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MINIMAL ORBITS AT INFINITY IN HOMOGENEOUS SPACES OF NONPOSITIVE CURVATURE

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Let M denote a simply connected, homogeneous space of nonpositive curvature and let G be the connected component of the identity of the isometry group of M.

In this paper we study the geometric consequences on M if $M(\infty)$, the boundary sphere of M, admits a G-orbit whose closure is a minimal set for G. A characterization of symmetric spaces of noncompact type in terms of the action of G in $M(\infty)$, is obtained. As an application we give some conditions, in terms of the Lie algebra of a simply transitive and solvable subgroup of G that is in standard position, which are equivalent to the fact that M is a symmetric space.

Introduction. Let M denote a simply connected, homogeneous space of nonpositive curvature $(K \leq 0)$ and let G be the connected component of the identity in $I(M)$, the isometry group of M.

In this paper we study the geometric consequences on M if $M(\infty)$, the boundary sphere of M, admits a G-orbit whose closure is a minimal set for G. In particular, we obtain a characterization of symmetric spaces of noncompact type in terms of the action of G in $M(\infty)$. As an application, some conditions in terms of properties of the Lie algebra of a simply transitive, solvable subgroup of G that is in standard position, which are equivalent to the fact that M is a symmetric space, are obtained.

In §1 we give a characterization of symmetric spaces in terms of the G-minimality of the closure of some orbits of G in $M(\infty)$, or equivalently in terms of K, the stability subgroup of G at any point in M, we obtain that M is a symmetric space of noncompact type if and only if $G(x) = K(x)$ for a particular x in $M(\infty)$ (Theorem 1).

In §2 we get a decomposition of $\mathscr{G}$, the Lie algebra of G, that coincides with the canonical one when M is symmetric. It is used to give, as an application of Theorem 1, a characterization of symmetric spaces of noncompact type in terms of properties of the Lie algebra of a simply transitive, solvable group of isometries of M that is in standard position (Theorem 2).

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Preliminaries. Let M be a complete, simply connected Riemannian manifold with nonpositive sectional curvature ($K \leq 0$). Let $I(M)$ and $I_0(M)$ denote the group of isometries of M and the connected component of the identity respectively. All geodesics of M will be assumed to have unit speed. Geodesics α and β of M are asymptotic if $d(\alpha(t), \beta(t)) \leq c$ for all $t \geq 0$ and some $c > 0$. $M(\infty)$ will denote the set of equivalence classes of asymptotic geodesics. $\overline{M} = M \cup M(\infty)$ equipped with the cone topology is a compactification of M and $M(\infty)$, with the induced topology from $\overline{M}$, is homeomorphic to the $(n-1)$-sphere, where $n = \dim M$. For a geodesic γ of M we let $\gamma(\infty)$, $\gamma(-\infty)$ denote the asymptotic equivalence classes of γ and $\gamma^{-1}(t \to \gamma(-t))$ respectively. Isometries of M extend to homeomorphisms of $M(\infty)$ by defining $g(\gamma(\infty)) = (g \circ \gamma)(\infty)$. Moreover, the map $(g, x) \to g(x)$ of $I(M) \times M(\infty)$ is continuous.

We say that distinct points x and y in $M(\infty)$ can be joined by a geodesic of M if there exists a geodesic γ of M such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. For each point p in M the geodesic symmetry $s_p \colon M \to M$ is defined by $s_p(\gamma(t)) = \gamma(-t)$ for all geodesics γ of M with $\gamma(0) = p$ and for all t in $\mathbb{R}$. The map s_p fixes p and is a diffeomorphism of M ($s_p = \exp_p \circ S \circ \exp_p^{-1}$, where $S(v) = -v$ for all v in T_pM). Let G^* denote the subgroup of diffeomorphisms of M generated by the geodesic symmetries $\{s_p \colon p \in M\}$. It is called the symmetry diffeomorphism group of M. The group G^* acts on $M(\infty)$ by homeomorphisms setting for each $p \in M$ and $x \in M(\infty)$, $s_p(x) = \gamma_{px}(-\infty)$ (where γ_{px} denotes the unique geodesic such that $\gamma_{px}(0) = p$ and $\gamma_{px}(\infty) = x$).

