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LIE ALGEBRAS OF TYPE D_4 OVER NUMBER FIELDS

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In this paper we show how to construct all central simple Lie algebras of type D_4 over an algebraic number field. The construction that we use is a special case of a modified version of a construction due to G. B. Seligman. The starting point for the construction is an 8-dimensional nonassociative algebra with involution $\mathrm{CD}(\mathscr{B}, \mu)$ that is obtained by the Cayley-Dickson doubling process from a 4-dimensional separable commutative associative algebra $\mathscr{B}$ and a nonzero scalar μ. The algebra $\mathrm{CD}(\mathscr{B}, \mu)$ is used as the coefficient algebra for a Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ that can be roughly described as the Lie algebra of 3×3-skew hermitian matrices with entries from $\mathrm{CD}(\mathscr{B}, \mu)$ relative to the involution $X \to \gamma^{-1}\overline{X}^t\gamma$, where γ is an invertible diagonal matrix with scalar entries. We show that any Lie algebra of type D_4 over a number field can be constructed as $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ for some choice of $\mathscr{B}$, μ and γ. We also give isomorphism conditions for two Lie algebras constructed in this way.

As background, we note that the problem of constructing all central simple Lie algebras of a given type over a field of characteristic 0 has previously been solved for types A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 5)$, G_2 and F_4 by W. Landherr, N. Jacobson, and M. L. Tomber ([**J5**, Chapter X], [**F&F**, Section 7]). Over number fields, this problem has been solved for types E_6, E_7 and E_8 by J. C. Ferrar using the 2nd Lie algebra construction of J. Tits and the Galois cohomological results of M. Kneser, G. Harder and V. I. Cernousov ([**F1**], [**F2**], [**F3**]).

Our main tool in this paper will be an associative algebra invariant $\mathscr{E}(\mathscr{L})$, which we call the Allen invariant, that can be associated to any Lie algebra $\mathscr{L}$ of type D_4 over a field of characteristic 0. $\mathscr{E}(\mathscr{L})$ was introduced for special D_4's by Jacobson [**J2**] and in general by H. P. Allen [**All1**]. Sections 2–6 of this paper are devoted to the study of the invariant $\mathscr{E}(\mathscr{L})$. The main result obtained in these sections is a characterization, using the corestriction of algebras, of the associative algebras that can arise as Allen invariants of Lie algebras of type D_4 over a number field. In §7 (and in an appendix—§12), we use the cohomological results of Harder and Kneser to prove a general isomorphism theorem for Lie algebras of type D_4 over number fields. Section 8 then contains the proof of the main results mentioned previously regarding the construction $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ over number fields. In §§9 and 10, we apply our results to describe anisotropic and Jordan D_4's over number fields. In §9 we also obtain a local global principle for strongly isotropic D_4's. Finally, in §11, we describe how to use the construction $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ to obtain all D_4's with a given Allen invariant $\mathscr{E}$ over a number field Φ. There are 2^k such D_4's up to isomorphism, where k is the number of real primes $\mathfrak{p}$ so that $\mathscr{E}_\mathfrak{p}$ is a full matrix algebra over its centre.

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Assumptions and notation. Throughout the paper *we assume that Φ is a field of characteristic zero.* With the exception of field extensions, *all algebras will be assumed to be finite dimensional.* Also, with the exception of Lie algebras, *all algebras are assumed to be unital* (and hence subalgebra means subalgebra containing 1). If $\mathscr{X}$ is any algebra we denote by $\mathscr{X}^{(n)} := \mathscr{X} \oplus \cdots \oplus \mathscr{X}$ the algebra direct sum of n copies of $\mathscr{X}$ and by $M_n(\mathscr{X})$ the algebra of $n \times n$-matrices with entries from $\mathscr{X}$. If $\mathscr{X}$ is an associative algebra over Φ, then $t_\mathscr{X}$ and $n_\mathscr{X}$ (or $t_{\mathscr{X}/\Phi}$ and $n_{\mathscr{X}/\Phi}$) will denote respectively the *generic trace and norm* on $\mathscr{X}$ [**J3**, Chapter VI]. We use the notation $\widetilde{\Phi}$ for a fixed algebraic closure of Φ and we let

$$G := \mathrm{Gal}(\widetilde{\Phi}/\Phi)$$

be the Galois group of $\widetilde{\Phi}/\Phi$ regarded as a topological group using the usual Krull topology. If $s \in G$ and $\alpha \in \widetilde{\Phi}$, we often write ${}^s\alpha := s\alpha$. Also, if P/Φ is any field extension, we use the notation $\widetilde{P}$ (or $P^\sim$) for an algebraic closure of P, and we use $P^\times$ for the multiplicative group of P. Finally, if P/Φ is an extension and $\mathscr{X}$ is an algebra over Φ, $\mathscr{X}_P$ will denote the P-algebra $P \otimes_\Phi \mathscr{X}$.

1. The Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$. Throughout this section, we assume that $\mathscr{B}$ is a *4-dimensional separable commutative associative algebra over* Φ (and so $\mathscr{B}_{\widetilde{\Phi}} \cong \widetilde{\Phi}^{(4)}$), $\mu \neq 0 \in \Phi$, *and* $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$ *is a* 3×3-*diagonal matrix with* $\gamma_1, \gamma_2, \gamma_3 \neq 0 \in \Phi$. In this section, we recall the definition of the Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ constructed from $\mathscr{B}$, μ and γ. This construction is a modified version [**A3**] of a special case of a construction due to Seligman [**Sel2**, §7.3].

We first look at the nonassociative algebra $\mathrm{CD}(\mathscr{B}, \mu)$ which is constructed from $\mathscr{B}$ and μ by the Cayley-Dickson process introduced in **[A&F1]**. Let $t_{\mathscr{B}}$ be the generic trace on $\mathscr{B}$. (So if we write $\mathscr{B}$ as the direct sum of field extensions of Φ, $t_{\mathscr{B}}$ is the direct sum of the corresponding field extension traces.) Define $\theta: \mathscr{B} \to \mathscr{B}$ by $b^\theta := -b + \frac{1}{2}t_{\mathscr{B}}(b)$. Now put

$$\mathscr{A} := \mathscr{B} \oplus s_0\mathscr{B},$$

where $s_0\mathscr{B}$ denotes another copy of the vector space $\mathscr{B}$, and define a product and involution on $\mathscr{A}$ by:

$$(b_1 + s_0 b_2)(b_3 + s_0 b_4) = b_1 b_3 + \mu(b_2 b_4^\theta)^\theta + s_0(b_1^\theta b_4 + (b_2^\theta b_3^\theta)^\theta)$$

and

$$\overline{b_1 + s_0 b_2} = b_1 - s_0 b_2^\theta.$$

Then, $(\mathscr{A}, -)$ is an 8-dimensional algebra with involution which we denote by $\mathrm{CD}(\mathscr{B}, \mu)$. We call $\mathrm{CD}(\mathscr{B}, \mu)$ the *quartic Cayley algebra determined by* $\mathscr{B}$ *and* μ.

We can now construct the Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ from $(\mathscr{A}, -) = \mathrm{CD}(\mathscr{B}, \mu)$ and γ. For $x, y \in \mathscr{A}$, define $D_{x,y} \in \mathrm{End}\,\mathscr{A}$ by

$$D_{x,y}z := \tfrac{1}{3}[[x, y] + [\overline{x}, \overline{y}], z] + [z, y, x] - [z, \overline{x}, \overline{y}],$$

where $[x, y] := xy - yx$ and $[x, y, z] := (xy)z - x(yz)$. Then, $D_{x,y}$ is a derivation of $(\mathscr{A}, -)$ for $x, y \in \mathscr{A}$ and

$$\mathrm{Inder}(\mathscr{A}, -) := \mathrm{span}\{D_{x,y} : x, y \in \mathscr{A}\}$$

is a 2-dimensional abelian Lie algebra under the commutator product $[\ ,\ ]$ ([**A3**, Theorem 7.2]). We next put

$$\mathscr{P} := \{X \in M_3(\mathscr{A}) : J_\gamma(X) = -X,\ \mathrm{tr}(X) = 0\},$$

where J_γ is the involution on $M_3(\mathscr{A})$ defined by $J_\gamma(X) = \gamma^{-1}\overline{X}^t\gamma$ and $\mathrm{tr}(X) = \sum_{i=1}^3 x_{ii}$ for $X = (x_{ij}) \in M_3(\mathscr{A})$. Finally, we put

$$\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma) := \mathrm{Inder}(\mathscr{A}, -) \oplus \mathscr{P},$$

and define a product $[\ ,\ ]$ on $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ by

(1.1) $[(D, X), (E, Y)] := ([D, E] + \Delta_{X,Y}, DY - EX + [X, Y]_0)$.

Here if $X = (x_{ij})$, $Y = (y_{ij}) \in \mathscr{P}$ and $D \in \mathrm{Inder}(\mathscr{A}, -)$, we are using the notation

$$\Delta_{X,Y} := \frac{1}{2}\sum_{i,j=1}^{3} D_{x_{ij}, y_{ji}}, \qquad DX = (Dx_{ij}), \qquad \text{and}$$

$$[X, Y]_0 := XY - YX - \frac{1}{3}\mathrm{tr}(XY - YX)I.$$

Then, under the product (1.1), $\mathcal{K}(\mathrm{CD}(\mathcal{B}, \mu), \gamma)$ *is a central simple Lie algebra of type* D_4 *over* Φ [**A3**, Theorem 7.2]. That is, $\mathcal{K}(\mathrm{CD}(\mathcal{B}, \mu), \gamma)_{\widetilde{\Phi}}$ is the simple Lie algebra of type D_4 over $\widetilde{\Phi}$.

Our main goal in this paper is to show that if Φ is a number field then any Lie algebra of type D_4 over Φ is obtained from the construction just described.

2. The Lie algebra $\widetilde{\mathcal{L}}$ and its automorphisms. In preparation for our investigation of Lie algebras of type D_4 over Φ, we need to recall in this section some facts due to Jacobson about automorphisms of the simple Lie algebra of type D_4 over $\widetilde{\Phi}$. We will use the specific realization $\widetilde{\mathcal{L}}$ of that Lie algebra that was introduced by Jacobson in [**J2**].

Let $(\widetilde{\mathcal{C}}, -)$ be the Cayley algebra over $\widetilde{\Phi}$ with its canonical involution. Let $\tilde{n}$ and $\tilde{t}$ be the norm and trace on $\widetilde{\mathcal{C}}$ respectively. Define a $\widetilde{\Phi}$-trilinear form $\langle\ ,\ \rangle$ on $\widetilde{\mathcal{C}}$ by

$$\langle x, y, z\rangle := \tfrac{1}{2}\tilde{t}(x(yz)).$$

Then,

$$\langle x, y, z\rangle = \langle z, x, y\rangle = \langle \bar{y}, \bar{x}, \bar{z}\rangle \quad \text{for } x, y, z \in \widetilde{\mathcal{C}}.$$

Denote by $\mathfrak{o}(\tilde{n})$ the orthogonal Lie algebra of $\tilde{n}$ consisting of all skew-symmetric elements of $\mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathcal{C}})$ relative to $\tilde{n}$. Put

$$\widetilde{\mathcal{L}} := \{(L_1, L_2, L_3) \in \mathfrak{o}(\tilde{n})^{(3)} : \langle L_1x, y, z\rangle + \langle x, L_2y, z\rangle + \langle x, y, L_3z\rangle = 0 \text{ for } x, y, z \in \widetilde{\mathcal{C}}\}.$$

Then, $\widetilde{\mathcal{L}}$ *is a simple Lie algebra of type* D_4 *over* $\widetilde{\Phi}$, *and the projection mappings* $(L_1, L_2, L_3) \to L_i$, $i = 1, 2, 3$, *give the three distinct* 8-*dimensional irreducible representations of* $\widetilde{\mathcal{L}}$ [**L2**, Lemmas 1 and 2].

Next put

$$\widetilde{\mathcal{E}} := (\mathrm{End}_{\widetilde{\Phi}}\widetilde{\mathcal{C}})^{(3)} = \widetilde{\mathcal{E}}_1 \oplus \widetilde{\mathcal{E}}_2 \oplus \widetilde{\mathcal{E}}_3,$$

where

$$\widetilde{\mathcal{E}}_1 = \{(X, 0, 0) : X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathcal{C}})\}, \quad \widetilde{\mathcal{E}}_2 = \{(0, X, 0) : X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathcal{C}})\}$$

and

$$\widetilde{\mathcal{E}}_3 = \{(0, 0, X) : X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathcal{C}})\},$$

and so $\widetilde{\mathcal{E}}_i \cong \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathcal{C}}) \cong M_8(\widetilde{\Phi})$, $i = 1, 2, 3$. We then have $\widetilde{\mathcal{L}} \subseteq \widetilde{\mathcal{E}}$, and in fact $\widetilde{\mathcal{E}}$ is the $\widetilde{\Phi}$-associative algebra generated by $\widetilde{\mathcal{L}}$. The centre $\widetilde{\mathscr{Z}}$ of $\widetilde{\mathscr{E}}$ is

$$\widetilde{\mathscr{Z}} = \tilde{\Phi}E_1 \oplus \tilde{\Phi}E_2 \oplus \tilde{\Phi}E_3,$$

where $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$. Let $\tilde{J}$ be the involution of $\widetilde{\mathscr{E}}$ defined by $\tilde{J}(X_1, X_2, X_3) := (X_1^*, X_2^*, X_3^*)$, where X^* denotes the adjoint of X relative to $\tilde{n}$. Thus,

$$\tilde{J}(L) = -L$$

for $L \in \widetilde{\mathscr{L}}$. Also, $\tilde{J}$ fixes the elements of $\widetilde{\mathscr{Z}}$.

