$$

Similarly, we have:

$$[S_X, \eta_{i_0}] = [X, \eta_{i_0}].$$

Hence it follows that:

$$[S_X, u] = [X, u], \quad \forall u \in \mathbf{U}(\mathscr{H})_\zeta.$$

Finally by using the commutation relations:

$$\begin{aligned} &[p_iq_j, p_kq_l] = \delta_{il}p_kq_j - \delta_{jk}p_iq_l, \\ &[p_iq_j, p_kp_l] = -\delta_{jk}p_ip_l - \delta_{jl}p_ip_k, \\ &[p_iq_j, q_kq_l] = \delta_{ik}q_lq_j + \delta_{il}q_kq_j, \\ &[p_ip_j, q_kq_l] = \delta_{ik}q_lp_j + \delta_{il}q_kp_j + \delta_{jk}p_kq_l + \delta_{jl}p_ip_k, \end{aligned}$$

we see that

$$[S_X, S_Y] - S_{[x,y]}, \quad \forall X, Y \in \mathrm{sp}(\mathscr{H}). \qquad \square$$

REMARK. The above expression of S_X is just the expression of D_n in [3] rewritten in the terminology of enveloping algebras instead of that of symmetric algebras as in [3].

Now let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ where $\mathscr{H}$ is an H-algebra with one-dimensional center $\mathscr{Z} = \mathbf{R}\zeta$. Assume that $\mathscr{Z}$ centralizes $\mathscr{S}$ and that $\mathscr{H}$ contains an $\mathscr{S}$-invariant subspace $\overline{\mathscr{H}}$ complementing $\mathscr{Z}$. Let $\mathscr{N}$ be the greatest nilpotent ideal of $\mathscr{H}$. Assume also that the center of $\mathscr{N}$ is equal to $\mathscr{Z}$ and that there exists an abelian ideal $\mathscr{K}$ of $\mathscr{G}$ contained in $\mathscr{H}$ such that $\mathscr{K}/\mathscr{Z}$ is central in $\mathscr{N}/\mathscr{Z}$. Put $\overline{\mathscr{K}} = \mathscr{K} \cap \overline{\mathscr{H}}$, $\overline{\mathscr{N}} = \mathscr{N} \cap \overline{\mathscr{H}}$. Let $\eta_1, \dots, \eta_m$ be a basis of $\overline{\mathscr{K}}$. Put:

$$\mathscr{H}_0\{X \in \mathscr{H}\,;\, [X, \overline{\mathscr{K}}] \subset \overline{\mathscr{K}}\}.$$

Then $\mathscr{N}_0 = \mathscr{H}_0 \cap \mathscr{N}$ is precisely the centralizer of $\mathscr{K}$ in $\mathscr{N}$. Let $\overline{\mathscr{H}}_0 = \mathscr{H}_0 \cap \overline{\mathscr{H}}$ and $\overline{\mathscr{N}}_0 = \mathscr{N}_0 \cap \overline{\mathscr{H}}$. Let $m = \dim \overline{\mathscr{K}}$ and $n = \frac{1}{2}\dim(\mathscr{H}/\mathscr{Z})$. Then we have

Proposition 3.2. *Let the notation be as above. Let $X_1, \dots, X_{2n}$ be any basis of $\overline{\mathscr{H}}$. Then there exist a Weyl subalgebra A_m of $\mathbf{U}(\mathscr{G})_\zeta$ with Gelfand-Kirillov basis $\mathscr{W} = \{p_i, q_i\,;\, 1 \le i \le m\}$ and a Lie algebra homomorphism χ from $\mathscr{H}_0 \odot \mathscr{S}$ onto a Lie subalgebra $\widetilde{\mathscr{G}}$ of $\mathbf{U}(\mathscr{G})_\zeta$ satisfying the following properties:*

1. *$\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$ can be identified with a subalgebra of $\mathbf{U}(\mathscr{G})_\zeta$ commuting with A_m such that*

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\widetilde{\mathscr{G}})_\zeta \otimes A_m.$$

Moreover the restriction of the principal anti-automorphism τ of $\mathbf{U}(\mathscr{G})_\zeta$ to $\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$ coincides with the principal anti-automorphism of the latter, and:

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i, \qquad 1 \le i \le m.$$

2. *Let $\widetilde{\mathscr{H}} = \chi(\mathscr{H}_0)$; then χ induces an isomorphism from $\mathscr{H}_0/\overline{\mathscr{K}}$ onto $\widetilde{\mathscr{H}}$. Moreover there exists a basis $\widetilde{X}_1, \dots, \widetilde{X}_{2n-2m}$ of $\widetilde{\mathscr{H}}$ such that each X_i may be expressed as:*

$$X_i = \zeta G_i(q, \zeta^{-1}\widetilde{X}, \zeta^{-1}p), \qquad 1 \le i \le 2n \tag{1}$$

where $q = (q_1, \dots, q_m)$, $\zeta^{-1}\widetilde{X} = (\zeta^{-1}\widetilde{X}_1, \dots, \zeta^{-1}\widetilde{X}_{2n-2m})$, $\zeta^{-1}p = (\zeta^{-1}p_1, \dots, \zeta^{-1}p_m)$ and each G_i is a polynomial of $2n$ indeterminates $\theta = (\theta_1, \dots, \theta_m)$, $\psi = (\psi_1, \dots, \psi_{2n-2m})$, $\omega = (\omega_1, \dots, \omega_m)$ which are in fact linear combinations of 1, ψ, ω with coefficients in $\mathbf{R}[\theta]$ such that the mapping $(\theta, \psi, \omega) \mapsto (G_i(\theta, \psi, \omega))_{1\le i\le 2n}$ is an automorphism of the polynomial ring $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

3. *χ is, in fact, an isomorphism from $\mathscr{S}$ onto $\widetilde{\mathscr{S}} = \chi(\mathscr{S})$ and the action of $\widetilde{\mathscr{S}}$ on $\widetilde{\mathscr{H}}$ is induced from that of $\mathscr{S}$ on $\mathscr{H}_0/\overline{\mathscr{K}}$. Moreover for each $Y \in \mathscr{S}$, $\chi(Y)$ can be expressed as:*

$$\chi(Y) = Y - \zeta S_Y(q, \zeta^{-1}\widetilde{X}, \zeta^{-1}p), \tag{2}$$

where the polynomial $S_Y(\theta, \psi, \omega)$ *is a linear combination of* 1, ψ, ω *with coefficients in* $\mathbf{R}[\theta]$.

Proof. By making a preliminary change of basis if necessary, we may assume that the basis has the form: $\{\eta_1, \dots, \eta_m, X_1, \dots, X_{2n-2m}, \xi_1, \dots, \xi_m\}$ where:

- $\eta_1, \dots, \eta_m$ is a basis of $\overline{\mathscr{Z}}$,
- $X_1, \dots, X_r$ is a basis of $\overline{\mathscr{N}}_0 \bmod \overline{\mathscr{Z}}$,
- $X_{r+1}, \dots, X_{2n-2m}$ is a basis of $\overline{\mathscr{H}}_0 \bmod \overline{\mathscr{N}}_0$,
- $\xi_1, \dots, \xi_m$ is a basis of $\overline{\mathscr{N}} \bmod \overline{\mathscr{N}}_0$.