Let Γ denote any subgroup of $I(M)$. Two points x and y in $M(\infty)$, not necessarily distinct, are said to be Γ-dual if there exists a sequence $\{g_n\} \subset \Gamma$ such that $g_n(p) \to x$ and $g_n^{-1}(p) \to y$ as $n \to \infty$ for some (or any) point p of M. The set of points in $M(\infty)$ that are Γ-dual to a given point $x \in M(\infty)$ is closed in $M(\infty)$ and invariant under Γ. The limit set $L(\Gamma)$ is defined by $L(\Gamma) = \Gamma(p)^- \cap M(\infty)$ ($p \in M$) where $\Gamma(p)^-$ is the closure of the Γ-orbit of p in $\overline{M}$. A closed subset $X \subseteq M(\infty)$ is said to be a minimal set for Γ if $\Gamma(x)^-$ (the closure of the Γ-orbit of x in $M(\infty)$) coincides with X for every $x \in X$.

Assume that M is homogeneous. Then M admits a solvable Lie group S acting simply and transitively on M (see [**1**, Proposition 2.5]). Let $\mathfrak{s}$ denote the Lie algebra of S. We know that S with the left invariant metric associated to the p-inner product on $\mathfrak{s}$, which is induced by the action of $I(M)$ on M $(g \to g(p)$, p is any fixed point in $M)$ is isometric to M. Moreover, $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{a}$ where $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$, is an abelian subalgebra of $\mathfrak{s}$ (see [**1**, Theorem 5.2]). For each $H \in \mathfrak{a}$, $\gamma_H(t) = \exp tH(p)$ is a geodesic of M since $\exp tH$ is a geodesic of S (see in [**2**, §3] the expression of the Riemannian connection associated to a left invariant metric). The connected Lie subgroup $A = \exp(\mathfrak{a})$ with Lie algebra $\mathfrak{a}$ is a flat totally geodesic submanifold of S.

Let $\lambda \in (\mathfrak{a}^{\mathrm{c}})^*$. λ is said to be a root of $\mathfrak{a}$ in $\mathfrak{s}$ if $\mathfrak{s}^{\mathrm{c}}_{\lambda} = \{U \in \mathfrak{s}^{\mathrm{c}} \colon (\mathrm{ad}_H - \lambda(H)I)^k U = 0$ for some $k \geq 1$ and all $H \in \mathfrak{a}\}$ is nonzero. Here, $\mathfrak{a}^{\mathrm{c}}$ and $\mathfrak{s}^{\mathrm{c}}$ denote the complexification of $\mathfrak{a}$ and $\mathfrak{s}$ respectively (see [**2**, §5]).

If $G = I_0(M)$ and K is any maximal compact subgroup of G, by the maximality of K and the Cartan fixed point theorem there exists a point $p \in M$ such that $K = G_p$, the stability subgroup of G at p (G_p is compact by Theorem 2.5 (Ch.IV) of [**8**]). Hence for any $p \in M$, G_p is a maximal compact subgroup of G since the stability subgroups of G are conjugate in G.

1. The orbits of $G = I_0(M)$ as minimal sets for G in $M(\infty)$. Let M be a simply connected, homogeneous space of nonpositive sectional curvature. In this section we give a characterization of symmetric spaces of noncompact type in terms of the G-minimality of the closure of some orbits of $G = I_0(M)$ in $M(\infty)$. For any $z \in M(\infty)$ let G_z denote the subgroup of G defined by $G_z = \{g \in G \colon g(z) = z\}$.

The proof of the following lemma can be found in [**3**, Lemma 2.4a]. We state it here because it will be used often.