The semi-linear automorphisms of $\widetilde{\mathscr{L}}$ have the following description which follows easily from Jacobson's description in **[J2]**.

PROPOSITION 2.1. *Let ϕ be an s-semilinear automorphism of $\widetilde{\mathscr{L}}$, where $s \in G$. Then there exists a permutation $p \in S_3$ and a triple $U = (U_1, U_2, U_3)$ of s-semilinear vector space automorphisms of $\widetilde{\mathscr{C}}$ so that*

$$\tilde{n}(U_i x) = {}^s\tilde{n}(x) \quad \text{for } x \in \widetilde{\mathscr{C}},\ i = 1, 2, 3, \tag{2.2}$$

$$\langle U_1 x, U_2 y, U_3 z\rangle = \begin{cases} {}^s\langle x, y, z\rangle & \text{if } p \text{ is even} \\ {}^s\langle y, x, z\rangle & \text{if } p \text{ is odd} \end{cases} \quad \text{for } x, y, z \in \widetilde{\mathscr{C}}, \tag{2.3}$$

and

$$\phi(L_1, L_2, L_3) = (U_1 L_{p1} U_1^{-1}, U_2 L_{p2} U_2^{-1}, U_3 L_{p3} U_3^{-1}) \tag{2.4}$$

for $(L_1, L_2, L_3) \in \widetilde{\mathscr{L}}$. Moreover, p is uniquely determined and U_1, U_2, U_3 are uniquely determined up to multiplication by three scalars from $\{-1, 1\}$ whose product is 1.

Proof. In **[J2]**, Jacobson works with the split Lie algebra of type D_4 over a finite Galois extension of Φ rather than $\widetilde{\mathscr{L}}$. The same arguments work here. By **[J2**, p. 139**]** there exists s-semilinear automorphisms T_1, T_2, T_3 of $\widetilde{\mathscr{C}}$ so that $\tilde{n}(T_i x) = \mu_i\, {}^s\tilde{n}(x)$, $\langle T_1 x, T_2 y, T_3 z\rangle = \nu\, {}^s\langle x, y, z\rangle$, and

$$\phi(L_1, L_2, L_3) = (T_1\tau^j L_{p1}\tau^{-j}T_1^{-1}, T_2\tau^j L_{p2}\tau^{-j}T_2^{-1}, T_3\tau^j L_{p3}\tau^{-j}T_3^{-1}),$$

where $\tau = {}^-$, $j = 0$ or 1 according as p is even or odd, $\mu_i, \nu \in \tilde{\Phi}^\times$, p is uniquely determined and T_1, T_2, T_3 are determined up to multiplication by scalars in $\tilde{\Phi}^\times$. Replacing T_i by a multiple, we can assume $\mu_i = 1$. But by **[J2**, Lemma 3**]** and the argument on p. 139 of **[J2]**, it follows that $(\tau T_1 \tau)(xy) = \nu^{-1}(T_2 x)(T_3 y)$ for $x, y \in \widetilde{\mathscr{C}}$.

Taking $\tilde{n}$ of both sides yields $\nu^2 = 1$. Thus, replacing T_3 by $-T_3$ if necessary, we can assume $\nu = 1$. Finally, put $U_i = T_i\tau^j$, $i = 1, 2, 3$. □

REMARK 2.5. (a) We denote the permutation p in Proposition 2.1 by $p(\phi)$ and call it the *permutation in* S_3 *determined by* ϕ.

(b) Conversely, if $p \in S_3$ and $U = (U_1, U_2, U_3)$ is a triple of s-semilinear vector space automorphisms of $\tilde{\mathscr{E}}$ so that (2.2) and (2.3) hold, then (2.4) defines an s-semilinear automorphism of $\widetilde{\mathscr{L}}$ that we call the *semilinear automorphism determined by the pair* (p, U).

COROLLARY 2.6. *The connected component of the algebraic group* $\mathrm{Aut}(\widetilde{\mathscr{L}})$ *is given by*

$$\mathrm{Aut}(\widetilde{\mathscr{L}})^0 = \{\phi \in \mathrm{Aut}(\widetilde{\mathscr{L}}) : p(\phi) = (1)\}.$$

Proof. Let A be the right-hand side. Then A is the image of an algebraic group under a morphism of algebraic groups and hence A is closed in $\mathrm{Aut}(\widetilde{\mathscr{L}})$. Also, $p(\phi\zeta) = p(\zeta)p(\phi)$ and so $\phi \to p(\phi)^{-1}$ defines a group homomorphism of $\mathrm{Aut}(\widetilde{\mathscr{L}})$ into S_3 with kernel A. This map is clearly onto and hence A has index 6 in $\mathrm{Aut}(\widetilde{\mathscr{L}})$. Thus, $\mathrm{Aut}(\widetilde{\mathscr{L}})^0 \subseteq A$ [**B**, p. 86]. But $\mathrm{Aut}(\widetilde{\mathscr{L}})^0$ has index 6 in $\mathrm{Aut}(\widetilde{\mathscr{L}})$ [**J5**, Remark on p. 281 and Exercise 9 on p. 287] and so we have the desired equality. □

3. The Allen invariant. In this section, we suppose that $\mathscr{L}$ *is a Lie algebra of type* D_4 *over* Φ. The basic tool used in the study of $\mathscr{L}$ will be its Allen invariant $\mathscr{E}(\mathscr{L})$. We recall here the definition and some properties of $\mathscr{E}(\mathscr{L})$ due to Allen [**All1**], and then prove that the corestriction of $\mathscr{E}(\mathscr{L})$ over its centre is trivial.

We recall first the notion of a Φ-form of an algebra over $\tilde{\Phi}$. If $\widetilde{\mathscr{X}}$ is an algebra over $\tilde{\Phi}$, a Φ-form of $\widetilde{\mathscr{X}}$ is a Φ-subalgebra $\mathscr{X}$ of $\widetilde{\mathscr{X}}$ so that the natural map $\mathscr{X}_{\tilde{\Phi}} \to \widetilde{\mathscr{X}}$ is a $\tilde{\Phi}$-algebra isomorphism. In that case, we usually identify $\mathscr{X}_{\tilde{\Phi}}$ and $\widetilde{\mathscr{X}}$. Then, if P/Φ is a subextension of $\tilde{\Phi}/\Phi$, $\mathscr{X}_P$ is a P-form of $\widetilde{\mathscr{X}}$. Also $\mathrm{End}_\Phi(\mathscr{X})$ naturally identifies as a Φ-form of $\mathrm{End}_{\tilde{\Phi}}(\widetilde{\mathscr{X}})$.

Now since $\mathscr{L}$ is a Lie algebra of type D_4 over Φ, we have $\mathscr{L}_{\tilde{\Phi}} \cong \widetilde{\mathscr{L}}$. Hence, we can and do *identify* $\mathscr{L}$ *as a* Φ*-form of* $\widetilde{\mathscr{L}}$. Then,

$$\mathscr{L} \subseteq \widetilde{\mathscr{L}} \subseteq \tilde{\mathscr{E}}.$$

We define the *Allen invariant* $\mathscr{E}(\mathscr{L})$ of $\mathscr{L}$ to be the associative Φ-algebra generated by $\mathscr{L}$ in $\tilde{\mathscr{E}}$. $\mathscr{E}(\mathscr{L})$ is a Φ-form of $\tilde{\mathscr{E}}$. (See [**All1**,

p. 255] or the proof of Proposition 3.3 below.) Thus, *$\mathscr{E}(\mathscr{L})$ is a 192-dimensional separable associative algebra over Φ.* It follows easily from Proposition 2.1 (for automorphisms) that the isomorphism class of $\mathscr{E}(\mathscr{L})$ is independent of our identification of $\mathscr{L}$ as a Φ-form of $\widetilde{\mathscr{L}}$.

We denote by $\mathscr{Z}(\mathscr{L})$ the centre of $\mathscr{E}(\mathscr{L})$. Since $\mathscr{E}(\mathscr{L})$ is a Φ-form of $\widetilde{\mathscr{E}}$, $\mathscr{Z}(\mathscr{L})$ is a Φ-form of $\widetilde{\mathscr{Z}}$. Thus, *$\mathscr{Z}(\mathscr{L})$ is a 3-dimensional separable commutative associative algebra.*

Let J be the restriction of $\widetilde{J}$ to $\mathscr{E}(\mathscr{L})$. Then, J is an involution of $\mathscr{E}(\mathscr{L})$ that fixes the elements of $\mathscr{Z}(\mathscr{L})$. Thus, *each of the simple summands of $\mathscr{E}(\mathscr{L})$ has exponent 1 or 2 in the Brauer group over its centre.* (In a separable associative algebra, the simple summands are just the simple ideals.)

To prove the next property of $\mathscr{E}(\mathscr{L})$, we will need the notion of D_4-type **[J2]**.

Let $\alpha = (\alpha_s)_{s\in G}$ be the *Galois precocycle determined by the Φ-form $\mathscr{L}$ of $\widetilde{\mathscr{L}}$.* Thus, by definition, for $s \in G$, α_s is the unique s-semilinear automorphism of $\widetilde{\mathscr{L}}$ that fixes the elements of $\mathscr{L}$. Let $p_s := p(\alpha_s)$ be the permutation in S_3 determined by α_s. Since $\alpha_{st} = \alpha_s\alpha_t$, it follows that

$$p_{st} = p_t p_s \quad \text{for } s, t \in G. \tag{3.1}$$

Thus, $\{p_s : s \in G\}$ is a subgroup of S_3, We say that $\mathscr{L}$ has *type* $D_{4\mathrm{I}}$, $D_{4\mathrm{II}}$, $D_{4\mathrm{III}}$ or $D_{4\mathrm{VI}}$ according as this subgroup has order 1, 2, 3 or 6. The D_4-type of $\mathscr{L}$ is independent of our identification of $\mathscr{L}$ in $\widetilde{\mathscr{L}}$ (by Proposition 2.1). Put

$$H := \{s \in G : p_s = (1)\} \quad \text{and} \quad \Gamma := \mathrm{Fix}(H),$$

where $\mathrm{Fix}(H) := \{\alpha \in \widetilde{\Phi} : {}^h\alpha = \alpha \text{ for all } h \in H\}$. Then, H is a closed normal subgroup of G of index 1, 2, 3 or 6 and Γ/Φ is a Galois extension of degree 1, 2, 3 or 6 according to D_4-type. Γ/Φ is called the *canonical $D_{4\mathrm{I}}$-extension of* Φ. It is the smallest subextension of $\widetilde{\Phi}/\Phi$ so that $\mathscr{L}_\Gamma$ has type $D_{4\mathrm{I}}$ **[All1**, p. 256].

Finally, we need the notion of corestriction of associative algebras. If P/Φ is a finite extension and $\mathscr{X}$ is a central simple associative algebra over P, then the *corestriction* $c_{P/\Phi}(\mathscr{X})$ of $\mathscr{X}$ is a central simple associative algebra over Φ of dimension $(\dim_P \mathscr{X})^{[P:\Phi]}$. The reader is referred to **[R]** or **[Tig]** for the definition and main properties of this construction. The property that we will use in the next proposition is the following. The assignment $\mathscr{X} \to c_{P/\Phi}(\mathscr{X})$ induces a homomorphism $\mathrm{Br}(P) \xrightarrow{c_{P/\Phi}} \mathrm{Br}(\Phi)$ of Brauer groups so that if we identify P/Φ in $\widetilde{\Phi}/\Phi$ and set $K := \mathrm{Gal}(\widetilde{\Phi}/P)$ then the diagram

$$\begin{array}{ccc} \mathrm{Br}(P) & \xrightarrow{\sim} & H^2(K, \widetilde{\Phi}^\times) \\ \downarrow c_{P/\Phi} & & \downarrow \mathrm{cor}_{G/K} \\ \mathrm{Br}(\Phi) & \xrightarrow{\sim} & H^2(G, \widetilde{\Phi}^\times) \end{array} \tag{3.2}$$

commutes. Here the horizontal maps are the usual isomorphisms [**Ser2**, p. 159], and $\mathrm{cor}_{G/K}$ is the corestriction map of group cohomology [**Ser1**, p. I–11]. This property is Theorem 11 of [**R**]. We note also that if $P = \Phi$, then $c_{P/\Phi}(\mathscr{X}) = \mathscr{X}$.

More generally if $\mathscr{X}$ is a separable algebra over Φ with centre $\mathscr{Z}$, we may write $\mathscr{X} = \mathscr{X}_1 \oplus \cdots \oplus \mathscr{X}_m$ and $\mathscr{Z} = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, where $\mathscr{X}_i$ is simple over Φ with centre Λ_i, $i = 1, \ldots, m$. We then define

$$c_{\mathscr{Z}/\Phi}(\mathscr{X}) := c_{\Lambda_1/\Phi}(\mathscr{X}_1) \otimes_\Phi \cdots \otimes_\Phi c_{\Lambda_m/\Phi}(\mathscr{X}_m).$$

We call $c_{\mathscr{Z}/\Phi}(\mathscr{X})$ the *corestriction of $\mathscr{X}$ over its centre.*

The following proposition was proved by Jacobson [**J2**, Theorem 4] for type $D_{4\mathrm{I}}$ and by Tamagawa [**Ta**, Theorem 2] for type $D_{4\mathrm{II}}$. For types $D_{4\mathrm{III}}$ and $D_{4\mathrm{VI}}$, the result was noticed first by Tamagawa (unpublished). The proposition is now a consequence of more general results on representations of algebraic groups due to Tits [**T2**, Corollaire 3.5 and Proposition 5.1]. Since we will need some of the notation and arguments in the rest of the paper, we present here for the convenience of the reader an elementary proof that generalizes Jacobson's argument in [**J2**].