Moreover it follows from Proposition 3.1 of [**2**] (see also Proposition 4.2 of [**1**]) that $\xi_1, \dots, \xi_m$ may be chosen so that:

$$[\xi_i, \eta_j] = \delta_{ij}\zeta, \qquad 1 \le i \le m.$$

Put $q_i = \zeta^{-1}\eta_i$, $1 \le i \le m$. Now for every $X \in \mathscr{H}_0 \odot \mathscr{S}$ there exists a real $m \times m$-matrix S^X such that:

$$[X, \eta_i] = -\sum_{j=1}^{m} S_{ij}^X \eta_j, \qquad 1 \le i \le m.$$

Note that $S^X = 0$ if $X \in \mathscr{N}_0$. Let l be the linear form on $\mathscr{H}$ such that $l(\zeta) = 1$, $l(\overline{\mathscr{H}}) = 0$, and let B_l be the associated skew-symmetric bilinear form on $\mathscr{H}$. For $X \in \mathscr{H}_0 \odot \mathscr{S}$ and $1 \le i,\ j \le m$ we have:

$$B_l([X, \xi_i], \eta_j) + B_l(\xi_i, [X, \eta_j]) = l([X, [\xi_i, \eta_j]]) = 0.$$

Hence

$$[X, \xi_j] = \sum_{i=1}^{m} S_{ij}^X \xi_i \pmod{\mathscr{N}_0}.$$

Put

$$S_X = -\frac{1}{2} \sum_{i,j=1}^{m} S_{ij}^X (\xi_i q_j + q_j \xi_i).$$

Then $X - S_X$ commutes with the q_i's. Moreover for $1 \le i \le m$ we have:

$$[X - S_X, \xi_i] = 0 \quad \left(\bmod \mathscr{N}_0 + \sum_{j=1}^{m} q_j \mathscr{N}_0\right). \tag{3}$$

It follows that for $X, Y \in \mathscr{H}_0 \odot \mathscr{S}$ we have:

$$[X, S_Y] = [S_X, Y] = [S_X, S_Y] \quad \left(\bmod \sum_{i=1}^{m} q_i \mathscr{N}_0 + \sum_{i,j=1}^{m} q_i q_j \mathscr{N}_0\right).$$

Hence

$$[X - S_X, Y - S_Y] = [X, Y] - [X, S_Y] - [S_X, Y] + [S_X, S_Y]$$
$$= [X, Y] - [S_X, S_Y] \quad \left(\operatorname{mod} \sum_{i=1}^{m} q_i \mathscr{N}_0 + \sum_{i,j=1}^{m} q_i q_j \mathscr{N}_0\right).$$

On the other hand, by a similar computation as in the proof of Lemma 3.1 we see that:

$$[S_X, S_Y] = S_{[X,Y]} \quad \left(\operatorname{mod} \sum_{i,j=1}^{m} q_i q_j \mathscr{N}_0\right).$$

Hence

(4) $[X - S_X, Y - S_Y]$

$$= [X, Y] - S_{[X,Y]} \quad \left(\operatorname{mod} \sum_{i=1}^{m} q_i \mathscr{N}_0 + \sum_{i,j=1}^{m} q_i q_j \mathscr{N}_0\right).$$

Let $Y_1, \dots, Y_t$ be a basis of $\mathscr{S}$. Then it follows from (3) that

$$\begin{aligned}
\mathbf{U}(\mathscr{G})_\zeta &= \mathbf{U}(\mathscr{N}_0)_\zeta[\xi_1, \dots, \xi_m][X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}] \\
&\quad \cdot [Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}] \\
&= \mathbf{U}(\mathscr{N}_0)_\zeta[X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}] \\
&\quad \cdot [Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}][\xi_1, \dots, \xi_m] \\
&= \mathbf{A}[\xi_1, \dots, \xi_m]
\end{aligned}$$

where

$$\mathbf{A} = \mathbf{U}(\mathscr{N}_0)_\zeta[X_{r+1} - S_{X_{r+1}}, \dots, X_{2n-2m} - S_{X_{2n-2m}}][Y_1 - S_{Y_1}, \dots, Y_t - S_{Y_t}].$$

Put $p_1 = \xi_1$, and for $1 \le i \le m-1$ put

$$p_{i+1} = \sum_{j_1, \dots, j_i} \frac{(-1)^{j_1 + \dots + j_i}}{j_1! \cdots j_i!} (\operatorname{ad} \xi_1)^{j_1} \cdots (\operatorname{ad} \xi_i)^{j_i} (\xi_{i+1}) q_1^{j_1} \cdots q_i^{j_i}.$$

On the other hand for $Y \in \mathbf{A}$ put

$$\nu(Y) = \sum_{j_1, \dots, j_m} \frac{(-1)^{j_1 + \dots + j_m}}{j_1! \cdots j_m!} (\operatorname{ad} \xi_1)^{j_1} \cdots (\operatorname{ad} \xi_m)^{j_m} (Y) q_1^{j_1} \cdots q_m^{j_m}.$$

Now by applying successively Lemma 4.7.6 of **[5]** we see that ν is a homomorphism from $\mathbf{A}$ onto a subalgebra $\widetilde{\mathbf{A}}$ of $\mathbf{U}(\mathscr{G})_\zeta$ commuting with the p_i's and q_i's so that

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \widetilde{\mathbf{A}} \otimes A_m.$$

Note that it follows also from Lemma 4.7.5 of [5] that ν induces an isomorphism from $\mathbf{A}/\overline{\mathscr{K}}\mathbf{A}$ onto $\widetilde{\mathbf{A}}$. On the other hand it follows from (4) that $\{X - S_X + \overline{\mathscr{K}}\mathbf{A};\, X \in \mathscr{H}_0\}$ (resp. $\{Y - S_Y + \overline{\mathscr{K}}\mathbf{A};\, Y \in \mathscr{S}\}$) is a Lie subalgebra of $\mathbf{A}/\overline{\mathscr{K}}\mathbf{A}$ isomorphic to $\mathscr{H}_0/\overline{\mathscr{K}}$ (resp. $\mathscr{S}$). Thus $X \mapsto \chi(X) = \nu(X - S_X)$ is a Lie algebra homomorphism from $\mathscr{H}_0 \odot \mathscr{S}$ onto a Lie subalgebra $\widetilde{\mathscr{G}}$ of $\widetilde{\mathbf{A}}$ which induces an isomorphism from $(\mathscr{H}_0/\overline{\mathscr{K}}) \odot \mathscr{S}$ onto $\widetilde{\mathscr{G}}$. Note that $\widetilde{\mathbf{A}}$ can be identified with $\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$. Moreover let $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{S}}$ be the images of $\mathscr{H}_0$ and $\mathscr{S}$ respectively; then the action of $\mathscr{S}$ on $\mathscr{H}_0/\overline{\mathscr{K}}$ is tranformed into the action of $\widetilde{\mathscr{S}}$ on $\widetilde{\mathscr{H}}$.