LEMMA 1.1. *Let Γ be any group of isometries of M. Let $x \in M(\infty)$ and let γ be a geodesic in M such that $x = \gamma(\infty)$. If $y = \gamma(-\infty)$ and z is Γ-dual to y then $z \in \Gamma(x)^-$.*

PROPOSITION 1.2. *Let Γ be a subgroup of $I(M)$ acting transitively on M. Assume that $\Gamma(y)^-$, the closure of the Γ-orbit of y in $M(\infty)$, is a minimal set for Γ. If x is a point in $M(\infty)$ which is joined to y by a geodesic of M then x is Γ-dual to y.*

Proof. Let γ be a geodesic of M with end points $x = \gamma(\infty)$ and $y = \gamma(-\infty)$ and set $p = \gamma(0)$. Since $L(\Gamma) = M(\infty)$ (Γ acts transitively on M) we can find a sequence $\{g_n\} \subset \Gamma$ such that $g_n(p) \to x$ as $n \to \infty$. Passing to a subsequence if necessary, $g_n^{-1}(p)$ converges to a point $z \in M(\infty)$ as $n \to \infty$. By Lemma 1.1, $z \in \Gamma(y)^-$ since z is Γ-dual to x and y is joined to x by γ. By hypothesis, $\Gamma(z)^- = \Gamma(y)^-$ and hence $y \in \Gamma(z)^-$. We note that $\Gamma(z)^-$ is contained in the set of points which are Γ-dual to x since this set is closed, invariant under Γ and z is Γ-dual to x. Thus, y is Γ-dual to x or x is Γ-dual to y.

THEOREM 1. *Let M be an irreducible, simply connected and nonflat homogeneous space of nonpositive sectional curvature. Set $G = I_0(M)$. Let $x \in M(\infty)$ be a point such that G_x acts transitively on M. If $y \in M(\infty)$ is a point that can be joined to x by a geodesic of M, then the following properties are equivalent.*

(1) *$G(y)^-$ is a minimal set for G in $M(\infty)$.*
(2) *$G(y) = K(y)$ for any maximal compact subgroup K of G.*
(3) *$G(y)$ is a closed subset of $M(\infty)$.*
(4) *M is a symmetric space of noncompact type.*
(5) *G_y acts transitively on M.*

REMARK. If M is a simply connected, homogeneous space of nonpositive curvature then M admits a simply transitive, solvable group S of isometries that has a fixed point in $M(\infty)$ by Theorem 3.4 of **[5]** (M has no flat de Rham factor). Moreover, if S is a transitive group of isometries of M that does not have a fixed point in $M(\infty)$, then M must be symmetric of noncompact type by [7, Proposition 4.4.7].

Proof of Theorem 1. (1) $\Rightarrow$ (2) Let $K \subsetneq G$ be any maximal compact subgroup. Then there exists a point $p \in M$ such that $K = G_p$, and hence $G = K \cdot G_x$ since G_x acts transitively on M. Let $y \in M(\infty)$ be a point that can be joined to x by a geodesic of M. Then,

(i) $G(x) = K(x)$.

Let $p \in M$ be the point above, and let $x^* = s_p(y) = \gamma_{py}(-\infty)$. Then x^* is G-dual to y by Proposition 1.2 and the fact that $G(y)^-$ is a minimal set for G in $M(\infty)$. Hence $x^* \in G(x)^- = K(x)$ by (i) and Lemma 1.1, and we obtain

(ii) $x = k^*(x^*)$ for some $k^* \in K$.

Let $g \in G$ be given, and let $z = s_p(g(y)) = \gamma_{pg(y)}(-\infty)$. Then y can be joined to $g^{-1}(z)$ by a geodesic of M, and hence y and $g^{-1}(z)$ are G-dual by Proposition 1.2. Therefore y and z are G-dual, and it follows that $z \in G(x)^- = K(x)$ by (i) and Lemma 1.1. From (ii) we obtain

(iii) $z = k(x^*)$ for some $k \in K$.

Finally, $y = \gamma_{px^*}(-\infty)$ and therefore $k(y) = \gamma_{k(p)k(x^*)}(-\infty) = \gamma_{pz}(-\infty) = g(y)$ by (iii) and the definitions of p and z. Hence $G(y) = K(y)$.

(2) $\Rightarrow$ (3) This is obvious since K is compact.