PROPOSITION 3.3. *Suppose $\mathscr{L}$ is a Lie algebra of type D_4 over Φ with Allen invariant $\mathscr{E}(\mathscr{L})$. Then,*

$$c_{\mathscr{Z}(\mathscr{L})/\Phi}(\mathscr{E}(\mathscr{L})) \sim \Phi,$$

where $\mathscr{Z}(\mathscr{L})$ is the centre of $\mathscr{E}(\mathscr{L})$ and $\sim$ denotes similarity of central simple algebras.

Proof. For $s \in G$, there exists $p_s \in S_3$ and a triple $U(s) = (U_1(s), U_2(s), U_3(s))$ of s-semilinear vector space automorphisms of $\widetilde{\mathscr{C}}$ so that

$$\tilde{n}(U_i(s)x) = {}^s\tilde{n}(x) \quad \text{for } x \in \widetilde{\mathscr{C}}, \ i = 1, 2, 3, \tag{3.4}$$

(3.5) $$\langle U_1(s)x, U_2(s)y, U_3(s)z\rangle = \begin{cases} {}^s\langle x, y, z\rangle & \text{if } p_s \text{ is even} \\ {}^s\langle y, x, z\rangle & \text{if } p_s \text{ is odd} \end{cases} \quad \text{for } x, y, z \in \widetilde{\mathscr{C}},$$

and

(3.6) $$\alpha_s(L_1, L_2, L_3) = (U_1(s)L_{p_s 1}U_1(s)^{-1}, U_2(s)L_{p_s 2}U_2(s)^{-1}, U_3(s)L_{p_s 3}U_3(s)^{-1})$$

for $(L_1, L_2, L_3) \in \widetilde{\mathscr{L}}$. p_s is uniquely determined and $U_1(s)$, $U_2(s)$, $U_3(s)$ are uniquely determined up to multiplication by three scalars from $\{-1, 1\}$ whose product is 1.

Let $\mathscr{C}$ be a Φ-form of the algebra $\widetilde{\mathscr{C}}$. Since $\widetilde{\Phi}\mathscr{E}(\mathscr{L}) = \widetilde{\mathscr{E}}$ and $\widetilde{\Phi}\,\mathrm{End}_\Phi(\mathscr{C})^{(3)} = \widetilde{\mathscr{E}}$, we may choose a finite Galois extension P/Φ so that $\Gamma \subseteq P \subseteq \widetilde{\Phi}$ and

(3.7) $$P\mathscr{E}(\mathscr{L}) = \mathrm{End}_P(\mathscr{C}_P)^{(3)}.$$

Let $K := \mathrm{Gal}(\widetilde{\Phi}/P)$. Then we can assume that the $U_i(s)$'s were chosen so that

(3.8) $$U_i(1) = 1, \qquad i = 1, 2, 3,$$

and

(3.9) $$s, t \in G,\ s^{-1}t \in K \Rightarrow U_i(s)^{-1}U_i(t)|_{\mathscr{C}_P} = 1, \qquad i = 1, 2, 3.$$

Then, enlarging P if necessary, we may assume that

(3.10) $$U_i(s)\mathscr{C}_P \subseteq \mathscr{C}_P,$$

for $s \in G$, $i = 1, 2, 3$.

Next from the remark made above about the uniqueness of the $U_i(s)$'s, we have

(3.11) $$U_i(s)U_{p_s i}(t) = \rho^{(i)}_{s,t}U_i(st), \qquad i = 1, 2, 3, \qquad \text{where}$$

(3.12) $$\rho^{(i)}_{s,t} = \pm 1, \qquad i = 1, 2, 3, \qquad \text{and} \qquad \rho^{(1)}_{s,t}\rho^{(2)}_{s,t}\rho^{(3)}_{s,t} = 1$$

for $s, t \in G$. But then since $U_i((rs)t) = U_i(r(st))$, we get using (3.1) and (3.11) that

(3.13) $$\rho^{(p_r i)}_{s,t}\rho^{(i)}_{r,st} = \rho^{(i)}_{rs,t}\rho^{(i)}_{r,s}, \qquad i = 1, 2, 3,$$

for $r, s, t \in G$. Finally, using (3.8)–(3.13), one can show without difficulty that $\rho^{(i)}_{s,t}$ is constant on cosets of K in G, and hence *the* *map* $(s, t) \to \rho_{s,t}^{(i)}$ *of* $G \times G$ *into* $\widetilde{\Phi}^\times$ *is continuous,* $i = 1, 2, 3$. (Here as usual $\widetilde{\Phi}^\times$ has the discrete topology.)

For $s \in G$, define $\beta_s : \widetilde{\mathscr{E}} \to \widetilde{\mathscr{E}}$ by

$$\begin{aligned}&\beta_s(X_1, X_2, X_3)\\ &\quad = (U_1(s)X_{p_s 1}U_1(s)^{-1}, U_2(s)X_{p_s 2}U_2(s)^{-1}, U_3(s)X_{p_s 3}U_3(s)^{-1}).\end{aligned}$$

Then, by (3.1) and (3.11), $\beta_{st} = \beta_s\beta_t$, $s, t \in G$. Thus, by (3.8), (3.9) and [**B**, AG.14.2], $\{X \in \widetilde{\mathscr{E}} : \beta_s X = X \text{ for } s \in G\}$ is a Φ-form of $\widetilde{\mathscr{E}}$. But this Φ-form contains $\mathscr{E}(\mathscr{L})$ and $\widetilde{\Phi}\mathscr{E}(\mathscr{L}) = \widetilde{\mathscr{E}}$. Thus, $\mathscr{E}(\mathscr{L}) = \{X \in \widetilde{\mathscr{E}} : \beta_s X = X \text{ for } s \in G\}$ and $\mathscr{E}(\mathscr{L})$ *is a* Φ*-form of* $\widetilde{\mathscr{E}}$. Hence,

$$(3.14)\quad \begin{aligned}\mathscr{E}(\mathscr{L}) = \{(X_1, X_2, X_3) \in \widetilde{\mathscr{E}} : U_i(s)X_{p_s i}U_i(s)^{-1}\\ = X_i \text{ for } s \in G, i = 1, 2, 3\}.\end{aligned}$$

We now consider cases. Suppose first that $\mathscr{L}$ has type $D_{4\mathrm{I}}$. So $H = G$ and $\Gamma = \Phi$. Then, $p_s = (1)$ for $s \in G$, and so, by (3.14),

$$\mathscr{E}(\mathscr{L}) = \mathscr{E}_1 \oplus \mathscr{E}_2 \oplus \mathscr{E}_3,$$

where $\mathscr{E}_i := \mathscr{E}(\mathscr{L}) \cap \widetilde{\mathscr{E}}_i$ is a Φ-form of $\widetilde{\mathscr{E}}_i$, $i = 1, 2, 3$. Thus, $\mathscr{E}_i$ is a 64-dimensional central simple algebra over Φ, $i = 1, 2, 3$. Also, by (3.14), the projection maps for the decomposition $\widetilde{\mathscr{E}} := \widetilde{\mathscr{E}}_1 \oplus \widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3$ restrict to isomorphisms

$$\mathscr{E}_i \cong \{X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathscr{C}}) : U_i(s)XU_i(s)^{-1} = X \text{ for } s \in G\}, \qquad i = 1, 2, 3.$$

By (3.13), $(\rho_{s,t}^{(i)})_{s,t\in G}$ is a continuous 2-cocycle in $\widetilde{\Phi}^\times$ which therefore determines an element $\rho^{(i)}$ of $H^2(G, \widetilde{\Phi}^\times)$. Also, for $s, t \in G$, $U_i(s)U_i(t) = \rho_{s,t}^{(i)}U_i(st)$. Hence, $[\mathscr{E}_i]$ maps to $\rho^{(i)}$ under the isomorphism $\mathrm{Br}\,\Phi \to H^2(G, \widetilde{\Phi}^\times)$, $i = 1, 2, 3$. Thus, by (3.12), $\mathscr{E}_1 \otimes_\Phi \mathscr{E}_2 \otimes_\Phi \mathscr{E}_3 \sim \Phi$.

Suppose next that $\mathscr{L}$ has type $D_{4\mathrm{II}}$. So $(G : H) = 2$ and Γ is a quadratic extension of Φ. We may assume that $p_{t_1} = (23)$ for some $t_1 \in G$. Then, by (3.14), we have

$$\mathscr{E}(\mathscr{L}) = \mathscr{F} \oplus \mathscr{G},$$

where $\mathscr{F} = \mathscr{E}(\mathscr{L}) \cap \widetilde{\mathscr{E}}_1$ and $\mathscr{G} = \mathscr{E}(\mathscr{L}) \cap (\widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3)$ are Φ-forms of $\widetilde{\mathscr{E}}_1$ and $\widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3$ respectively. The projection maps onto the first and second factors in the decomposition $\widetilde{\mathscr{E}} := \widetilde{\mathscr{E}}_1 \oplus \widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3$ restrict to isomorphisms

$$\mathscr{F} \cong \{X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathscr{E}}) : U_1(s)XU_1(s)^{-1} = X \text{ for } s \in G\} \quad \text{and}$$
$$\mathscr{G} \cong \{X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathscr{E}}) : U_2(s)XU_2(s)^{-1} = X \text{ for } s \in H\}$$

respectively. Thus, $\mathscr{F}$ is a 64-dimensional central simple algebra over Φ and $\mathscr{G}$ is a 128-dimensional simple algebra over Φ with centre Γ. As in the previous case, $(\rho^{(1)}_{s,t})_{s,t\in G}$ determines an element $\rho^{(1)}$ of $H^2(G, \widetilde{\Phi}^\times)$ which is the image of $[\mathscr{F}]$ under the isomorphism $\mathrm{Br}(\Phi) \to H^2(G, \widetilde{\Phi}^\times)$. Similarly, $(\rho^{(2)}_{s,t})_{s,t\in H}$ determines an element $\rho^{(2)}$ of $H^2(H, \widetilde{\Phi}^\times)$ which is the image of $[\mathscr{G}]$ under the isomorphism $\mathrm{Br}(\Gamma) \to H^2(H, \widetilde{\Phi}^\times)$.

Now let $B := M^H_G(\widetilde{\Phi}^\times)$ in the notation of **[Ser1**, p. I-12**]**. Thus, by definition, B is the G-module consisting of all continuous maps $\alpha^*: G \to \widetilde{\Phi}^\times$ so that $a^*(hs) = {}^h a^*(s)$ for $h \in H$, $s \in G$. The G-action on B is given by $({}^s a^*)(t) = a^*(ts)$. If $(\alpha, \beta) \in \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times$, then there is a unique element a^* of B so that $a^*(1) = \alpha$ and $a^*(t_1) = \beta$. Every element of B is of this form and so we have an identification $B = \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times$. If $\alpha, \beta = \pm 1$ and $h \in H$, the G-action on B satisfies

$$(3.15) \qquad {}^h(\alpha, \beta) = (\alpha, \beta) \quad \text{and} \quad {}^{t_1}(\alpha, \beta) = (\beta, \alpha).$$

Now by **[Ser1**, p. I-12 to I-13**]**, the projection map $B \to \widetilde{\Phi}^\times$ onto the first factor induces an isomorphism $H^2(G, B) \xrightarrow{\phi_1} H^2(H, \widetilde{\Phi}^\times)$, while the map $B \to \widetilde{\Phi}^\times$ defined by $(\alpha, \beta) \to \alpha({}^{t_1^{-1}}\beta)$ induces a homomorphism $H^2(G, B) \xrightarrow{\phi_2} H^2(G, \widetilde{\Phi}^\times)$. Then, by definition, $\mathrm{cor}_{G/H} = \phi_2 \circ \phi_1^{-1}$.

Define $\pi_{s,t} := (\rho^{(2)}_{s,t}, \rho^{(3)}_{s,t}) \in B$ for $s, t \in G$. Then, it follows from (3.13) and (3.15) that $(\pi_{s,t})_{s,t\in G}$ is a continuous 2-cocycle in B which therefore determines an element π of $H^2(G, B)$. But, $\phi_1(\pi) = \rho^{(2)}$ and so $\mathrm{cor}_{G/H}(\rho^{(2)}) = \phi_2(\pi)$ is represented by the 2-cocycle $(\rho^{(2)}_{s,t}\rho^{(3)}_{s,t})_{s,t\in G}$. Hence, by (3.2) and (3.12), $\mathscr{F} \otimes_\Phi c_{\Gamma/\Phi}(\mathscr{G}) \sim \Phi$.

Finally, suppose $\mathscr{L}$ has type $D_{4\text{III}}$ or $D_{4\text{VI}}$. Choose $s_0 \in G$ so that $p_{s_0} = (123)$. If $\mathscr{L}$ has type $D_{4\text{VI}}$, choose $t_1 \in G$ so that $p_{t_1} = (23)$. Put $F = H$ in type $D_{4\text{III}}$ and $F = \langle H, t_1 \rangle$ in type $D_{4\text{VI}}$. Then, F is a subgroup of G of index 3 with coset representatives $1, s_0, s_0^2$. Put $\Lambda = \mathrm{Fix}(F)$. Then, $[\Lambda : \Phi] = 3$. In fact $\Lambda = \Gamma$ in the case of type $D_{4\mathrm{III}}$, while Λ is one of the cubic subfields of Γ in the case of type $D_{4\mathrm{VI}}$. Next the first projection map for the decomposition $\widetilde{\mathscr{E}} := \widetilde{\mathscr{E}}_1 \oplus \widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3$ restricts to an isomorphism

$$\mathscr{E}(\mathscr{L}) \cong \{X \in \mathrm{End}_{\widetilde{\Phi}}(\widetilde{\mathscr{E}}) : U_1(s)XU_1(s)^{-1} = X \text{ for } s \in F\}.$$

Thus, $\mathscr{E}(\mathscr{L})$ is a 192-dimensional simple algebra with centre Λ. Moreover, $(\rho^{(1)}_{s,t})_{s,t\in F}$ determines an element $\rho^{(1)}$ of $H^2(F, \widetilde{\Phi}^\times)$ which is the image of $[\mathscr{E}(\mathscr{L})]$ under the isomorphism $\mathrm{Br}(\Lambda) \to H^2(F, \widetilde{\Phi}^\times)$.