Now it is clear that $\widetilde{X}_j = \chi(X_j)$ $(1 \le j \le 2n-2m)$ and p_j $(1 \le j \le m)$ may be expressed in the form

$$\widetilde{X}_j = \sum_{i=1}^{r} \mathbf{a}_{ij} X_i + e_j \zeta, \qquad 1 \le j \le r, \tag{5}$$

$$\widetilde{X}_{r+j} = X_{r+j} + \sum_{i=1}^{r} \mathbf{b}_{ij} X_i + \sum_{i=1}^{m} \mathbf{c}_{ij} \xi_i + e_{r+j}\zeta, \qquad 1 \le j \le 2n-2m-r,$$

$$p_j = \xi_j + \sum_{i=1}^{r} \mathbf{d}_{ij} X_i + f_j \zeta, \qquad 1 \le j \le m,$$

where $e_1, \dots, e_{2n-2m}, f_1, \dots, f_m \in \mathbf{R}[q]$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are matrices with coefficients in $\mathbf{R}[q]$ of dimension $r \times r$, $r \times (2n-2m-r)$, $m \times (2n-2m-r)$ and $r \times m$ respectively. Moreover since $\mathscr{N}$ is nilpotent, we may choose X_i, $1 \le i \le r$ so that $\mathbf{a}$ is a unipotent matrix and hence $\det(a) = 1$. For an arbitrary basis $\{X_i,\, 1 \le i \le r\}$ of $\overline{\mathscr{N}}_0$ we can make a change of basis for $\{\widetilde{X}_i\}$ with real matrix coefficients which preserves $\det(\mathbf{a})$. Therefore $\mathbf{a}^{-1}$ is also a matrix with coefficients in $\mathbf{R}[q]$. Hence it follows that the η_i's, X_i's and ξ_i's may be expressed in the form (1) with

- $G_i(\theta, \psi, \omega) = \theta_i$, $1 \le i \le m$,
- for $1 \le i \le r$, $G_{m+i}(\theta, \psi, \omega)$ is a linear combination of 1, $\psi_1, \dots, \psi_r$ with coefficients in $\mathbf{R}[\theta]$,
- for $1 \le i \le 2n-2m-r$, $G_{m+r+i}(\theta, \psi, \omega) - \psi_{r+i}$ is a linear combination of $1, \psi_1, \dots, \psi_r, \omega$ with coefficients in $\mathbf{R}[\theta]$,
- for $1 \le i \le m$, $G_{2n-m+i}(\theta, \psi, \omega) - \omega_i$ is a linear combination of $1, \psi_1, \dots, \psi_r$ with coefficients in $\mathbf{R}[\theta]$.

Hence it is clear that the polynomial map defined by the G_i's is an automorphism of the polynomial ring $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

Finally (2) follows immediately from the definition of χ and a similar computation as above. Note that $X_1, \ldots, X_r$ commute with the q_i's so that

$$\widetilde{X}_j = \sum_{i=1}^{r} X_i \mathbf{a}_{ij} + e_j\zeta, \qquad 1 \le j \le r.$$

Therefore

$$\tau(\widetilde{X}_j) = \sum_{i=1}^{r} \tau(\mathbf{a}_{ij})\tau(X_i) + \tau(e_j\zeta) = -\sum_{i=1}^{r} \mathbf{a}_{ij}X_i - e_j\zeta = -\widetilde{X}_j.$$

This together with (5) imply that the restriction of τ to $\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$ is precisely the principal anti-automorphism of $\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$. □

THEOREM 3.3. *Let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ where $\mathscr{H}$ is a nilpotent H-algebra with one-dimensional center $\mathscr{Z} = \mathbf{R}_\zeta$. Assume that $\mathscr{Z}$ centralizes $\mathscr{S}$ and that $\mathscr{H}$ contains an $\mathscr{S}$-invariant subspace $\overline{\mathscr{H}}$ complementing $\mathscr{Z}$. Let $n = \frac{1}{2}\dim(\mathscr{H}/\mathscr{Z})$.*

1. *Under these conditions, for an arbitrary basis $X_1, \ldots, X_{2n}$ of $\overline{\mathscr{H}}$, there exists a Gelfand-Kirillov basis $\mathscr{W} = \{p_i, q_i; 1 \le i \le n\}$ of $\mathbf{U}(\mathscr{H})_\zeta$ such that*

(i) *$\tau(p_i) = -p_i$, $\tau(q_i) = q_i$, $1 \le i \le n$ where τ is the principal anti-automorphism of $\mathbf{U}(\mathscr{H})_\zeta$;*

(ii) *for $1 \le i \le 2n$, $\zeta^{-1}X_i$ is a linear combination of 1, $\zeta^{-1}p_1, \ldots, \zeta^{-1}p_n$ with coefficients in $\mathbf{R}[q]$ and the corresponding polynomials of $2n$ indeterminates $\theta_1, \ldots, \theta_n$, $\omega_1, \ldots, \omega_n$ define an automorphism of the polynomial ring $\mathbf{R}[\theta, \omega]$ with Jacobian 1.*

2. *For each $Y \in \mathscr{S}$ there exists a polynomial $a_Y(\theta, \omega)$ which is a polynomial of degree ≤ 2 in $\omega_1, \ldots, \omega_n$ with coefficients in $\mathbf{R}[\theta]$ such that:*

(i) *$Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from $\mathscr{S}$ into $\mathbf{U}(\mathscr{H})_\zeta$;*

(ii) *$a_Y(q, \zeta^{-1}p)$ is symmetric and*

$$[Y, u] = [\zeta a_Y(q, \zeta^{-1}p), u], \quad \forall u \in \mathbf{U}(\mathscr{H})_\zeta;$$

(iii) *the mapping $Y \mapsto Y - \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra isomorphism from $\mathscr{S}$ onto a Lie subalgebra $\mathscr{S}'$ of $\mathbf{U}(\mathscr{G})_\zeta$ so that*

$$\begin{aligned}\mathbf{U}(\mathscr{G})_\zeta &\simeq \mathbf{U}(\mathscr{H})_\zeta \otimes \mathbf{U}(\mathscr{S}') \\ &\simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathscr{S}).\end{aligned}$$

Proof. The proof is carried out by induction on $\dim(\mathscr{H})$. If $\mathscr{H}$ is isomorphic to a Heisenberg algebra with center $\mathscr{Z}$ then the theorem follows from Lemma 3.1. Otherwise there is always an abelian ideal $\mathscr{K}$ of $\mathscr{G}$ contained in $\mathscr{H}$ satisfying the conditions of Proposition 3.2 (see Proposition 2.3 of **[1]**). By making a preliminary change of basis if necessary we may assume that

$$X_i = \eta_i, \qquad 1 \le i \le m,$$
$$X_{2n-m+i} = \xi_i, \qquad 1 \le i \le m,$$

where $m = \dim(\mathscr{K}/\mathscr{Z})$. Hence it follows from Proposition 3.2 that there exist a Lie algebra homomorphism χ from $\mathscr{H}_0 \odot \mathscr{S}$ onto a Lie subalgebra $\widetilde{\mathscr{G}}$ of $\mathbf{U}(\mathscr{G})_\zeta$ and elements p_i, $1 \le i \le m$ of $\mathbf{U}(\mathscr{H})_\zeta$ satisfying the following properties.