(3) $\Rightarrow$ (4) We set $K = G_p$ where $p = \gamma(0)$ and γ is the geodesic in M such that $\gamma(-\infty) = y$ and $\gamma(\infty) = x$. If $G(y)$ is closed then $G(y)^- = G(y)$ is a minimal set for G in $M(\infty)$, and hence $G(y) = K(y)$ by (1) $\Rightarrow$ (2) since K is a maximal compact subgroup of G.

If $X = K(x) \cup K(y)$ then $X = G(x) \cup G(y)$ is a closed, G-invariant subset of $M(\infty)$. It then follows that X is invariant under the symmetry diffeomorphism group G^* since $s_p(k(x)) = k(y)$, $s_p(k(y)) = k(x)$ for any $k \in K$ ($k \circ \gamma$ joins $k(x)$ and $k(y)$ through p) and $s_{g(p)} = g \circ s_p \circ g^{-1}$ $(M = G(p))$.

Suppose that $X = M(\infty)$. Since $M(\infty)$ is homeomorphic to the $(n-1)$-sphere, it follows from Baire's Theorem that $G(x)$ (or $G(y)$) has interior nonempty. Then $G(x)$ (or $G(y)$) is an open set in $M(\infty)$ which is also closed, and consequently $G(x) = M(\infty)$. In this case, by applying Proposition 4.12 of [3], M is a symmetric space of rank one.

If $X \subsetneqq M(\infty)$, it follows from Theorem 3.2 of [6] that M is a symmetric space of noncompact type of rank ≥ 2 since it is irreducible.

We remark that in the proof above we only needed a geodesic γ of M satisfying $\gamma(0) = p$ and $G(\gamma(\pm\infty)) = K(\gamma(\pm\infty))$.

(4) $\Rightarrow$ (5) Note that if $K = G_p$ $(p \in M)$, $G(y) = K(y)$ by Theorem 4.5 of [3], and it follows immediately that $G = K \cdot G_y = G_y \cdot K$. Hence $M = G(p) = G_y(p)$ since G acts transitively on M.

(5) $\Rightarrow$ (1) If G_y acts transitively on M we have that $G = K \cdot G_y = G_y \cdot K$, where $K = G_p$ $(p \in M)$. Thus $G(y) = K(y)$ is a closed subset of $M(\infty)$, and hence $G(y)^- = G(y)$ is a minimal set for G in $M(\infty)$.

This completes the proof of Theorem 1.

Note that Theorem 5.4 of [3] and Proposition 4.7.1 of [7] show that if M is simply connected and homogeneous with sectional curvature $K \leq 0$, then M is symmetric of noncompact type if and only if $G(y)^-$ is a minimal set for G for *every* $y \in M(\infty)$. Thus, by the remark above, Theorem 1 gives us a strengthened version of this result.

2. A canonical decomposition of the Lie algebra of $I_0(M)$. Let M be a simply connected, homogeneous space of nonpositive sectional curvature. We assume that M has no flat de Rham factor. We denote by B the Killing form on $\mathfrak{g}$, the Lie algebra of $G = I_0(M)$.

In this section, a decomposition of $\mathfrak{g}$ that coincides with the canonical one when M is symmetric of noncompact type, is obtained. As an application of Theorem 1, we get some algebraic conditions in terms of $\mathfrak{g}$ and the data $\mathfrak{s}$, the Lie algebra of a subgroup S of G that acts simply transitively on M and is in standard position, in order to ensure that M is a symmetric space of noncompact type.

A closed subgroup S of G is said to be in standard position if

(i) S acts simply transitively on M.

(ii) For some point $p \in M$, $B(H, U) = 0$ for all $H \in \mathfrak{a}$ and $U \in \mathfrak{k}$, where $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ relative to the p-inner product on $\mathfrak{s}$ and $\mathfrak{k}$ is the Lie algebra of K, the stability subgroup of G at p.

We remark on the following facts about groups that are in standard position:

(1) If $B(\mathfrak{a}, \mathfrak{k}) = 0$ for *one* point $p \in M$ then $B(\mathfrak{a}, \mathfrak{k}) = 0$ for *every* $p \in M$.