Let $B := M_F^G(\widetilde{\Phi}^\times)$. Then, for $(\alpha, \beta, \gamma) \in \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times$, there is a unique element a^* of B so that $a^*(1) = \alpha$, $a^*(s_0) = \beta$ and $a^*(s_0^2) = \gamma$. This gives an identification $B = \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times \times \widetilde{\Phi}^\times$. If $\alpha, \beta, \gamma = \pm 1$ and $f \in F$, the action of G on B satisfies

$$^f(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma), \quad ^{s_0}(\alpha, \beta, \gamma) = (\beta, \gamma, \alpha)$$
$$\text{and, in type } D_{4\mathrm{VI}}, \quad ^{t_1}(\alpha, \beta, \gamma) = (\alpha, \gamma, \beta).$$

Again the projection map $B \to \widetilde{\Phi}^\times$ onto the first factor induces an isomorphism $H^2(G, B) \xrightarrow{\phi_1} H^2(F, \widetilde{\Phi}^\times)$, while the map $B \to \widetilde{\Phi}^\times$ defined by $(\alpha, \beta, \gamma) \to \alpha(^{s_0^{-1}}\beta)(^{s_0^{-2}}\gamma)$ induces a homomorphism $H^2(G, B) \xrightarrow{\phi_2} H^2(G, \widetilde{\Phi}^\times)$. By definition, $\mathrm{cor}_{G/H} = \phi_2 \circ \phi_1^{-1}$. But then $\pi_{s,t} := (\rho^{(1)}_{s,t}, \rho^{(2)}_{s,t}, \rho^{(3)}_{s,t})$, $s, t \in G$, defines a continuous 2-cocycle which determines an element $\pi \in H^2(G, B)$ such that $\phi_1(\pi) = \rho^{(1)}$. Thus, $\mathrm{cor}_{G/F}(\rho^{(1)}) = \phi_2(\pi)$ is represented by the 2-cocycle $(\rho^{(1)}_{s,t}\rho^{(2)}_{s,t}\rho^{(3)}_{s,t})_{s,t\in G}$. Hence, by (3.2) and (3.12), $c_{\Lambda/\Phi}(\mathscr{E}(\mathscr{L})) \sim \Phi$. □

REMARK 3.16. For convenient later reference, we summarize the case-by-case information observed so far. (See also **[J2]**, **[All1]** and **[Ta]**.) If $\mathscr{L}$ has type $D_{4\mathrm{I}}$, then

$$\mathscr{E}(\mathscr{L}) = \mathscr{E}_1 \oplus \mathscr{E}_2 \oplus \mathscr{E}_3,$$

where $\mathscr{E}_1$, $\mathscr{E}_2$, and $\mathscr{E}_3$ are 64-dimensional central simple,

(3.17) $[\mathscr{E}_1][\mathscr{E}_2][\mathscr{E}_3] = 1$ and $[\mathscr{E}_1]^2 = [\mathscr{E}_2]^2 = [\mathscr{E}_3]^2 = 1$ in $\mathrm{Br}(\Phi)$.

If $\mathscr{L}$ has type $D_{4\mathrm{II}}$, then

$$\mathscr{E}(\mathscr{L}) = \mathscr{F} \oplus \mathscr{G},$$

where $\mathscr{F}$ is 64-dimensional central simple, $\mathscr{G}$ is 128-dimensional simple with centre Γ of degree 2 over Φ,

(3.18) $[\mathscr{F}][c_{\Gamma/\Phi}(\mathscr{G})] = 1$ in $\mathrm{Br}(\Phi)$, $[\mathscr{F}]^2 = 1$ in $\mathrm{Br}(\Phi)$ and $[\mathscr{G}]^2 = 1$ in $\mathrm{Br}(\Gamma)$.

If $\mathscr{L}$ has type $D_{4\mathrm{III}}$ or $D_{4\mathrm{VI}}$, then $\mathscr{E}(\mathscr{L})$ is 192-dimensional simple with centre Λ of degree 3 over Φ,

(3.19) $[c_{\Lambda/\Phi}(\mathscr{E}(\mathscr{L}))] = 1$ in $\mathrm{Br}(\Phi)$ and $[\mathscr{E}(\mathscr{L})]^2 = 1$ in $\mathrm{Br}(\Lambda)$.

Here $\Lambda = \Gamma$ in the case of type $D_{4\mathrm{III}}$, while Λ is one of the (isomorphic) cubic subfields of Γ in the case of type $D_{4\mathrm{VI}}$.

REMARK 3.20. Suppose $\mathscr{L}$ is a Lie algebra of type D_4 over Φ. We call $\mathscr{L}$ *orthogonal* if $\mathscr{L}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(q)$ of an 8-dimensional nondegenerate quadratic form q. The following characterization of orthogonal D_4's holds:

(3.21) $\mathscr{L}$ is orthogonal $\Leftrightarrow \mathscr{E}(\mathscr{L})$ has a simple summand isomorphic to $M_8(\Phi)$.

Indeed the implication "$\Rightarrow$" follows from the fact that the projection mappings $(L_1, L_2, L_3) \to L_i$ give all 3-distinct 8-dimensional irreducible modules for $\widetilde{\mathscr{L}}$. Conversely, if $\mathscr{E}(\mathscr{L})$ has a simple summand $\mathscr{Y}$ that is isomorphic to $M_8(\Phi)$, then $\mathscr{Y} = \widetilde{\mathscr{E}}_i \cap \mathscr{E}(\mathscr{L})$ for some $i \in \{1, 2, 3\}$, and so the ith projection map $\widetilde{\mathscr{E}} \to \widetilde{\mathscr{E}}_i$ restricts to an isomorphism of $\mathscr{L}$ onto $\mathscr{S}(\mathscr{Y}, J_{\mathscr{Y}}) := \{X \in \mathscr{Y} : J_{\mathscr{Y}} X = -X\}$, where $J_{\mathscr{Y}} = J|_{\mathscr{Y}}$. But then $\mathscr{L}$ is orthogonal [**J1**, §§6 and 7].

4. The Allen invariant of $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$. Suppose in this section that $\mathscr{B}$, μ, γ are as in §1 and $\mathscr{K} := \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$. In this section we recall the results from [**A2**] and [**A3**] that we will need regarding the Allen invariant of $\mathscr{K}$ and its use in the description of isotropic D_4's.

Quaternion algebras will play a fundamental role in our discussion here and in the rest of the paper. If $\mathscr{Z}$ is a separable commutative associative algebra over Φ, an algebra $\mathscr{D}$ over Φ is called a *quaternion algebra over $\mathscr{Z}$* if $\mathscr{D}$ has centre $\mathscr{Z}$ and $\mathscr{D} \cong (\alpha, \beta/\mathscr{Z})$ as $\mathscr{Z}$-algebras for some units α, β of $\mathscr{Z}$. Here, as is usual, $(\alpha, \beta/\mathscr{Z})$ or $(\frac{\alpha, \beta}{\mathscr{Z}})$ denotes the associative $\mathscr{Z}$-algebra $\mathscr{Z}1 \oplus \mathscr{Z}\mathbf{i} \oplus \mathscr{Z}\mathbf{j} \oplus \mathscr{Z}\mathbf{ij}$ that is the free $\mathscr{Z}$-module with $\mathscr{Z}$-basis 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{ij}$ satisfying the relations $\mathbf{i}^2 = \alpha 1$, $\mathbf{j}^2 = \beta 1$, $\mathbf{ij} = -\mathbf{ji}$. If we write $\mathscr{Z} = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, where Λ_i is a field, then the quaternion algebras over $\mathscr{Z}$ are precisely the algebras of the form $\mathscr{D}_1 \oplus \cdots \oplus \mathscr{D}_m$, where $\mathscr{D}_i$ is a quaternion algebra over Λ_i (in the usual sense), $i = 1, \ldots, m$.

Let $\bigwedge^2 \mathscr{B}$ be the second exterior power of $\mathscr{B}$. For $b \in \mathscr{B}$, define $F_b \colon \bigwedge^2 \mathscr{B} \to \bigwedge^2 \mathscr{B}$ by $F_b(c \wedge d) = (bc) \wedge d + c \wedge (bd)$. Let $\mathscr{Q}$ be the associative subalgebra of $M_2(\operatorname{End} \bigwedge^2 \mathscr{B})$ generated by the matrices $\left[\begin{smallmatrix} F_b & 0 \\ 0 & F_{b^\theta} \end{smallmatrix}\right]$, $b \in \mathscr{B}$, and $\left[\begin{smallmatrix} 0 & \mu I \\ I & 0 \end{smallmatrix}\right]$. The centre of $\mathscr{Q}$ is $\{\left[\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right] : R \in \mathscr{R}\}$, where $\mathscr{R} := \operatorname{span}\{F_b F_c : b, c \in \mathscr{B}_0\}$ and $\mathscr{B}_0 := \{b \in \mathscr{B} : t_{\mathscr{B}}(b) = 0\}$ [**A3**, Proposition 6.7]. We *identify* $\mathscr{R}$ *with the centre of* $\mathscr{Q}$ by the map $R \to \left[\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right]$. $\mathscr{R}$ is a 3-dimensional separable commutative associative algebra and $\mathscr{Q}$ is a quaternion algebra over $\mathscr{R}$ (see Proposition 4.4 below). $\mathscr{R}$ is called the *cubic resolvent algebra of* $\mathscr{B}$, and $\mathscr{Q}$ is called *the quaternion algebra determined by* $\mathscr{B}$ *and* μ. These algebras are important for our purposes because of the following result which is Theorem 8.10 of [**A3**]:

PROPOSITION 4.1. *If* $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ *then* $\mathscr{E}(\mathscr{K}) \cong M_4(\mathscr{Q})$ *and* $\mathscr{Z}(\mathscr{K}) \cong \mathscr{R}$, *where* $\mathscr{Q}$ *is the quaternion algebra determined by* $\mathscr{B}$ *and* $\mathscr{R}$ *is the cubic resolvent algebra of* $\mathscr{B}$.

We next describe generators and relations for $\mathscr{Q}$ and $\mathscr{R}$. To do this, we select a generator b_0 for $\mathscr{B}$ with minimum polynomial $f(x)$ of the form

$$f(x) = x^4 + \beta_2 x^2 + \beta_1 x + \beta_0, \tag{4.2}$$

where $\beta_i \in \Phi$ and $\beta_1 \neq 0$. (Such a choice is always possible.) Let

$$h(x) := x^3 + 2\beta_2 x^2 + (\beta_2^2 - 4\beta_0)x - \beta_1^2. \tag{4.3}$$

(The polynomial $-h(-x)$ is classically called the *cubic resolvent* of $f(x)$.) If $f(x)$ has roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in $\widetilde{\Phi}$, then $h(x)$ has roots $(\lambda_1 + \lambda_4)^2$, $(\lambda_2 + \lambda_4)^2$, $(\lambda_3 + \lambda_4)^2$ in $\widetilde{\Phi}$. In both cases the roots are necessarily distinct. With this notation, we have the following description of $\mathscr{Q}$ and $\mathscr{R}$ which is part of Propositions 6.2 and 6.7 of [**A3**].

PROPOSITION 4.4. $\mathscr{R}$ *has a generator* ν *with minimum polynomial* $h(x)$ *so that* $\mathscr{Q} \cong \left(\frac{\nu, \mu}{\mathscr{R}}\right)$.

In the following corollary of Propositions 4.1 and 4.4, we compute the D_4-type of $\mathscr{K}$ and determine when $\mathscr{K}$ is orthogonal (see Remark 3.20). This last determination will be useful later in the description of anisotropic D_4's over number fields. If K is a group, we use the term *K-cubic* (resp. *K-quartic*) to refer to a degree 3 (resp. degree 4) extension of Φ whose minimum Galois splitting field has Galois group isomorphic to K.