- $\mathbf{A}_m$ be the subalgebra generated by $\mathscr{W}_1 = \{p_i, q_i; 1 \le i \le m\}$ which is in fact a Weyl algebra with Gelfand-Kirillov basis $\mathscr{W}_1$. Then

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\widetilde{\mathscr{G}})_\zeta \otimes \mathbf{A}_m.$$

- Let τ be the principal anti-automorphism of $\mathbf{U}(\mathscr{G})_\zeta$. Then the restriction of τ to $\mathbf{U}(\widetilde{\mathscr{G}})_\zeta$ coincides with the principal anti-automorphism of the latter, and furthermore

$$\tau(p_i) = -p_i, \quad \tau(q_i) = q_i \qquad 1 \le i \le m,$$

- χ induces an isomorphism from $\mathscr{H}_0/\overline{\mathscr{K}}$ onto $\widetilde{\mathscr{H}} = \chi(\mathscr{H}_0)$ and

$$\mathbf{U}(\mathscr{H})_\zeta \simeq \mathbf{U}(\widetilde{\mathscr{H}})_\zeta \otimes \mathbf{A}_m.$$

- For $m+1 \le i \le 2n$, $\zeta^{-1}X_i$ may be expressed as a linear combination of $1, \zeta^{-1}\widetilde{X}_1, \ldots, \zeta^{-1}\widetilde{X}_{2n-2m}, \zeta^{-1}p_1, \ldots, \zeta^{-1}p_m$ with coefficients in $\mathbf{R}[q_1, \ldots, q_m]$, where $\widetilde{X}_1, \ldots, \widetilde{X}_{2n-2m}$ is a basis of $\widetilde{\mathscr{H}}$ as described in Proposition 3.2. Let $G_i(\theta, \psi, \omega)$ be the corresponding real polynomials. Then the mapping

$$(\theta, \psi, \omega) \mapsto (\theta, G_{m+1}(\theta, \psi, \omega), \ldots, G_{2n}(\theta, \psi, \omega))$$

is an automorphism of $\mathbf{R}[\theta, \psi, \omega]$ with Jacobian 1.

- χ is in fact an isomorphism from $\mathscr{S}$ onto $\widetilde{\mathscr{S}}$ such that

$$\chi(Y) = Y - \zeta \hat{s}_Y(\hat{q}, \zeta^{-1}\widetilde{X}, \zeta^{-1}\hat{p}), \quad \forall Y \in \mathscr{S}$$

where $\hat{q} = (q_1, \ldots, q_m)$, $\hat{p} = (p_1, \ldots, p_m)$, $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_{2n-2m})$, and $\hat{s}_Y(\theta, \psi, \omega)$ is a linear combination of $1, \psi_1, \ldots, \psi_{2n-2m}, \omega_1, \ldots, \omega_m$ with coefficients in $\mathbf{R}[\theta]$.

Now by the induction hypothesis $\mathbf{U}(\widetilde{\mathscr{H}})_\zeta$ is isomorphic to a Weyl algebra with Gelfand-Kirillov basis $\widetilde{\mathscr{W}} = \{\tilde{p}_i, \tilde{q}_i; 1 \le i \le n-m\}$ where the following hold

(α) $\tau(\tilde{p}_i) = -\tilde{p}_i$, $\tau(\tilde{q}_i) = \tilde{q}_i$, $1 \le i \le n-m$,

(β) For $1 \le i \le 2n-2m$, $\zeta^{-1}\widetilde{X}_i$ is a linear combination of $1, \zeta^{-1}\tilde{p}_1, \dots, \zeta^{-1}\tilde{p}_{n-m}$ with coefficients in $\mathbf{R}[\tilde{q}]$ such that the corresponding polynomials $\widetilde{F}_i(\tilde{\theta}, \tilde{\omega})$, $1 \le i \le 2n-2m$ of $2n-2m$ indeterminates $(\tilde{\theta}, \tilde{\omega}) \equiv (\tilde{\theta}_1, \dots, \tilde{\theta}_{n-m}, \tilde{\omega}_1, \dots, \tilde{\omega}_{n-m})$ determine an automorphism $\widetilde{F}$ of $\mathbf{R}[\tilde{\theta}, \tilde{\omega}]$ with Jacobian 1. Put

$$F_i(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \begin{cases} \theta_i, & 1 \le i \le m, \\ G_i(\theta, \widetilde{F}(\tilde{\theta}, \tilde{\omega}), \omega), & m+1 \le i \le 2n. \end{cases}$$

Then the mapping

$$(\theta, \tilde{\theta}, \tilde{\omega}, \omega) \mapsto (F_1(\theta, \tilde{\theta}, \tilde{\omega}, \omega), \dots, F_{2n}(\theta, \tilde{\theta}, \tilde{\omega}, \omega))$$

is an automorphism of $\mathbf{R}[\theta, \tilde{\theta}, \tilde{\omega}, \omega]$ with Jacobian 1. Moreover we have

$$X_i = \zeta F_i(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}), \qquad 1 \le i \le 2n.$$

On the other hand it follows also from the induction hypothesis that for each $\widetilde{Y} \in \widetilde{\mathscr{S}}$ there exists a polynomial $\tilde{a}_{\widetilde{Y}}(\tilde{\theta}, \tilde{\omega})$ which is in fact a polynomial of degree ≤ 2 in $\tilde{\omega}_1, \dots, \tilde{\omega}_{n-m}$ with coefficients in $\mathbf{R}[\tilde{\theta}]$ such that $\widetilde{Y} \mapsto \zeta\tilde{a}_{\widetilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p})$ is a Lie algebra homomorphism from $\widetilde{\mathscr{S}}$ into $\mathbf{U}(\widetilde{\mathscr{H}})_\zeta$ and moreover

$$[\widetilde{Y}, \tilde{u}] = [\zeta\tilde{a}_{\widetilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}), \tilde{u}], \quad \forall \tilde{u} \in \mathbf{U}(\widetilde{\mathscr{H}})_\zeta. \tag{6}$$

Put

$$a_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \tilde{a}_{\chi(Y)}(\tilde{\theta}, \tilde{\omega}) + s_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega), \quad \forall Y \in \mathscr{S}$$

where $S_Y(\theta, \tilde{\theta}, \tilde{\omega}, \omega) = \hat{s}_Y(\theta, \widetilde{F}(\tilde{\theta}, \tilde{\omega}), \omega)$. Then for $\widetilde{Y} = \chi(Y)$ we have

$$\begin{aligned} Y - \zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}) & \\ &= Y - \zeta\tilde{a}_{\widetilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}) - \zeta s_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}) \\ &= \widetilde{Y} - \zeta\tilde{a}_{\widetilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p}). \end{aligned}$$