(2) There is a simply transitive, solvable group of isometries of M that is in standard position. If a simply transitive, solvable group S is in standard position, then gSg^{-1} is also in standard position for any $g \in G$.

(3) If S_1 and S_2 are two simply transitive, solvable groups of isometries on M in standard position, then they are conjugate by an element of G.

(4) If M is a symmetric space of noncompact type and $G = K \cdot A \cdot N$ is an Iwasawa decomposition of G, then $S = A \cdot N$ is a simply transitive, solvable group of isometries of M in standard position.

We refer the reader to §6 (pages 45–57) of [2] for a more complete discussion. The definition and facts mentioned above are explicitly stated there (6.4, 6.5-(a), 6.5-(c), Theorem 6.7 and Corollary 6.10).

Let S be a solvable Lie subgroup of G that acts simply-transitively on M and is in standard position. Let K be the stability subgroup of G at p, a point in M chosen arbitrarily, and let $\mathfrak{p} = \{X \in \mathfrak{g} : B(X, U) = 0$ for every $U \in \mathfrak{k}\}$.

PROPOSITON 2.1. *$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a direct sum decomposition of $\mathfrak{g}$ such that* $\mathrm{Ad}(k)(\mathfrak{p}) \subseteq \mathfrak{p}$. *Moreover, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.*

Proof. We first show that B restricted to $\mathfrak{k}\times\mathfrak{k}$ is negative definite. Although the proof of this fact is the same as that in the symmetric case, we include it for the sake of completeness.

Since K is compact and acts on $\mathfrak{g}$ by the adjoint representation $\mathrm{Ad}(K)\subset \mathrm{Gl}(\mathfrak{g})$, $\mathfrak{g}$ admits an inner product $(\ ,\ )$ such that $\mathrm{Ad}(k)$ are isometries for all $k\in K$. Thus, ad_X is skew symmetric with respect to $(\ ,\ )$ for every $X\in\mathfrak{k}$. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\ ,\ )$. For $X\in\mathfrak{g}$,

$$\begin{aligned} B(X, X) = \mathrm{tr}(\mathrm{ad}_X\circ \mathrm{ad}_X) &= \sum_i(\mathrm{ad}_X^2 X_i, X_i)\\ &= -\sum_i(\mathrm{ad}_X X_i, \mathrm{ad}_X X_i)\le 0, \end{aligned}$$

and the equality holds if and only if $X\in\mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$. By Theorem 2.1 and Proposition 2.3 of [3], $\mathfrak{z}(\mathfrak{g})=0$ since it is the Lie algebra of the center of G. Thus, $B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite.

Next we will prove the proposition. It is clear that $\mathfrak{p}$ is a subspace of $\mathfrak{g}$ which is $\mathrm{Ad}(K)$ invariant since $\mathfrak{k}$ and B are both invariant under $\mathrm{Ad}(K)$. From the assertion above, we have that $\mathfrak{k}\cap\mathfrak{p}=0$. It remains to show that $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Let $\{X_i\}$ be a basis for $\mathfrak{k}$ so that $B(X_i, X_j)=-\delta_{ij}$ ($B|_{\mathfrak{k}\times\mathfrak{k}}$ is negative definite). If $X\in\mathfrak{g}$, we set $Y = X-\sum_i B(X, X_i)/B(X_i, X_i)X_i$. $Y\in\mathfrak{p}$, and hence $X = \sum_i B(X, X_i)/B(X_i, X_i)X_i + Y\in\mathfrak{k}+\mathfrak{p}$.

Since $B(\mathfrak{a}, \mathfrak{k})=0$, we have that $\mathfrak{a}\subset\mathfrak{p}$. The last assertion follows from Lemma 2.2 below since $C_{\mathfrak{p}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{p}$, is $\mathfrak{a}$.

LEMMA 2.2. $C_{\mathfrak{p}}(\mathfrak{a}) = \{X\in\mathfrak{p}\colon [X, H]=0$ *for all* $H\in\mathfrak{a}\}=\mathfrak{a}$.