COROLLARY 4.5. *Let* $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$. *Then, the following table gives the* D_4*-type of* $\mathscr{K}$ *and indicates whether or not* $\mathscr{K}$ *is orthogonal for each possible choice of* $\mathscr{B}$ *(up to isomorphism)*:

$\mathscr{B}$	D_4-type	$\mathscr{K}$ orthogonal
$\Phi^{(4)}$ or $E^{(2)}$ with E/Φ quadratic	$D_{4\mathrm{I}}$	Yes
a $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ quartic	$D_{4\mathrm{I}}$	Yes iff μ is a norm for one of the quadratic subextensions of $\mathscr{B}/\Phi$
$\Phi^{(2)} \oplus E$ with E/Φ quadratic	$D_{4\mathrm{II}}$	Yes
$E_1 \oplus E_2$ with E_1/Φ and E_2/Φ nonisomorphic quadratics	$D_{4\mathrm{II}}$	Yes
a $\mathbb{Z}/(4)$-quartic or a dihedral quartic	$D_{4\mathrm{II}}$	Yes iff μ is a norm for the quadratic subextension of $\mathscr{B}/\Phi$
an A_4-quartic or $\Phi \oplus E$ with E/Φ a $\mathbb{Z}/(3)$-cubic	$D_{4\mathrm{III}}$	No
an S_4-quartic or $\Phi \oplus E$ with E/Φ an S_3-cubic	$D_{4\mathrm{VI}}$	No

Proof. Let $P_{\mathscr{B}}/\Phi$ and $P_{\mathscr{R}}/\Phi$ denote respectively the minimum Galois splitting field of $\mathscr{B}$ and $\mathscr{R}$ in $\widetilde{\Phi}/\Phi$. Then, by Proposition 4.4, since $\sum_{i=1}^{4} \lambda_i = 0$,

$$P_{\mathscr{B}} = \Phi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad \text{and}$$
$$P_{\mathscr{R}} = \Phi((\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3), (\lambda_2 + \lambda_4)(\lambda_1 + \lambda_3), (\lambda_3 + \lambda_4)(\lambda_1 + \lambda_2)).$$

Put $G_{\mathscr{B}} := \mathrm{Gal}(P_{\mathscr{B}}/\Phi)$ and $G_{\mathscr{R}} := \mathrm{Gal}(P_{\mathscr{R}}/\Phi)$. Identifying $G_{\mathscr{B}}$ as a subgroup of S_4, we have

$$G_{\mathscr{R}} \cong G_{\mathscr{B}}/G_{\mathscr{B}} \cap V_4, \tag{4.6}$$

where $V_4 = \{(1), (14)(23), (24)(13), (34)(12)\}$. But since $\mathscr{R} \cong \mathscr{Z}(\mathscr{K})$, it follows from Remark 3.16 that $\mathscr{K}$ has type $D_{4\mathrm{I}}$, $D_{4\mathrm{II}}$, $D_{4\mathrm{III}}$ or $D_{4\mathrm{VI}}$ according as $G_{\mathscr{R}}$ has order 1, 2, 3 or 6.

The rest of the argument is a case-by-case check. We consider the most complicated case and leave the others to the reader. Suppose $\mathscr{B}$

is a dihedral quartic. We may identify $\mathscr{B} = \Phi[\lambda_1]$ and relabel λ_2, λ_3, λ_4 if necessary so that $\mathscr{B}$ is the fixed field of (34) in $P_{\mathscr{B}}$. In that case, $G_{\mathscr{B}} = \langle(1324), (34)\rangle$. Hence, by (4.6), $G_{\mathscr{B}}$ has order 2 and so $\mathscr{K}$ has type D_{II}. Also, $\nu_3 := (\lambda_3 + \lambda_4)^2$ is a root of $h(x)$ in Φ. Thus, there is a homomorphism of $\mathscr{R}$ onto Φ so that $\nu \to \nu_3$, which induces a homomorphism of $\mathscr{Q}$ onto $(\nu_3, \mu/\Phi)$. Since $\mathscr{K}$ has type $D_{4\text{II}}$ and $\mathscr{E}(\mathscr{K}) \cong M_4(\mathscr{Q})$, $(\nu_3, \mu/\Phi)$ is the unique 4-dimensional simple summand of $\mathscr{Q}$. Hence, by Remark 3.20, $\mathscr{L}$ is orthogonal iff $(\nu_3, \mu/\Phi)$ splits, which holds iff μ is a norm for $\Phi[\lambda_3 + \lambda_4]$ **[Lam**, Theorem 2.7, p. 58]. □

Recall next that a Lie algebra $\mathscr{L}$ of type D_4 over Φ is said to be *isotropic* if $\mathscr{L}$ has a nonzero element X so that $\text{ad}(X)$ is diagonalizable over Φ. Otherwise, $\mathscr{L}$ is said to be *anisotropic.* We say that $\mathscr{L}$ is *strongly isotropic* if $\mathscr{L}$ is isotropic and $\mathscr{L}$ is not isomorphic to the orthogonal Lie algebra $\mathfrak{o}(q)$ of an 8-dimensional nondegenerate quadratic form of Witt index 1. We now see using results from **[A2]** and **[A3]** that the D_4's that are strongly isotropic all come from the construction in §1, and that they are determined up to isomorphism by their Allen invariants.

PROPOSITION 4.7. *Let* $\gamma_0 := \text{diag}(1, -1, 1)$. *If* $\mathscr{L}$ *is a Lie algebra of type* D_4 *over* Φ, *then* $\mathscr{L}$ *is strongly isotropic if and only if* $\mathscr{L} \cong \mathscr{K}(\text{CD}(\mathscr{B}, \mu), \gamma_0)$ *for some* $\mathscr{B}$, μ *as in* §1. *Moreover, if* $\mathscr{L} \cong \mathscr{K}(\text{CD}(\mathscr{B}, \mu), \gamma_0)$ *and* $\mathscr{L}' \cong \mathscr{K}(\text{CD}(\mathscr{B}', \mu'), \gamma_0)$, *then*

$$\mathscr{L} \cong \mathscr{L}' \Leftrightarrow \mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{L}') \Leftrightarrow \mathscr{Q} \cong \mathscr{Q}',$$

where $\mathscr{Q}$ *(resp.* $\mathscr{Q}'$*) is the quaternion algebra determined by* $\mathscr{B}$, μ *(resp.* $\mathscr{B}'$, μ'*).*

Proof. We use the fact that $\mathscr{K}(\text{CD}(\mathscr{B}, \mu), \gamma_0) \cong \mathscr{K}(\text{CD}(\mathscr{B}, \mu))$, where $\mathscr{K}(\text{CD}(\mathscr{B}, \mu))$ is the Lie algebra constructed from $\text{CD}(\mathscr{B}, \mu)$ using I. L. Kantor's Lie algebra construction **[A3**, Theorem 2.2]. With that fact in mind, the present proposition is part of Theorems 5.1 and 8.1 of **[A2]**. □

REMARK 4.8. Theorem 5.1 of **[A2]** (used above) is proved using the description of finite dimensional central simple structurable algebras given in **[A1]**. Recently O. N. Smirnov **[Sm]** has pointed out that there is a missing class of 35-dimensional algebras in that description and has corrected its proof. The proof of Theorem 5.1 of **[A2]** then goes through without any changes. (See **[A&F2**, §5] for details.)

5. Real and $\mathfrak{p}$-adic D_4's. The classification of real and $\mathfrak{p}$-adic Lie algebras of type D_4 is well known ([**Ve**], [**J2**, §7], [**All1**, §4]). It is important though for our purposes to understand those Lie algebras in terms of the construction $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$.

If $\mathscr{L}$ is a Lie algebra of type D_4 over $\mathbb{R}$, we denote the signature of the Killing form of $\mathscr{L}$ by $\mathrm{sig}(\mathscr{L})$ and call it the *signature of* $\mathscr{L}$.

The first proposition computes the Allen invariant and signature of the Lie algebra $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ in the case when Φ is the real field $\mathbb{R}$. In the table, $\mathbb{C}$ and $\mathbb{H}$ denote respectively the complex field and the real quaternion division algebra. The top row lists the possibilities for $\mathscr{B}$, and the first column lists the possibilities for μ and γ. "γ_i same sign" covers the cases when γ_1, γ_2, γ_3 are all positive or all negative, while "γ_i diff. sign" covers the remaining cases.

PROPOSITION 5.1. *Let* $\Phi = \mathbb{R}$ *and let* $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$, *where* $\mathscr{B}$, μ, γ *are as in* §1. *Then, the Allen invariant* $\mathscr{E}(\mathscr{K})$ *and the signature of* $\mathscr{K}$ *are given in the following table*:

	$\mathbb{R}^{(2)} \oplus \mathbb{C}$	$\mathbb{C}^{(2)}$	$\mathbb{R}^{(4)}$
$\mu > 0$	$M_8(\mathbb{R}) \oplus M_8(\mathbb{C})$, 2	$M_8(\mathbb{R})^{(3)}$, 4	$M_8(\mathbb{R})^{(3)}$, 4
$\mu < 0$, γ_i diff. sign	$M_8(\mathbb{R}) \oplus M_8(\mathbb{C})$, 2	$M_8(\mathbb{R}) \oplus M_4(\mathbb{H})^{(2)}$, -4	$M_8(\mathbb{R})^{(3)}$, 4
$\mu < 0$, γ_i same sign	$M_8(\mathbb{R}) \oplus M_8(\mathbb{C})$, -14	$M_8(\mathbb{R}) \oplus M_4(\mathbb{H})^{(2)}$, -4	$M_8(\mathbb{R})^{(3)}$, -28

Proof. We use the notation of §4. Also, let B be the Killing form of $\mathscr{K}$ and let $q_{m,n}$ denote the symmetric bilinear form with matrix $\left[\begin{smallmatrix} I_m & 0 \\ 0 & -I_n \end{smallmatrix}\right]$. In [**A3**, Theorem 7.2], we calculated the following formula for B:

$$(5.2)\quad \frac{2}{3}B \cong (-\delta_1 t_{\mathscr{B}}) \perp (\delta_1 \mu t_{\mathscr{B}}) \perp (-\delta_2 t_{\mathscr{B}}) \perp (\delta_2 \mu t_{\mathscr{B}}) \perp (-\delta_3 t_{\mathscr{B}}) \perp (\delta_3 \mu t_{\mathscr{B}}) \perp (\mu q_{1,0}) \perp (\mu t_{\mathscr{R}}),$$

where $\delta_1 := \gamma_2 \gamma_3^{-1}$, $\delta_2 := \gamma_3 \gamma_1^{-1}$, $\delta_3 := \gamma_1 \gamma_2^{-1}$.

Suppose first that $\mathscr{B} = \mathbb{R}^{(2)} \oplus \mathbb{C}$. Let $b_0 = (0, -2, 1+i)$, where $i^2 = -1$ in $\mathbb{C}$. Then, $f(x)$ has roots 0, -2, $1+i$, $1-i$ and so $h(x)$ has roots $-2i$, $2i$, 4. By Proposition 4.4, there exists $\nu \in \mathscr{R}$ with minimum polynomial $h(x)$ so that $\mathscr{R} = \Phi[\nu] \cong \mathbb{C} \oplus \mathbb{R}$ and

$$\mathscr{Q} \cong \left(\frac{\nu, \mu}{\mathbb{R}}\right) \cong \left(\frac{-2i, \mu}{\mathbb{C}}\right) \oplus \left(\frac{4, \mu}{\mathbb{R}}\right) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{R}).$$

Since $\mathscr{E}(\mathscr{K}) \cong M_4(\mathscr{C})$, we have the Allen invariants in the first column. Also, $t_{\mathscr{B}} \cong q_{3,1}$ and $t_{\mathscr{R}} \cong q_{2,1}$. If $\mu > 0$, we then get $B \cong q_{15,13}$ (by (5.2)) and so B has signature 2. So suppose $\mu < 0$. If the γ_i have different signs, then exactly two of the δ_i's are negative and we get $B \cong q_{15,13}$ again. Finally, if the γ_i's have the same sign, then the δ_i's are all positive and so $B \cong q_{7,21}$.

If $\mathscr{B} = \mathbb{C}^{(2)}$ or $\mathscr{B} = \mathbb{R}^{(4)}$, we may choose $b_0 = (1+i, -1+2i)$ or $(1, 2, -3, 0)$ respectively. The rest of the calculations are similar to the ones just described and so we omit them. □

It follows from Propositions 4.7 and 5.1 that *there are exactly* 3 *real Lie algebras of type* D_4 *that are strongly isotropic and these have the distinct signatures* 4, 2, -4. But there is a unique anisotropic (=compact [**Sel1**, p. 292]) real D_4 and it has signature -28. Also, there is a unique 8-dimensional real nondegenerate quadartic form of Witt index 1. The corresponding orthogonal Lie algebra must have signature -14 (since -14 occurs in Proposition 5.1). Thus, we recover the very well known fact that *there are exactly* 5 *real* D_4*'s up to isomorphism and these are distinguished by their signatures.* We also obtain:

Corollary 5.3. *Any Lie algebra of type* D_4 *over* $\mathbb{R}$ *is isomorphic to* $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ *for some* $\mathscr{B}, \mu, \gamma$.

We now look at $\mathfrak{p}$-adic D_4's. By a $\mathfrak{p}$-*adic field* we mean a completion of a number field at a finite prime. If Φ is a $\mathfrak{p}$-adic field, then by a theorem of Kneser (see [**Kn1**, Satz 3] or [**B&T**, Proposition 6]) any Lie algebra of type D_4 over Φ is isotropic. Moreover, there are no nondegenerate 8-dimensional quadratic forms of Witt index 1 over a $\mathfrak{p}$-adic field [**Lam**, p. 156]. Thus, *every Lie algebra of type* D_4 *over a* $\mathfrak{p}$*-adic field is strongly isotropic.* Hence, by Proposition 4.7, we have the following:

Proposition 5.4. *Suppose* Φ *is a* $\mathfrak{p}$*-adic field and let* $\gamma_0 :=$ diag$(1, -1, 1)$. *Then, any Lie algebra* $\mathscr{L}$ *of type* D_4 *over* Φ *is isomorphic to* $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma_0)$ *for some* $\mathscr{B}, \mu$. *Moreover, if* $\mathscr{L}$ *and* $\mathscr{L}'$ *are Lie algebras of type* D_4 *over* Φ, *then*

$$\mathscr{L} \cong \mathscr{L}' \Leftrightarrow \mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{L}'). \tag{5.5}$$

6. The Allen invariant over a number field. In this section, we characterize the associative algebras that can occur as Allen invariants over a number field Φ.