Hence

$$[Y, \tilde{u}] - [\zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}\hat{p}), \tilde{u}] = 0, \quad \forall \tilde{u} \in \mathbf{U}(\widetilde{\mathscr{H}})_\zeta.$$

On the other hand since $\widetilde{Y}$ and $\zeta\tilde{a}_{\widetilde{Y}}(\tilde{q}, \zeta^{-1}\tilde{p})$ commute with $\{p_i, q_i; 1 \le i \le m\}$ we have

$$[Y, u] = [\zeta a_Y(\hat{q}, \tilde{q}, \zeta^{-1}\tilde{p}, \zeta^{-1}, \hat{p}), u], \quad \forall u \in \mathbf{U}(\mathscr{H})_\zeta. \tag{7}$$

Now it follows from (6) that

$$\begin{aligned}[\zeta\tilde{a}_{\widetilde{Y}_1}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})\,,\,\zeta\tilde{a}_{\widetilde{Y}_2}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})] &= [\widetilde{Y}_1\,,\,\zeta\tilde{a}_{\widetilde{Y}_2}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})] \\ &= [\zeta\tilde{a}_{\widetilde{Y}_1}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})\,,\,\widetilde{Y}_2].\end{aligned}$$

Hence

$$\begin{aligned}&[\widetilde{Y}_1 - \zeta\tilde{a}_{\widetilde{Y}_1}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})\,,\,\widetilde{Y}_2 - \zeta\tilde{a}_{\widetilde{Y}_2}(\tilde{q}\,,\,\zeta^{-1}\tilde{p})] \\ &\quad = [\widetilde{Y}_1\,,\,\widetilde{Y}_2] - \zeta\tilde{a}_{[\widetilde{Y}_1\,,\,\widetilde{Y}_2]}(\tilde{q}\,,\,\zeta^{-1}\tilde{p}) \quad \forall \widetilde{Y}_1\,,\,\widetilde{Y}_2 \in \widetilde{\mathscr{S}}.\end{aligned}$$

Put $p = (\hat{p}\,,\,\tilde{p})$, $q = (\hat{q}\,,\,\tilde{q})$. Then for $Y_1\,,\,Y_2 \in \mathscr{S}$ and $\widetilde{Y}_i = \chi(Y_i)$, $i = 1\,,\,2$, we have

$$\begin{aligned}&[Y_1 - \zeta a_{Y_1}(q\,,\,\zeta^{-1}p)\,,\,Y_2 - \zeta a_{Y_2}(q\,,\,\zeta^{-1}p)] \\ &\quad = [\widetilde{Y}_1\,,\,\widetilde{Y}_2] - \zeta\tilde{a}_{[\widetilde{Y}_1\,,\,\widetilde{Y}_2]}(\tilde{q}\,,\,\zeta^{-1}\tilde{p}) \\ &\quad = \chi([Y_1\,,\,Y_2]) - \zeta\tilde{a}_{\chi([Y_1\,,\,Y_2])}(\tilde{q}\,,\,\zeta^{-1}\tilde{p}) \\ &\quad = [Y_1\,,\,Y_2] - \zeta a_{[Y_1\,,\,Y_2]}(q\,,\,\zeta^{-1}p)\end{aligned}$$

i.e. $Y \mapsto Y - \zeta a_Y(q\,,\,\zeta^{-1}p)$ is a Lie algebra homomorphism which is in fact an isomorphism. Let $\mathscr{S}'$ be the image of $\mathscr{S}$ by this isomorphism. Then $\mathscr{S}'$ commutes with $\mathbf{U}(\mathscr{H})_\zeta$ and hence

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\mathscr{H})_\zeta \otimes \mathbf{U}(\mathscr{S}') \simeq A_n \otimes \mathbf{R}[\zeta\,,\,\zeta^{-1}] \otimes \mathbf{U}(\mathscr{S}).$$

Finally (7) implies that

$$\begin{aligned}&[\zeta a_{Y_1}(q\,,\,\zeta^{-1}p)\,,\,\zeta a_{Y_2}(q\,,\,\zeta^{-1}p)] \\ &\quad = [Y_1\,,\,\zeta a_{Y_2}(q\,,\,\zeta^{-1}p)] \\ &\quad = [\zeta a_{Y_1}(q\,,\,\zeta^{-1}p)\,,\,Y_2], \qquad Y_1\,,\,Y_2 \in \mathscr{S}.\end{aligned}$$

Hence

$$\begin{aligned}&[Y_1\,,\,Y_2] - \zeta a_{[Y_1\,,\,Y_2]}(q\,,\,\zeta^{-1}p) \\ &\quad = [Y_1\,,\,Y_2] - [\zeta a_{Y_1}(q\,,\,\zeta^{-1}p)\,,\,\zeta a_{Y_2}(q\,,\,\zeta^{-1}p)]\end{aligned}$$

i.e. $Y \mapsto \zeta a_Y(q\,,\,\zeta^{-1}p)$ is a Lie algebra homomorphism from $\mathscr{S}$ into $\mathbf{U}(\mathscr{H})_\zeta$. □

REMARK. This theorem contains Lemma 3.2 and Theorem 3.5 of **[6]** as special cases.

4. The general case. Let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ as in Theorem 1 of the Introduction. Assume also that there exists a nilpotent ideal $\mathscr{N}$ of $\mathscr{G}$ contained in $\mathscr{H}$ such that

- $\mathscr{N}$ is an H-algebra with center $\mathscr{Z}$,
- the action of $\mathscr{S}$ on $\mathscr{H}/\mathscr{N}$ is trivial.

Then it follows from Theorem 2.9 and Lemma 2.3 of [**2**] that there exists a Heisenberg subalgebra $\mathscr{H}_1$ of $\mathscr{H}$ with center $\mathscr{Z}$ such that $\mathscr{H} = \mathscr{N} + \mathscr{H}_1$, $\mathscr{N} \cap \mathscr{H}_1 = \mathscr{Z}$ and $[\mathscr{H}_1, \overline{\mathscr{N}}] \subset \overline{\mathscr{N}}$. Moreover $\mathscr{H}_1$ commutes with $\mathscr{S}$.

Now by applying Theorem 3.3 for $\mathscr{G}_1 = \mathscr{N} \odot (\mathrm{ad}_{\overline{\mathscr{N}}}(\mathscr{H}_1) \times \mathscr{S})$ we see that for any basis $X_1, \dots, X_{2m}$ of $\mathscr{N}$ there exists a Gelfand-Kirillov basis $\mathscr{W}_1 = \{p_i, q_i; 1 \le i \le m\}$ of $\mathbf{U}(\mathscr{N})_\zeta$ satisfying the following properties.

(i) For $1 \le i \le m$, p_i is skew-symmetric (resp q_i is symmetric).