Proof. Let H be an element in $\mathfrak{a}$ satisfying $\alpha(H)>0$ for all $\alpha\in\mathfrak{a}^*$ such that $\alpha+i\beta$ is a root of $\mathfrak{a}$ in $\mathfrak{s}'=[\mathfrak{s}, \mathfrak{s}]$. Such an H exists since M has no flat de Rham factor (see [**1**, Proposition 5.6]).

Let X be a unit vector in $\mathfrak{p}$ such that $[X, H]=0$. If $X(t)$ is the variation vector field $\partial f/\partial s\ (0, t)$ on $\gamma_H(t)=\exp tH(p)$, where $f\colon \mathbb{R}\times\mathbb{R}\to M$ is the geodesic variation of γ_H given by $f(s, t)=\exp sX\exp tH(p)$, then X is a Jacobi vector field on γ_H with $X(0)=d\varphi_e X$ ($\varphi\colon G\to M$ is defined by $\varphi(g)=g(p)$). Moreover, $f(s, t)=(\exp tH\exp sX)(p)$ since $[H, X]=0$. Therefore, $X(t)=d(\exp tH)_p(d\varphi_e X)$ and $|X(t)|=|d\varphi_e X|$. Since X is a Jacobi vector field on γ_H, it follows that it is also parallel on γ_H

(the convex function $g(t) = |X(t)|^2$ is constant and hence, $g''(t) = |\nabla_{\gamma_H'} X|^2_{\gamma_H(t)} - K(\gamma_H'(t), X(t)) = 0$). Hence, X induces a parallel Jacobi vector field J on the geodesic $\exp tH$ in S such that $p(J(0)) = X$, where p denotes the projection from $\mathfrak{g}$ onto $\mathfrak{p}$ associated to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. By the same argument as in the proof of Theorem 1.3 of [**4**], it follows that $J(0) \in \mathfrak{a}$ and hence, $X \in \mathfrak{a}$ since $\mathfrak{a} \subset \mathfrak{p}$.

We observe that we have actually shown that $C_{\mathfrak{p}}(H)$, the centralizer of H in $\mathfrak{p}$, is $\mathfrak{a}$. Here, H is chosen as in the beginning of the proof of Lemma 2.2.

THEOREM 2. *Let M be a simply connected, homogeneous space of nonpositive curvature with no flat de Rham factor. Let $\mathfrak{g}$ be the Lie algebra of $G = I_0(M)$ and let S be a subgroup of G that acts simply transitively on M and is in standard position. Let $\mathfrak{s}$ and $\mathfrak{k}$ be the Lie algebra of S and K respectively, where K is the stability subgroup of G at a point p in M chosen arbitrarily. If $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form B on $\mathfrak{g}$ and $\mathfrak{a}$ is the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$, relative to the inner product on $\mathfrak{s}$ induced by p, then the following properties are equivalent.*

(1) $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{k}$.

(2) $\mathfrak{p} = \bigcup\{\mathrm{Ad}(k)(\mathfrak{a}) : k \in K\}$.

(3) *The geodesics through the point p are orbits* $\exp tX(p)$ *for every* $X \in \mathfrak{p}$.

(4) *M is a symmetric space of noncompact type.*

Proof. (1) $\Rightarrow$ (2) (See [**8**, Lemma 6.3 (iii), Ch. V].) Let H be an element in $\mathfrak{a}$ such that $C_{\mathfrak{p}}(H) = \mathfrak{a}$ (see the remark at the end of Lemma 2.2). Let $X \in \mathfrak{p}$ be fixed and let $f: K \to \mathbb{R}$ be the map defined by $f(k) = B(H, \mathrm{Ad}(k)X)$. We will show that $\mathrm{Ad}(k_0)X \in \mathfrak{a}$ whenever k_0 is a critical point of f. In fact, for such a k_0 (it exists since K is compact) and any $U \in \mathfrak{k}$ the function of $t \in \mathbb{R}$, $f_U(t) = f(\exp tUk_0)$ has a critical point at $t = 0$. Hence,

$$\begin{aligned} 0 = f_U'(t) &= B(H, [U, \mathrm{Ad}(k_0)X]) = B([H, U], \mathrm{Ad}(k_0)X) \\ &= -B(U, [H, \mathrm{Ad}(k_0)X]) \end{aligned}$$