We recall some number theoretic notation that we will use here and frequently in the rest of the paper. If Φ is a number field we denote by $S(\Phi)$ the set of all primes of Φ (finite or infinite) and by $S_{\mathbb{R}}(\Phi)$ the set of all real primes of Φ. If $\mathfrak{p} \in S(\Phi)$, $\Phi_{\mathfrak{p}}$ will denote the completion of Φ at $\mathfrak{p}$ and, if $\mathscr{X}$ is an algebra over Φ, we write $\mathscr{X}_{\mathfrak{p}} := \mathscr{X}_{\Phi_{\mathfrak{p}}} := \Phi_{\mathfrak{p}} \otimes_{\Phi} \mathscr{X}$. If $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, we denote by $\sigma_{\mathfrak{p}}$ an embedding of Φ into $\mathbb{R}$ which induces the prime $\mathfrak{p}$. Its extension to an isomorphism $\Phi_{\mathfrak{p}} \to \mathbb{R}$ of valued fields will also be denoted by $\sigma_{\mathfrak{p}}$. If $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$ and $\alpha \in \Phi_{\mathfrak{p}}$, we say that α is *positive at* $\mathfrak{p}$ (resp. *negative at* $\mathfrak{p}$) if $\sigma_{\mathfrak{p}}(\alpha) > 0$ (resp. if $\sigma_{\mathfrak{p}}(\alpha) < 0$). This is written as $\alpha >_{\mathfrak{p}} 0$ (resp. $\alpha <_{\mathfrak{p}} 0$).

PROPOSITION 6.1. *Let Φ be a number field. Suppose $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ over Φ. Then, the following statements are equivalent:*

(i) $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$,

(ii) $\mathscr{D} \cong (\nu, \mu/\mathscr{Z})$ *for some generator* ν *of* $\mathscr{Z}$ *so that* $n_{\mathscr{Z}}(\nu) \in \Phi^{\times 2}$ *and some* $\mu \in \Phi^{\times}$.

Moreover, in that case μ *can be chosen to be totally negative* (*i.e.* $\mu <_{\mathfrak{p}} 0$ *for all* $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$).

Proof. Write

$$\mathscr{D} = \mathscr{D}_1 \oplus \cdots \oplus \mathscr{D}_m \quad \text{and} \quad \mathscr{Z} = \Lambda_1 \oplus \cdots \oplus \Lambda_m,$$

where $\mathscr{D}_i$ is a quaternion algebra over its centre Λ_i, and Λ_i is a field, $i = 1, \ldots, m$.

"(i) $\Rightarrow$ (ii)" For each i, $\mathscr{D}_i \otimes_{\Lambda_i} \Lambda_{i\mathfrak{P}} \cong M_2(\Lambda_{i\mathfrak{P}})$ as $\Lambda_{i\mathfrak{P}}$-algebras for all but a finite number of primes $\mathfrak{P}$ of Λ_i [**P**, p. 358]. Here, $\Lambda_{i\mathfrak{P}}$ denotes the completion of Λ_i at $\mathfrak{P}$. Thus, we may choose a finite nonempty set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ of finite primes of Φ so that

$$\mathscr{D}_i \otimes_{\Lambda_i} \Lambda_{i\mathfrak{P}} \cong M_2(\Lambda_{i\mathfrak{P}}) \quad \text{as } \Lambda_{i\mathfrak{P}}\text{-algebras} \tag{6.2}$$

for $i = 1, \ldots, m$ and all finite primes $\mathfrak{P}$ of Λ_i such that $\mathfrak{P} \cap \Phi \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$. Now for fixed $j \in \{1, \ldots, l\}$, we have

$$\sum_{i=1}^{m} \sum_{\mathfrak{P}} [\Lambda_{i\mathfrak{P}} : \Phi_{\mathfrak{p}_j}] = 3,$$

where the inner sum runs over all finite primes $\mathfrak{P}$ of Λ_i so that $\mathfrak{P} \cap \Phi = \mathfrak{p}_j$. Thus, at most one term $[\Lambda_{i\mathfrak{P}} : \Phi_{\mathfrak{p}_j}]$ in the double sum equals 2. Since $\Phi_{\mathfrak{p}_j}$ has more than one quadratic extension [**Lam**, Theorem

2.22, p. 161], we may choose a quadratic extension $\Phi_{\mathfrak{p}_j}(\sqrt{\mu_j})/\Phi_{\mathfrak{p}_j}$ so that, as extensions of $\Phi_{\mathfrak{p}_j}$,

(6.3) $$\Phi_{\mathfrak{p}_j}(\sqrt{\mu_j}) \text{ is not isomorphic to } \Lambda_{i\mathfrak{P}},$$

for $i = 1, \dots, m$ and all primes $\mathfrak{P}$ of Λ_i such that $\mathfrak{P} \cap \Phi = \mathfrak{p}_j$.

By the strong approximation theorem [**C**, p. 67] and the local square theorem [**Lam**, Theorem 2.19, p. 160], we may choose $\mu \neq 0 \in \Phi$ so that

(6.4) $$\mu\mu_j \in \Phi_{\mathfrak{p}_j}^{\times 2}, \quad j = 1, \dots, l, \qquad \text{and} \qquad \mu <_{\mathfrak{p}} 0 \text{ for all } \mathfrak{p} \in S_{\mathbb{R}}(\Phi).$$

Then, μ is totally negative.

Put $K_i = \Lambda_i(\sqrt{\mu})$, $i = 1, \dots, m$. We next claim that for $i = 1, \dots, m$

(6.5) $$\mathscr{D}_i \otimes_{\Lambda_i} K_i \cong M_2(K_i) \quad \text{as } K_i\text{- algebras}.$$

To see this it suffices, by the Albert-Hasse-Brauer-Noether theorem [**P**, p. 354], to show that

(6.6) $$\mathscr{D}_i \otimes_{\Lambda_i} K_{i\mathfrak{Q}} \cong M_2(K_{i\mathfrak{Q}}) \quad \text{as } K_{i\mathfrak{Q}}\text{-algebras}$$

for all primes $\mathfrak{Q}$ of K_i. If $\mathfrak{Q}$ is infinite, this is clear, since μ is totally negative. So suppose $\mathfrak{Q}$ is finite and put $\mathfrak{P} = \mathfrak{Q} \cap \Lambda_i$. Then,

(6.7) $$\mathscr{D}_i \otimes_{\Lambda_i} K_{i\mathfrak{Q}} \cong (\mathscr{D}_i \otimes_{\Lambda_i} \Lambda_{i\mathfrak{P}}) \otimes_{\Lambda_{i\mathfrak{P}}} K_{i\mathfrak{Q}} \quad \text{as } K_{i\mathfrak{Q}}\text{-algebras}.$$

Thus, by (6.2), we may assume that $\mathfrak{P} \cap \Phi = \mathfrak{p}_j$ for some $j \in \{1, \dots, l\}$. But then $K_{i\mathfrak{Q}} = \Lambda_{i\mathfrak{P}}(\sqrt{\mu}) = \Lambda_{i\mathfrak{P}}(\sqrt{\mu_j})$ (by (6.4)) and hence, by (6.3), $K_{i\mathfrak{Q}}$ is a quadratic extension of $\Lambda_{i\mathfrak{P}}$. But any quadratic extension splits a quaternion algebra over a $\mathfrak{p}$-adic field [**Lam**, Lemma 2.14, p. 517]. Thus, by (6.7), we have (6.6) and hence (6.5).

Since $K_i = \Lambda_i(\mu)$ it follows from (6.5) that we may write

$$\mathscr{D}_i = \left(\frac{\alpha_i, \mu}{\Lambda_i}\right)$$

for some $\alpha_i \neq 0 \in \Lambda_i$ [**P**, Corollary on p. 241], $i = 1, \dots, m$. Then, by the projection formula [**Tig**, Theorem 3.2], we have $c_{\Lambda_i/\Phi}(\mathscr{D}_i) \sim (n_i(\alpha_i), \mu/\Phi)$, where $n_i := n_{\Lambda_i}$, $i = 1, \dots, m$. Hence, $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim (\delta, \mu/\Phi)$, where $\delta = \prod_{i=1}^m n_i(\alpha_i)$. Thus, by our hypothesis that $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$, we have $(\delta, \mu/\Phi) \sim \Phi$. Hence, for each i, $[(\delta, \mu/\Lambda_i)] = 1$ in $\operatorname{Br}(\Lambda_i)$ and so

$$[\mathscr{D}_i] = \left[\left(\frac{\alpha_i, \mu}{\Lambda_i}\right)\right]\left[\left(\frac{\delta, \mu}{\Lambda_i}\right)\right] = \left[\left(\frac{\alpha_i\delta, \mu}{\Lambda_i}\right)\right],$$

which implies that $\mathscr{D}_i \cong (\alpha_i\delta, \mu/\Lambda_i)$ over Λ_i. Since $\sum_{i=1}^m [\Lambda_i : \Phi] = 3$, we may choose $\beta_i \neq 0 \in \Lambda_i$ so that $\nu_i := \alpha_i\delta\beta_i^2$ generates Λ_i over Φ, $i = 1, \ldots, m$, and, if $m = 3$, ν_1, ν_2, ν_3 are distinct. Then,

$$\mathscr{D}_i \cong \left(\frac{\nu_i, \mu}{\Lambda_i}\right), \qquad i = 1, \ldots, m.$$

Thus, putting $\nu = \sum_{i=1}^m \nu_i \in \mathscr{D}$, we have $\mathscr{D} \cong (\nu, \mu/\mathscr{Z})$, ν generates $\mathscr{Z}$, and

$$\begin{aligned} n_{\mathscr{Z}}(\nu)\Phi^{\times 2} &= \prod_{j=1}^m n_j(\nu_j)\Phi^{\times 2} \\ &= \prod_{j=1}^m (n_j(\alpha_j)\delta^{[\Lambda_j : \Phi]})\Phi^{\times 2} = \delta^4\Phi^{\times 2} = 1\Phi^{\times 2}. \end{aligned}$$

"(ii) $\Rightarrow$ (i)" Suppose $\mathscr{D} \cong (\nu, \mu/\mathscr{Z})$ as in (ii). Write $\nu = \nu_1 + \cdots + \nu_m$, where $\nu_i \in \Lambda_i$, $i = 1, \ldots, m$. Then, $\prod_{i=1}^m n_i(\nu_i) = n_{\mathscr{Z}}(\nu) = \eta^2$ for some $\eta \in \Phi^\times$. Also, $\mathscr{D}_i \cong (\nu_i, \mu/\Lambda_i)$, and so, by the projection formula, $c_{\Lambda_i/\Phi}(\mathscr{D}_i) \sim (n_i(\nu_i), \mu/\Phi)$, $i = 1, \ldots, m$. Thus, $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim (\eta^2, \mu/\Phi) \sim \Phi$. □

The following lemma follows immediately from Propositions 4.1 and 4.4, and it is valid over any field of characteristic 0.

LEMMA 6.8 (*$\mathscr{B}$-construction lemma*). *Suppose $\mathscr{Z}$ is a 3-dimensional separable commutative associative algebra and $\mathscr{D} \cong (\nu, \mu/\mathscr{Z})$, where ν is a generator of $\mathscr{Z}$ such that $n_{\mathscr{Z}}(\nu) \in \Phi^{\times 2}$ and $\mu \in \Phi^\times$. Let*

$$h(x) = x^3 + \alpha_2 x^2 + \alpha_1 x - \eta^2$$

be the minimal polynomial of ν over Φ, where $\alpha_2, \alpha_1 \in \Phi$, $\eta \in \Phi^\times$ and $\eta^2 = n_{\mathscr{Z}}(\nu)$. Put

$$f(x) = x^4 + \frac{1}{2}\alpha_2 x^2 + \eta x + \frac{1}{16}(\alpha_2^2 - 4\alpha_1) \quad \text{and} \quad \mathscr{B} = \Phi[b_0],$$

where b_0 has minimum polynomial $f(x)$ over Φ. Then, $\mathscr{D}$ is isomorphic to the quaternion algebra determined by $\mathscr{B}$ and μ. Hence, for any $\gamma = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_1, \gamma_2, \gamma_3 \in \Phi^\times$, $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ has Allen invariant isomorphic to $M_4(\mathscr{D})$.

THEOREM 6.9. *Let Φ be a number field. Suppose $\mathscr{E}$ is an associative algebra over Φ. Then the following statements are equivalent:*

(i) *$\mathscr{E}$ is isomorphic to the Allen invariant of a Lie algebra of type D_4 over Φ.*

(ii) $\mathscr{E}_{\tilde{\Phi}} \cong M_8(\tilde{\Phi})^{(3)}$, *the simple summands of* $\mathscr{E}$ *have exponent* 1 *or* 2 *in the Brauer groups over their centres, and* $c_{\mathscr{Z}/\Phi}(\mathscr{E}) \sim \Phi$, *where* $\mathscr{Z}$ *is the centre of* $\mathscr{E}$.

(iii) $\mathscr{E} \cong M_4(\mathscr{D})$, *where* $\mathscr{D}$ *is a quaternion algebra over a* 3*-dimensional separable algebra* $\mathscr{Z}$ *over* Φ *and* $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$.

Proof. (i) $\Rightarrow$ (ii) follows from §3 (in particular Proposition 3.3).