(ii) For $1 \le i \le 2m$, $\zeta^{-1}X_i$ is a linear combination of 1, $\zeta^{-1}p_1$, $\dots, \zeta^{-1}p_m$ with coefficients in $\mathbf{R}[q]$. Furthermore the corresponding polynomials F_i, $1 \le i \le 2m$, of $2m$ indeterminates (θ, ω) define an automorphism of $\mathbf{R}[\theta, \omega]$ with Jacobian 1.

(iii) For every $Y \in \mathscr{S}_1 = \mathrm{ad}_{\overline{\mathscr{N}}}(\mathscr{H}_1) \times S$ there exists a polynomial $a_Y(\theta, \omega)$ which may be expressed as a polynomial of deg ≤ 2 in ω with coeffcients in $\mathbf{R}[\theta]$ such that $a_Y(q, \zeta^{-1}p)$ is symmetric and:

- $Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from $\mathscr{S}_1$ into $\mathbf{U}(\mathscr{N})_\zeta$.
- $[Y, u] = [\zeta a_Y(q, \zeta^{-1}p), u]$, $\forall u \in \mathbf{U}(\mathscr{N})_\zeta$.
- $Y \mapsto Y - \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra isomorphism from $\mathscr{S}_1$ into $\mathbf{U}(\mathscr{G}_1)_\zeta$.

Let ζ, ξ_i, η_i, $1 \le i \le n-m$, be the standard Heisenberg basis of $\mathscr{H}_1$, i.e.

$$[\xi_i, \eta_j] = \delta_{ij}\zeta, \qquad 1 \le i, j \le n-m.$$

For $1 \le i \le n-m$ put

$$\tilde{p}_i = \xi_i - \zeta a_{\mathrm{ad}\,\xi_i}(q, \zeta^{-1}p),$$
$$\tilde{q}_i = \zeta^{-1}\eta_i - a_{\mathrm{ad}\,\eta_i}(q, \zeta^{-1}p).$$

Note that

$$[\zeta a_{\mathrm{ad}\,\xi_i}(q, \zeta^{-1}p), \zeta a_{\mathrm{ad}\,\eta_j}(q, \zeta^{-1}p)] = \zeta a_{\mathrm{ad}[\xi_i, \eta_j]}(q, \zeta^{-1}p) = 0.$$

Hence

$$[\tilde{p}_i, \tilde{q}_j] = [\xi_i, \zeta^{-1}\eta_j] = \delta_{ij}.$$

On the other hand for all $\tilde{u} \in \mathbf{U}(\mathscr{N})_\zeta$ we have

$$[\zeta a_{\mathrm{ad}\,\xi_i}(q\,,\,\zeta^{-1}p)\,,\,\tilde{u}] = \mathrm{ad}\,\xi_i(\tilde{u})$$

i.e.

$$[\tilde{p}_i\,,\,\tilde{u}] = 0\,, \qquad 1 \leq i \leq n-m.$$

Similarly, we have:

$$[\tilde{q}_i\,,\,\tilde{u}] = 0\,, \qquad 1 \leq i \leq n-m.$$

In particular $\mathscr{W} = \{p_i\,,\,\tilde{p}_j\,,\,q_i\,,\,\tilde{q}_j\,;\,1 \leq i \leq m\,,\,1 \leq j \leq n-m\}$ is a Gelfand-Kirillov basis for $\mathbf{U}(\mathscr{H})_\zeta$. Furthermore it is clear that the $\tilde{p}_i$'s are skew symmetric (resp. the $\tilde{q}_i$'s are symmetric). It follows that for an arbitrary basis $X_{2m+1}\,,\,\dots\,,\,X_{2n}$ of $\overline{\mathscr{H}}$ complementing $\overline{\mathscr{N}}$ there exist polynomials F_i, $2m+1 \leq i \leq 2n$, of $2n$ indeterminates $(\theta\,,\,\tilde{\theta}\,,\,\tilde{\omega}\,,\,\omega)$ of $\deg \leq 1$ in $\tilde{\omega}$ (resp. of $\deg \leq 2$ in ω) with coefficients in $\mathbf{R}[\theta\,,\,\tilde{\theta}]$ such that

$$X_i = \zeta F_i(q\,,\,\tilde{q}\,,\,\zeta^{-1}\tilde{p}\,,\,\zeta^{-1}p)\,, \qquad 2m+1 \leq i \leq 2n.$$

Moreover the mapping $(\theta\,,\,\tilde{\theta}\,,\,\tilde{\omega}\,,\,\omega) \mapsto (F(\theta\,,\,\omega)\,,\,\widetilde{F}(\theta\,,\,\tilde{\theta}\,,\,\tilde{\omega}\,,\,\omega))$ is an automorphism of $\mathbf{R}[\theta\,,\,\tilde{\theta}\,,\,\tilde{\omega}\,,\,\omega]$ with Jacobian 1, where $F = (F_i)_{1 \leq i \leq 2m}$, $\widetilde{F} = (F_i)_{2m+1 \leq i \leq 2n}$. Finally $Y \mapsto \zeta a_Y(q\,,\,\zeta^{-1}p)$ is a Lie algebra homomorphism from $\mathscr{S}$ into $\mathbf{U}(\mathscr{N})_\zeta$ such that $a_Y(q\,,\,\zeta^{-1}p)$ is symmetric and $\mathscr{S}' = \{Y - \zeta a_Y(q\,,\,\zeta^{-1}p)\,;\,Y \in \mathscr{S}\}$ is a Lie subalgebra of $\mathbf{U}(\mathscr{G})_\zeta$ isomorphic to $\mathscr{S}$ and commuting with the elements of $\mathscr{W}$ so that

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\mathscr{H})_\zeta \otimes \mathbf{U}(\mathscr{S}') \simeq A_n \otimes \mathbf{R}[\zeta\,,\,\zeta^{-1}] \otimes \mathbf{U}(\mathscr{S}).$$

Thus by changing slightly the notation we obtain the following

PROPOSITION 4.1. *Let $\mathscr{G}$, $\mathscr{H}$, $\mathscr{S}$, $\mathscr{N}$ be as above. Then for any basis $X_1\,,\,\dots\,,\,X_{2n}$ of $\overline{\mathscr{H}}$ such that $X_1\,,\,\dots\,,\,X_{2m}$ is a basis of $\overline{\mathscr{N}} = \overline{\mathscr{H}} \cap \mathscr{N}$, we may choose a Gelfand-Kirillov basis $\mathscr{W} = \{p_i\,,\,q_i\,;\,1 \leq i \leq n\}$ of $\mathbf{U}(\mathscr{H})_\zeta$ such that $\mathscr{W}_1 = \{p_i\,,\,q_i\,;\,1 \leq i \leq m\}$ is a Gelfand-Kirillov basis for $\mathbf{U}(\mathscr{N})_\zeta$ with skew-symmetric p_i (resp. symmetric q_i), $1 \leq i \leq n$. Moreover there exist polynomials F_i, $1 \leq i \leq 2n$ and a_Y, $Y \in \mathscr{S}$ of $2n$ indeterminates $(\theta\,,\,\omega)$ satisfying the same properties as those in Theorem* 3.3 *with the only exceptions*:

(i) *for $1 \leq i \leq 2m$, F_i is a polynomial of* deg ≤ 1 *in $\omega_1\,,\,\dots\,,\,\omega_m$ with coefficients in $\mathbf{R}[\theta_1\,,\,\dots\,,\,\theta_m]$*

(ii) *for $2m+1 \leq i \leq 2n$, F_i is a polynomial of* deg ≤ 2 *in $\omega_1\,,\,\dots\,,\,\omega_m$ (resp. of* deg ≤ 1 *in $\omega_{m+1}\,,\,\dots\,,\,\omega_n$) with coefficients in $\mathbf{R}[\theta_1\,,\,\dots\,,\,\theta_n]$*

(iii) *a_Y depends only on $(\theta_1, \dots, \theta_m, \omega_1, \dots, \omega_m)$. In particular*

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\mathscr{H})_\zeta \otimes \mathbf{U}(\mathscr{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathscr{S})$$

where $\mathscr{S}' = \{Y - \zeta a_Y(q_1, \dots, q_m, \zeta^{-1}p_1, \dots, \zeta^{-1}p_m);\ Y \in \mathscr{S}\}$ is a Lie subalgebra of $\mathbf{U}(\mathscr{G})_\zeta$ isomorphic to $\mathscr{S}$ and commuting with $\mathscr{W}$

REMARK. The following lemma, which follows immediately from Proposition V.2.5 of **[4]**, shows that the assumptions of Proposition 4.1 certainly hold if the greatest nilpotent ideal of $\mathscr{H}$ is an H-algebra with center $\mathscr{Z}$.

LEMMA 4.2. *Let $\mathscr{G}$ be a Lie algebra over a field of characteristic 0. Let $\mathscr{H}$ be a solvable ideal of $\mathscr{G}$ and $\mathscr{N}$ the greatest nilpotent ideal of $\mathscr{H}$. Then $[\mathscr{G}, \mathscr{H}] \subset \mathscr{N}$.*

We are now ready to state the

THEOREM 4.3. *Let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ and assume that there exists an $\mathscr{S}$-invariant subspace $\overline{\mathscr{H}}$ as usual. Then*

1. *For any basis $X_1, \dots, X_{2n}$ of $\overline{\mathscr{H}}$ we may choose a Gelfand-Kirillov basis $\mathscr{W} = \{p_i, q_i;\ 1 \le i \le n\}$ of $\mathbf{U}(\mathscr{H})_\zeta$ with skew-symmetric p_i (resp. symmetric q_i), $1 \le i \le n$ and polynomials F_i, $1 \le i \le 2n$ of $2n$ indeterminates (θ, ω) satisfying the following properties:*

(i) *for $1 \le i \le 2n$, F_i is in fact a polynomial of* deg ≤ 2 *in ω with coefficients in $\mathbf{R}[\theta]$;*

(ii) *the mapping $(\theta, \omega) \mapsto (F_i(\theta, \omega))_{1 \le i \le 2n}$ is an automorphism of $\mathbf{R}[\theta, \omega]$ with Jacobian* 1;

(iii) *$X_i = \zeta F_i(q, \zeta^{-1}p)$, $1 \le i \le 2n$;*

2. *for each $Y \in \mathscr{S}$ there exists a polynomial $a_Y(\theta, \omega)$ which is in fact of* deg ≤ 2 *in ω with coefficients in $\mathbf{R}[\theta]$ such that*

(i) *$a_Y(q, \zeta^{-1}p)$ is symemtric;*

(ii) *$Y \mapsto \zeta a_Y(q, \zeta^{-1}p)$ is a Lie algebra homomorphism from $\mathscr{S}$ into $\mathbf{U}(\mathscr{H})_\zeta$;*

(iii) *$\mathscr{S}' = \{Y - \zeta a_Y(q, \zeta^{-1}p);\ Y \in \mathscr{S}\}$ is Lie subalgebra of $\mathbf{U}(\mathscr{G})_\zeta$ isomorphic to $\mathscr{S}$ and commuting with $\mathscr{W}$ so that*

$$\mathbf{U}(\mathscr{G})_\zeta \simeq \mathbf{U}(\mathscr{H})_\zeta \otimes \mathbf{U}(\mathscr{S}') \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes \mathbf{U}(\mathscr{S}).$$

Proof. The proof is carried out by induction on $\dim \mathscr{H}$. Let $\mathscr{N}$ be the greatest nilpotent ideal of $\mathscr{H}$.

(α) If $\mathscr{N}$ is isomorphic to a Heisenberg algebra with center $\mathscr{Z}$ then the theorem follows from Proposition 4.1.

(β) Thus assume that $\mathscr{N}$ is not isomorphic to any Heisenberg algebra with center $\mathscr{Z}$ but the center of $\mathscr{N}$ is still $\mathscr{Z}$. In this case the proof is carried out exactly as in Theorem 3.3. The only difference is when applying the induction hypothesis we obtain polynomials $\widetilde{F}_i(\tilde{\theta}, \tilde{\omega})$, $1 \leq i \leq 2n-2m$ which have $\deg \leq 2$ (instead of $\deg \leq 1$) in $\tilde{\omega}$ with coefficients in $\mathbf{R}[\tilde{\theta}]$. Thus in the final results the polynomials F_i are of $\deg \leq 2$ in ω with coefficients in $\mathbf{R}[\theta]$ as stated in (1.i)

(γ) Finally assume that the center of $\mathscr{N}$ contains $\mathscr{Z}$ strictly. Let $\mathscr{K}$ be a minimal abelian ideal of $\mathscr{G}$ contained in the center of $\mathscr{N}$ such that $\mathscr{K} \neq \mathscr{Z}$. Since the action of $\mathscr{S}$ on $\mathscr{H}/\mathscr{N}$ is trivial by Lemma 4.2, by contragredient the action of $\mathscr{S}$ on $\mathscr{K}/\mathscr{Z}$ is also trivial. Therefore it follows from the proof of Theorem 2.9 of [2] that $\dim(\mathscr{K}/\mathscr{Z}) = 1$. Thus there exist $\xi \in \overline{\mathscr{H}}\backslash\overline{\mathscr{N}}$ and $\eta \in \overline{\mathscr{K}} = \overline{\mathscr{H}} \cap \mathscr{K}$ such that $[\xi, \eta] = \zeta$ and $[\mathscr{S}, \xi] = [\mathscr{S}, \eta] = \{0\}$. Put $q_1 = \zeta^{-1}\eta$ and

$$\mathscr{H}_0\{X \in \mathscr{H}; [X, \overline{\mathscr{K}}] \subset \overline{\mathscr{K}}\} = \mathrm{Cent}_{\mathscr{H}}(\eta).$$