(for any $Z \in \mathfrak{g}$, ad_Z is skew symmetric relative to B). Note that $[H, \mathrm{Ad}(k_0)X] \in \mathfrak{k}$ since $[\mathfrak{a}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant. Moreover, the result above is true for all $U \in \mathfrak{k}$. Now, from the fact that B is negative definite on $\mathfrak{k}$, it follows that $[H, \mathrm{Ad}(k_0)X] = 0$. Hence, $\mathrm{Ad}(k_0)X \in \mathfrak{a}$ or $X \in \mathrm{Ad}(k_0^{-1})(\mathfrak{a})$.

(2) $\Rightarrow$ (3) Note that under our hypothesis $K \neq \mathrm{id}$; otherwise, $\mathfrak{a} = \mathfrak{p}$ and M is Euclidean ($K(X, Y) = 0$ for all X and $Y \in \mathfrak{a}$). Given $X \in \mathfrak{p}$ choose $k \in K$ and $H \in \mathfrak{a}$ so that $X = \mathrm{Ad}(k)H$. Then $\exp tH(p)$ is a geodesic of M and hence so is $k\gamma(t) = k(\exp tHk^{-1})(p) = \exp tX(p)$.

(3) $\Rightarrow$ (4) Assume first that M is irreducible. Let γ_H be the geodesic of M defined by $\gamma_H(t) = \exp tH(p)$. We choose H a unit vector in $\mathfrak{a}$ such that $x = \gamma_H(\infty)$ is a fixed point of S (see Theorem 3.4 of [**5**]) and we will show that if $y = \gamma_H(-\infty)$ then $G(y) = K(y)$ for $K = G_p$. It will then follow from Theorem 1 that M is a symmetric space of noncompact type since $G(y)$ is closed in $M(\infty)$.

Let g be any element in G and set $g_n = \exp nHg^{-1}$. Since $y = \lim \exp -nH(p)$ as $n \to \infty$, we have that $g(y) = \lim g \exp -nH(p) = \lim g_n^{-1}(p)$ as $n \to \infty$. Suppose that $g_n(p) = \exp t_n X_n(p)$ with X_n a unit vector in $\mathfrak{p}$. Therefore, there exists $\{k_n\} \subset K$ so that $g_n^{-1} \exp t_n X_n = k_n$ and $g_n^{-1} = k_n \exp -t_n X_n$. By assuming that $k_n \to k$, choosing a subsequence if necessary, we get $g(y) = \lim g_n^{-1}(p) = k(y)$ as $n \to \infty$ since $X_n \to H$ $(g_n(p) \to x)$.

In the general case, assume that $M = M_1 \times M_2$ where M_1 and M_2 are irreducible. Since $G = G_1 \times G_2$ (direct product) with $G_i = I_0(M_i)$, if $p = (p_1, p_2)$ and K_i is the stability subgroup of G_i at p_i $(i = 1, 2)$, we have that $K = K_1 \times K_2$ and hence $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_i$ is the Lie algebra of K_i $(i = 1, 2)$. Thus, if $\mathfrak{p}_i$ is the orthogonal complement of $\mathfrak{k}_i$ with respect to the Killing form B_i on $\mathfrak{g}_i$, the Lie algebra of G_i, it follows that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ since $B = B_1 \oplus B_2$ (note that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum of ideals). Then the geodesics through the points p_i are orbits $\exp tX_i$ with $X_i \in \mathfrak{p}_i$ for $i = 1, 2$, and hence M_i is a symmetric space of noncompact type. Therefore M is symmetric.

(4) $\Rightarrow$ (1) We note that $\mathfrak{g}$ is semisimple (M has no flat de Rham factor) and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the canonical decomposition of $\mathfrak{g}$ associated to $M = G/K$. Hence, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ and (1) follows since $\mathfrak{a} \subset \mathfrak{p}$.

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