"(ii) $\Rightarrow$ (iii)" $\mathscr{Z}$ is 3-dimensional separable and we have $\mathscr{E} = \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_m$ and $\mathscr{Z} = \Lambda_1 \oplus \cdots \oplus \Lambda_m$, where $\mathscr{E}_i$ is simple with centre Λ_i and $\dim_{\Lambda_i} \mathscr{E}_i = 64$, $i = 1, \ldots, m$. Since index equals exponent in the Brauer group over a number field [**P**, p. 359], $\mathscr{E}_i \cong M_4(\mathscr{D}_i)$, where $\mathscr{D}_i$ is a quaternion algebra over Λ_i. Then, $\mathscr{E} \cong M_4(\mathscr{D})$, where $\mathscr{D} = \mathscr{D}_1 \oplus \cdots \oplus \mathscr{D}_m$, in which case $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim c_{\mathscr{Z}/\Phi}(\mathscr{E}) \sim \Phi$.

(iii) $\Rightarrow$ (i) follows from Proposition 6.1 ((i) $\Rightarrow$ (ii)) and the $\mathscr{B}$-construction lemma. □

REMARKS 6.10. (a) The equivalence of (i) and (ii) in the theorem answers, in the number field case, a question raised by T. Tamagawa after a lecture on an earlier version of this work.

(b) Theorem 6.2 is also true if $\Phi = \mathbb{R}$ or a $\mathfrak{p}$-adic field. Indeed, since index equals exponent in the Brauer group over those fields [**P**, p. 339], the proofs of "(i) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (iii)" are the same as above. If $\Phi = \mathbb{R}$, "(iii) $\Rightarrow$ (i)" follows from Proposition 5.1. Finally, if Φ is a $\mathfrak{p}$-adic field, then the implication "(i) $\Rightarrow$ (ii)" in Proposition 6.1 follows from the "local part" of the argument given in the number field case. Hence, the proof of "(iii) $\Rightarrow$ (i)" in the Theorem is also valid in the $\mathfrak{p}$-adic case.

(c) Suppose Φ is a $\mathfrak{p}$-adic field. Then, it is an easy matter to list the possible algebras $\mathscr{D}$ such that $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ and $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim 1$. Indeed, *let* $\mathbb{D}(E)$ *denote the unique quaternion division algebra over* E *for each finite extension* E/Φ. If $\mathscr{Z} \cong \Phi \oplus \Phi \oplus \Phi$, then we must have $\mathscr{D} \cong M_2(\Phi)^{(3)}$ or $M_2(\Phi) \oplus \mathbb{D}(\Phi)^{(2)}$. Suppose next that $\mathscr{Z} \cong \Phi \oplus \Gamma$, where Γ/Φ is a quadratic extension. Then $\mathscr{D} \cong M_2(\Phi) \oplus M_2(\Gamma)$ or $\mathscr{D} \cong \mathscr{D}_1 \oplus \mathbb{D}(\Gamma)$, where $\mathscr{D}_1 \sim c_{\Gamma/\Phi}(\mathbb{D}(\Gamma))$. But $\mathbb{D}(\Gamma)$ cannot be obtained by base field extension from a quaternion algebra over Φ (since Γ splits any quaternion algebra over Φ). Hence, by the Albert-Riehm theorem [**Sch**, Chapter 8, Theorems 9.5 and 11.2(ii)], $c_{\Gamma/\Phi}(\mathbb{D}(\Gamma))$ is not similar to Φ. Thus, $\mathscr{D} \cong M_2(\Phi) \oplus M_2(\Gamma)$ or $\mathscr{D} \cong \mathbb{D}(\Phi) \oplus \mathbb{D}(\Gamma)$. Finally, suppose that $\mathscr{Z} = \Lambda$, a cubic extension of Φ. Then $\mathscr{D} \cong M_2(\Lambda)$ or $\mathbb{D}(\Lambda)$. But $\mathbb{D}(\Phi)_\Lambda$ is a division algebra and so $\mathbb{D}(\Lambda) \cong \mathbb{D}(\Phi)_\Lambda$. Thus,

[Tig, Theorem 2.5], $c_{\Lambda/\Phi}(\mathbb{D}(\Lambda)) \cong \mathbb{D}(\Phi)\otimes_\Phi\mathbb{D}(\Phi)\otimes_\Phi\mathbb{D}(\Phi)$ which is not similar to Φ and so $\mathscr{D} \cong M_2(\Lambda)$. Thus, the list of quaternion algebras $\mathscr{D}$ over 3-dimensional separable algebras $\mathscr{Z}$ so that $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$ is:

$$(6.11)\qquad M_2(\Phi)^{(3)} \quad\text{and}\quad M_2(\Phi)\oplus\mathbb{D}(\Phi)^{(2)},$$
$$M_2(\Phi)\oplus M_2(\Gamma) \quad\text{and}\quad \mathbb{D}(\Phi)\oplus\mathbb{D}(\Gamma) \quad\text{for } [\Gamma:\Phi]=2, \quad\text{and}$$
$$M_2(\Lambda) \quad \text{for } [\Lambda:\Phi]=3.$$

By remark (b) and (5.5), *the Allen invariant induces a bijection from the set of isomorphism classes of D_4's over Φ onto the set of algebras $M_4(\mathscr{D})$, where $\mathscr{D}$ runs through the list* (6.11). (Compare **[J2**, §7] and **[All1**, §4].)

7. Isomorphism of D_4's over number fields. Now that the possible Allen invariants of Lie algebras of type D_4 over number fields have been identified, it is natural to ask how close the invariants come to determining the Lie algebras. In this section, we prove an isomorphism theorem that answers that question. We begin with some preliminary results.

If $\widetilde{\mathscr{M}}$ is a semisimple Lie algebra over $\widetilde{\Phi}$, an automorphism of $\widetilde{\mathscr{M}}$ is said to be *inner* if it lies in the connected component $\mathrm{Aut}(\widetilde{\mathscr{M}})^0$ of the algebraic group $\mathrm{Aut}(\widetilde{\mathscr{M}})$. Otherwise, the automorphism is said to be *outer*. If $\mathscr{L}$ is a Lie algebra of type D_4 over Φ, an automorphism of $\mathscr{L}$ is called *inner* or *outer* according as its extension to an automorphism of $\mathscr{L}_{\widetilde{\Phi}}$ is inner or outer.

LEMMA 7.1. *Suppose that $\Phi = \mathbb{R}$, $\mathbb{C}$ or a $\mathfrak{p}$-adic field, and $\mathscr{L}$ is a Lie algebra of type $D_{4\mathrm{I}}$, $\mathscr{D}_{\mathrm{II}}$ or $D_{4\mathrm{III}}$ over Φ. Then, $\mathscr{L}$ has an outer automorphism.*

Proof. Suppose first that $\mathscr{E}(\mathscr{L})$ has a simple summand that is isomorphic to $M_8(\Phi)$. Hence, by Remark 3.20, $\mathscr{L} \cong \mathfrak{o}(q)$ for some 8-dimensional quadratic form q. Regarding this isomorphism as an identification, we may take ϕ to be the automorphism of $\mathscr{L}$ defined by $\phi(X) = RXR^{-1}$, where R is an orthogonal reflection (relative to q) in a hyperplane. It follows from **[J4**, §4] that ϕ (extended to $\mathscr{L}_{\widetilde{\Phi}}$) lies outside a proper closed subgroup of finite index in $\mathrm{Aut}(\mathscr{L}_{\widetilde{\Phi}})$. Hence ϕ is outer.

This, by Proposition 5.1 and Corollary 5.3, completes the proof if $\Phi = \mathbb{R}$ or $\mathbb{C}$. Suppose then that Φ is a $\mathfrak{p}$-adic field. In that case

$\mathrm{Br}(\Phi)$ has exactly two elements of exponent 1 or 2, namely $[\Phi]$ and $[\mathbb{D}]$, where $\mathbb{D}$ is the unique quaternion division algebra over Φ [**Lam**, Theorem 2.10, p. 154].

If $\mathscr{L}$ has type $D_{4\mathrm{I}}$, then $\mathscr{E}_i \cong M_8(\Phi)$ for some i (by (3.17)) and so we're done by the above. Suppose next that $\mathscr{L}$ has type $D_{4\mathrm{II}}$. With the notation of Remark 3.16, $\mathscr{F} \cong M_8(\Phi)$ or $M_4(\mathbb{D})$. Thus, we may assume that $\mathscr{F} \cong M_4(\mathbb{D})$. But then, as in Remark 3.20, $\mathscr{L} \cong \mathscr{S}(\mathscr{F}, J_{\mathscr{F}})$. Hence, by [**J1**, §§6 and 7], $\mathscr{L} \cong \mathscr{S}(M_4(\mathbb{D}), J_S)$, where $J_S(X) = S\overline{X}^t S^{-1}$ and S is an invertible 4×4-diagonal matrix over $\mathbb{D}$ that is skew-hermitian with respect to the canonical involution $^-$ on $\mathbb{D}$. Identify $\mathscr{L} = \mathscr{S}(M_4(\mathbb{D}), J_S)$. Let $\delta\Phi^{\times 2}$ be the *discriminant* of S, i.e. $\delta\Phi^{\times 2}$ is the square class in $\Phi^\times/\Phi^{\times 2}$ represented by the reduced norm (=generic norm) of S in $M_4(\mathbb{D})$. The reduced norm $n_{\mathbb{D}}$ on $\mathbb{D}$ is universal [**Lam**, Corollary 2.12, p. 156] and so we may write $\delta = n_{\mathbb{D}}(x)$ for some $x \neq 0 \in \mathbb{D}$. Then, $x = s_1 s_2$ for some $s_1, s_2 \neq 0 \in \mathscr{S}(\mathbb{D}, {}^-)$. Hence, S has the same discriminant as $S' = \mathrm{diag}(s_1, s_1, s_1, s_2)$ and hence $(M_4(\mathbb{D}), J_S) \cong (M_4(\mathbb{D}), J_{S'})$ [**J1**, Theorem 9]. Thus, we may assume that $S = \mathrm{diag}(s_1, s_1, s_1, s_2)$. But since $\mathscr{L}$ has type $D_{4\mathrm{II}}$, δ is not a square. (See [**T1**, p. 57], or use base field extension and argue using [**J2**, top of p. 145].) Hence, $n_{\mathbb{D}}(s_1)\Phi^{\times 2} \neq n_{\mathbb{D}}(s_2)\Phi^{\times 2}$. Thus, putting $P_i := \Phi[s_i]$, $i = 1, 2$, P_1 and P_2 are not isomorphic. Thus, the norm groups $n_{P_1/\Phi}(P_1^\times)$ and $n_{P_2/\Phi}(P_2^\times)$ are distinct [**Ser2**, Chapter 14, §6]. But these norm groups are subgroups of $\Phi^\times$ of index 2 [**Ser2**, Proposition 9, p. 196]. Thus, $n_{P_1/\Phi}(P_1^\times)n_{P_2/\Phi}(P_2^\times) = \Phi^\times$. Hence, $n_{\mathbb{D}}(P_1^\times)n_{\mathbb{D}}(P_2^\times) = \Phi^\times$. Now fix $s_0 \neq 0 \in P_2^\perp$ ($\perp$ with respect to $n_{\mathbb{D}}$). Thus, we may choose $g_1 \in P_1$ and $g_0 \in P_2$ so that $n_{\mathbb{D}}(g_1)n_{\mathbb{D}}(g_0) = -n_{\mathbb{D}}(s_0)$. Put $g_2 = s_0 g_0^{-1}$. Then, $g_1 \in P_1$, $g_2 \in P_2^\perp$ and $n_{\mathbb{D}}(g_1) = -n_{\mathbb{D}}(g_2)$. Thus,

$$g_1 s_1 = s_1 g_1, \quad g_2 s_2 = -s_2 g_2 \quad \text{and} \quad n_{\mathbb{D}}(g_1) = -n_{\mathbb{D}}(g_2).$$

Put $R = \mathrm{diag}(g_1, g_1, g_1, g_2)$. Then, $(J_S R)R = \alpha I$, where $\alpha = n_{\mathbb{D}}(g_1)$. Thus, the map $\psi\colon M_4(\mathbb{D}) \to M_4(\mathbb{D})$ defined by $\psi(X) = RXR^{-1}$ is an automorphism of $(M_4(\mathbb{D}), J_S)$ which therefore restricts to an automorphism ϕ of $\mathscr{L}$. But R has reduced norm $-\alpha^4$ and $(J_S R)R = \alpha I$. Thus, using [**J4**, §4] it follows that ϕ (extended to $\mathscr{L}_{\tilde{\Phi}}$) lies outside a proper closed subgroup of finite index in $\mathrm{Aut}(\mathscr{L}_{\tilde{\Phi}})$. So ϕ is outer.