Let D_1 and D_2 be the nilpotent and semisimple parts of the derivation ad ξ so that D_1 may be extended to a locally nilpotent derivation of $\mathbf{U}(\mathscr{H}_0 \odot \mathscr{S})_\zeta$ such that $D_1 q_1 = 1$, $D_1(\mathscr{S}) = 0$. Now the action of $\mathbf{R}D_2 \times \mathscr{S}$ on $\mathscr{H}_0$ defines a semidirect product $\mathscr{G}_0 = \mathscr{H}_0 \odot (\mathbf{R}D_2 \times \mathscr{S})$ which contains $H_0 \odot \mathscr{S}$ as an ideal. Moreover by modifying $\overline{\mathscr{H}}$ outside of the subspace generated by $[\mathscr{S}, \mathscr{H}]$ if necessary we may assume that $\overline{\mathscr{H}}$ is also invariant under the action of D_2. For $X \in \mathbf{U}(\mathscr{G}_0)_\zeta$ put

$$\chi(X) = \sum_i \frac{(-1)^i}{i!} D_1^i(X) q_1^i.$$

Then it follows from Lemma 4.7.5 of [5] that χ is a homomorphism from $\mathbf{U}(\mathscr{G}_0)_\zeta$ onto a subalgebra $\mathbf{A}$ of $\mathbf{U}(\mathscr{G})_\zeta$ commuting with q_1 such that the action of D_1 on $\mathbf{A}$ is trivial. Moreover since D_1 commutes with $\mathbf{R}D_2 \times \mathscr{S}$ it is clear that the action of $\chi(\mathbf{R}D_2 \times \mathscr{S})$ on $\chi(\mathscr{H}_0) = \widetilde{\mathscr{H}_0}$ is induced from the action of $\mathbf{R}D_2 \times \mathscr{S}$ on $\mathscr{H}_0$. Note that $\overline{\mathscr{H}}_0$ is a Lie subalgebra of $\mathbf{A}$ isomorphic to $\mathscr{H}_0/\mathbf{R}\eta$. Again by some preliminary change of basis we may assume that $X_1 = \eta$, $X_{2n} = \xi$. Hence by using an argument similar to that in the proof of Theorem 3.3 and the induction hypothesis, we see that there exist a Gelfand-Kirillov basis $\mathscr{W}_1 = \{p_i, q_i; 2 \leq i \leq n\}$ of $\mathbf{A}$ with skew-symmetric p_i (resp. symmetric q_i) $2 \leq i \leq n$, and polynomials F_i, $2 \leq i \leq 2n-1$ of deg ≤ 2 in $\hat{\omega} \equiv (\omega_2, \ldots, \omega_n)$ with coefficients in $\mathbf{R}[\hat{\theta}] \equiv \mathbf{R}[\theta_2, \ldots, \theta_n]$ such that

- $X_i = \zeta F_i(\hat{q}, \zeta^{-1}\hat{p})$, $2 \leq i \leq 2n-1$,
- $(\hat{\theta}, \hat{\omega}) \mapsto (F_i(\hat{\theta}, \hat{\omega}))_{2 \leq i \leq 2n-1}$ is an automorphism of $\mathbf{R}[\hat{\theta}, \hat{\omega}]$ with Jacobian 1.

Moreover there exist polynomials d, a_Y $(Y \in \mathscr{S})$ of deg ≤ 2 in $\hat{\omega}$ with coefficients in $\mathbf{R}[\hat{\theta}]$ such that

$$(tD_2, Y) \mapsto t\zeta d(\hat{q}, \zeta^{-1}\hat{p}) + \zeta a_Y(\hat{q}, \zeta^{-1}p)$$

is a Lie algebra homomorphism from $\mathbf{R}D_2 \times \mathscr{S}$ into $\mathbf{A}$ and

$$\begin{cases} [D_2, u] = [\zeta d(\hat{q}, \zeta^{-1}\hat{p}), u], \\ [Y, u] = [\zeta a_Y(\hat{q}, \zeta^{-1}\hat{p}), u], \end{cases} \quad \forall u \in \mathbf{A}.$$

In particular

$$[D_2 - \zeta d(\hat{q}, \zeta^{-1}\hat{p}), u] = 0, \quad \forall u \in \mathbf{A}.$$

This shows that $p_1 \equiv \xi - \zeta d(\hat{q}, \zeta^{-1}\hat{p})$ commutes with $\mathbf{A}$ and

$$[p_1, q_1] = D_1(q_1) = 1,$$

i.e. $\mathbf{U}(\mathscr{G})_\zeta$ is isomorphic to a Weyl algebra with Gelfand-Kirillov basis $\mathscr{W} = \{p_i, q_i; 1 \leq i \leq n\}$. Finally by putting

$$F_i(\theta_1, \ldots, \theta_n, \omega_1, \ldots, \omega_n) = \begin{cases} \theta_1 & \text{if } i = 1, \\ F_i(\hat{\theta}, \hat{\omega}) & \text{if } 2 \leq i \leq 2n-1, \\ \omega_1 + d(\hat{\theta}, \hat{\omega}) & \text{if } i = 2n, \end{cases}$$

and

$$a_Y(\theta_1, \hat{\theta}, \omega_1, \hat{\omega}) = a_Y(\hat{\theta}, \hat{\omega})$$

we see that F_i, $1 \leq i \leq 2n$, and a_Y, $Y \in \mathscr{S}$, satisfy the statements (1) and (2) of the theorem. □

COROLLARY 4.4. *Let $\mathscr{G} = \mathscr{H} \odot \mathscr{S}$ as in Theorem 4.3. Let $\mathbf{Z}(\mathscr{G}_c)$ be the center of $\mathbf{U}(\mathscr{G}_c)$, where $\mathscr{G}_c$ is the complexification of $\mathscr{G}$. Then $\mathbf{Z}(\mathscr{G}_c)_\zeta$ is isomorphic to the localized polynomial ring $\mathbf{C}[Y_1, \ldots, Y_r, \zeta, \zeta^{-1}]$ where $Y_1, \ldots, Y_r$ is a basis of some Cartan subalgebra of $\mathscr{S}_c$.*

REMARK. This corollary gives a generalization of Theorem 2 in [3].

COROLLARY 4.5. *The Gelfand-Kirillov conjecture holds for the Lie algebras of connected unimodular solvable Lie groups having discrete series with one-dimensional center. In particular it also holds for H-algebras with one-dimensional center.*

Proof. We can apply the theorem with $\mathscr{S}$ abelian and get

$$\mathbf{U}(\mathscr{G})_\zeta \simeq A_n \otimes \mathbf{R}[\zeta, \zeta^{-1}] \otimes S(\mathscr{S})$$

where $S(\mathscr{S})$ is the symmetric algebra of $\mathscr{S}$. From this it follows that the (skew) field of quotients of $\mathbf{U}(\mathscr{G})$ is isomorphic to the (skew) field of quotients of the Weyl algebra A_n over the polynomial ring $\mathbf{R}[Y_1, \dots, Y_t, \zeta_a]$ where $Y_1, \dots, Y_t$ is a basis of $\mathscr{S}$. □

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