Suppose finally that $\mathscr{L}$ has type $D_{4\mathrm{III}}$. Then, by [**All1**, p. 264 and Theorem 5], $\mathscr{L} \cong \mathrm{Der}(\mathscr{J}/\Gamma)$, where $\mathscr{J}$ is the (split) exceptional simple Jordan algebra, Γ is a 3-dimensional subalgebra of $\mathscr{J}$

which is a Galois cubic extension of Φ, and $\mathrm{Der}(\mathscr{J}/\Gamma) := \{D \in \mathrm{Der}\,\mathscr{J} : D|_\Gamma = 0\}$. Identify $\mathscr{L} = \mathrm{Der}(\mathscr{J}/\Gamma)$. Let η be a generator of $\mathrm{Gal}(\Gamma/\Phi)$. Then, as in the proof of **[All1**, Corollary on p. 261], η can be extended to an automorphism R of $\mathscr{J}$. Define $\phi: \mathscr{L} \to \mathscr{L}$ by $\phi(X) = RXR^{-1}$. Then, since $R|_\Gamma \neq 1$, it follows from **[All1**, proof of Theorem 7 and the Note on p. 253] that ϕ (extended to $\mathscr{L}_{\widetilde{\Phi}}$) lies outside a proper closed subgroup of finite index in $\mathrm{Aut}(\mathscr{L}_{\widetilde{\Phi}})$. Thus, ϕ is outer. □

If $\mathscr{L}$ and $\mathscr{L}'$ are Φ-forms of $\widetilde{\mathscr{L}}$, we say that $\mathscr{L}$ are $\mathscr{L}'$ are *inner isomorphic*, written $\mathscr{L} \cong_0 \mathscr{L}'$, if there is an inner automorphism ϕ of $\widetilde{\mathscr{L}}$ so that $\phi\mathscr{L} = \mathscr{L}'$. We say that the Allen invariants $\mathscr{E}(\mathscr{L})$ and $\mathscr{E}(\mathscr{L}')$ are *inner isomorphic*, written $\mathscr{E}(\mathscr{L}) \cong_0 \mathscr{E}(\mathscr{L}')$, if there is an automorphism ψ of $\widetilde{\mathscr{E}}$ so that $\psi|\widetilde{\mathscr{Z}} = I$ and $\psi(\mathscr{E}(\mathscr{L})) = \mathscr{E}(\mathscr{L}')$. It follows from Corollary 2.6 that

$$\mathscr{L} \cong_0 \mathscr{L}' \Rightarrow \mathscr{L} \cong \mathscr{L}' \quad \text{and} \quad \mathscr{E}(\mathscr{L}) \cong_0 \mathscr{E}(\mathscr{L}').$$

We now see that the converse holds over $\mathbb{R}$, $\mathbb{C}$ and $\mathfrak{p}$-adic fields.

PROPOSITION 7.2. *Suppose $\Phi = \mathbb{R}$, $\mathbb{C}$ or a $\mathfrak{p}$-adic field and $\mathscr{L}$, $\mathscr{L}'$ are Φ-forms of $\widetilde{\mathscr{L}}$. Then,*

$$\mathscr{L} \cong_0 \mathscr{L}' \Leftrightarrow \mathscr{L} \cong \mathscr{L}' \quad \textit{and} \quad \mathscr{E}(\mathscr{L}) \cong_0 \mathscr{E}(\mathscr{L}').$$

Proof. Suppose that $\mathscr{L} \cong \mathscr{L}'$ and $\mathscr{E}(\mathscr{L}) \cong_0 \mathscr{E}(\mathscr{L}')$. Thus, there exist $\phi \in \mathrm{Aut}(\widetilde{\mathscr{L}})$ so that $\phi\mathscr{L} = \mathscr{L}'$ and $\psi \in \mathrm{Aut}(\widetilde{\mathscr{E}})$ so that $\psi|\widetilde{\mathscr{Z}} = I$ and $\psi\mathscr{E}(\mathscr{L}) = \mathscr{E}(\mathscr{L}')$. Then, ϕ is determined by some pair (p, U) (see Remark 2.5) where $p = p(\phi) \in S_3$ and $U = (U_1, U_2, U_3)$ satisfy (2.2)–(2.4) with $s = 1$. We now define $\omega \in \mathrm{Aut}(\widetilde{\mathscr{E}})$ by

$$\omega(X_1, X_2, X_3) = (U_1 X_{p1} U_1^{-1}, U_2 X_{p2} U_2^{-1}, U_3 X_{p3} U_3^{-1}).$$

Then, $\omega|\widetilde{\mathscr{L}} = \phi$ and hence $\omega(\mathscr{E}(\mathscr{L})) = \mathscr{E}(\mathscr{L}')$. Moreover,

$$\omega(\widetilde{\mathscr{E}}_i) = \widetilde{\mathscr{E}}_{p^{-1}i}, \qquad i = 1, 2, 3. \tag{7.3}$$

Also since $\psi(E_i) = E_i$, we have

$$\psi(\widetilde{\mathscr{E}}_i) = \widetilde{\mathscr{E}}_i, \qquad i = 1, 2, 3. \tag{7.4}$$

Now it suffices to find an automorphism $\eta \in \mathrm{Aut}(\mathscr{L})$ so that (denoting the extension of η to $\widetilde{\mathscr{L}}$ by η as well) the permutation $p(\eta)$ in S_3 determined by η is p. Indeed, in that case we would have

$(\phi\eta^{-1})(\mathscr{L}) = \mathscr{L}'$ and $\phi\eta^{-1} \in \mathrm{Aut}(\widetilde{\mathscr{L}})^0$ by Corollary 2.6. Thus, we certainly may assume that $p \neq (1)$. We now consider cases and use the notation of Remark 3.16 for $\mathscr{L}$ and $\mathscr{L}'$ (with primes in the latter case). We note that since $\mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{L}')$, $\mathscr{L}$ and $\mathscr{L}'$ have the same D_4-type (by Remark 3.16).

Suppose first that $\mathscr{L}$ has type $D_{4\mathrm{I}}$. Then, by (7.3) and (7.4),

$$(7.5) \qquad \omega(\mathscr{E}_i) = \mathscr{E}'_{p^{-1}i} \quad \text{and} \quad \mathscr{E}_i \cong \mathscr{E}'_i, \qquad i = 1, 2, 3.$$

Since $p \neq (1)$, (7.5) forces two distinct $\mathscr{E}_i$'s to be isomorphic, say $\mathscr{E}_2 \cong \mathscr{E}_3$. Thus, by (3.17), $\mathscr{E}_1 \sim \Phi$. Suppose now that $\mathscr{E}_2 \sim \Phi$. Then, $\mathscr{E}_1 \cong \mathscr{E}_2 \cong \mathscr{E}_3 \sim \Phi$. Thus, by **[J2**, Theorem 7], $\mathscr{L}$ isomorphic to $\mathfrak{o}(n)$, where n is the norm form of a Cayley algebra $\mathscr{C}$ over Φ. We may identify $\mathscr{C}$ as a Φ-form of $\widetilde{\mathscr{C}}$. Then, $\mathfrak{o}(n)$ is isomorphic to the following Φ-form of $\widetilde{\mathscr{L}}$:

$$\begin{aligned}\mathscr{L}'' := \{(L_1, L_2, L_3) \in \mathfrak{o}(n)^{(3)} : &\langle L_1 x, y, z\rangle \\ &+ \langle x, L_2 y, z\rangle + \langle x, y, L_3 z\rangle = 0 \text{ for } x, y, z \in \mathscr{C}\}\end{aligned}$$

[J2, Lemma 2]. Hence, $\mathscr{L}''$ is isomorphic to $\mathscr{L}$. It is clear from Remark 2.5(b) that $\mathscr{L}''$ has automorphisms which determine all 6 permutations in S_3. Hence, the same is true of $\mathscr{L}$ and we're done in the case when $\mathscr{E}_2 \sim \Phi$. So suppose that $\mathscr{E}_2$ is not similar to Φ. Then, by (7.5), $p = (23)$. But by Lemma 7.1, $\mathscr{L}$ has an outer automorphism η. Extending η to an automorphism ν of $\widetilde{\mathscr{E}}$ (just as we extended ϕ at the beginning of the proof), we see that $\nu\mathscr{E}_i = \mathscr{E}_{q^{-1}i}$, $i = 1, 2, 3$, where $q = p(\eta)$. Hence, $p(\eta) = (23)$.

Suppose next that $\mathscr{L}$ has type $D_{4\mathrm{II}}$. Then, by (7.4), we may assume that $\mathscr{E}(\mathscr{L}) = \mathscr{F} \oplus \mathscr{G}$ and $\mathscr{E}(\mathscr{L}') = \mathscr{F}' \oplus \mathscr{G}'$, where $\mathscr{F} = \widetilde{\mathscr{E}}_1 \cap \mathscr{E}(\mathscr{L})$, $\mathscr{G} = (\widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3) \cap \mathscr{E}(\mathscr{L})$, $\mathscr{F}' = \widetilde{\mathscr{E}}_1 \cap \mathscr{E}(\mathscr{L}')$ and $\mathscr{G}' = (\widetilde{\mathscr{E}}_2 \oplus \widetilde{\mathscr{E}}_3) \cap \mathscr{E}(\mathscr{L}')$. By (7.3), $p = (23)$. But $\mathscr{L}$ has an outer automorphism η, and again extending η to $\widetilde{\mathscr{E}}$, we see that $p(\eta) = (23)$.

Suppose finally that $\mathscr{L}$ has type $D_{4\mathrm{III}}$ or $D_{4\mathrm{VI}}$. Since $\psi|\widetilde{\mathscr{Z}} = I$, $\mathscr{Z}(\mathscr{L}) = \mathscr{Z}(\mathscr{L}')$. Thus, by (7.3), ω restricts to a nontrivial automorphism of $\mathscr{Z}(\mathscr{L})$ whose order is the order of p. Hence, $\mathscr{L}$ has type $D_{4\mathrm{III}}$ and $p = (123)$ or (132). But $\mathscr{L}$ has an outer automorphism η and extending η to $\widetilde{\mathscr{E}}$ we see that $p(\eta) = (123)$ or (132). Thus, $p(\eta) = p$ or $p(\eta^2) = p$. □

If A is an algebraic group defined over Φ (in the sense of **[B]**) and P/Φ is an extension, we denote by $H^i(P, A)$ the cohomology set $H^i(\mathrm{Gal}(\widetilde{P}/P), A(\widetilde{P}))$ whenever the latter makes sense. Here $\widetilde{P}$ is an algebraic closure of P and $A(\widetilde{P})$ is the group of $\widetilde{P}$-points of A. Then, $H^i(P, A)$ is functorial in P [**Ser1**, p. II-3].

THEOREM 7.6 (*Injectivity theorem*). *Suppose A is an almost simple adjoint algebraic group of type D_4 defined over a number field Φ. Then, the map*

$$H^1(\Phi, A) \to \prod_{\mathfrak{p}\in S(\Phi)} H^1(\Phi_\mathfrak{p}, A)$$

is injective.

The injectivity theorem will be proved using the corresponding result for simply connected groups due to Harder [**Ha**]. This involves a short excursion into Galois cohomology that is independent of the rest of the paper. We therefore postpone the proof until an appendix (§12). For the terminology used in the statement of the theorem see for example [**T1**].

We now use the injectivity theorem and Proposition 7.2 to prove the following result:

THEOREM 7.7 (*D_4-isomorphism theorem*). *Suppose that $\mathscr{L}$ and $\mathscr{L}'$ are Lie algebras of type D_4 over a number field Φ. Then,*

$$\mathscr{L} \cong \mathscr{L}' \Leftrightarrow \mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{L}') \text{ and } \mathscr{L}_\mathfrak{p} \cong \mathscr{L}'_\mathfrak{p} \quad \text{for all real primes } \mathfrak{p}.$$

Proof. We need only prove "$\Leftarrow$".

Choose an algebraically closed extension Ω/Φ of high enough transcendency degree to contain copies of $\Phi_\mathfrak{p}/\Phi$ for all $\mathfrak{p} \in S(\Phi)$. We identify $\Phi_\mathfrak{p}/\Phi$ in Ω/Φ for all $\mathfrak{p} \in S(\Phi)$, and we take $\widetilde{\Phi}$ (resp. $\Phi_\mathfrak{p}^\sim$) to be the algebraic closure of Φ (resp. $\Phi_\mathfrak{p}$) in Ω.

We identify $\widetilde{\mathscr{E}}$, $\widetilde{\mathscr{L}}$, $\widetilde{\mathscr{E}}$ and $\widetilde{\mathscr{Z}}$ as well as $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$, $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{L}}$, $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$ and $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{Z}}$ as subalgebras of $\Omega \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$, $\Omega \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{L}}$, $\Omega \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$ and $\Omega \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{Z}}$ respectively. We note that $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{L}}$, $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$ and $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{Z}}$ can be regarded as the algebras constructed from $\Phi_\mathfrak{p}^\sim \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{E}}$ exactly as $\widetilde{\mathscr{L}}$, $\widetilde{\mathscr{E}}$ and $\widetilde{\mathscr{Z}}$ were constructed from $\widetilde{\mathscr{E}}$ in §2.

Now identify $\mathscr{L}$ and $\mathscr{L}'$ as Φ-forms of $\widetilde{\mathscr{L}}$. Since $\mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{L}')$, we have an isomorphism, $\psi: \widetilde{\mathscr{E}} \to \widetilde{\mathscr{E}}$ so that $\psi(\mathscr{E}(\mathscr{L})) = \mathscr{E}(\mathscr{L}')$. Then, $\psi(E_i) = E_{qi}$, $i = 1, 2, 3$, for some $q \in S_3$. But then letting ϕ be any element of $\mathrm{Aut}(\widetilde{\mathscr{L}})$ so that $p(\phi) = q$ and extending

ϕ to an automorphism ω of $\widetilde{\mathscr{E}}$ (as in the proof of Proposition 7.2), we see that $\omega\psi|_{\widetilde{\mathscr{Z}}} = I$. Thus, replacing $\mathscr{L}$ by $\phi\mathscr{L}$ and ψ by $\omega\psi$, we may assume that *there exists an isomorphism* $\psi: \widetilde{\mathscr{E}} \to \widetilde{\mathscr{E}}$ *so that* $\psi(\mathscr{E}(\mathscr{L})) = \mathscr{E}(\mathscr{L}')$ *and* $\psi|_{\widetilde{\mathscr{Z}}} = I$. That is $\mathscr{E}(\mathscr{L}) \cong_0 \mathscr{E}(\mathscr{L}')$. Hence, since with our identifications we have $\mathscr{E}(\mathscr{L}_{\mathfrak{p}}) = \mathscr{E}(\mathscr{L})_{\mathfrak{p}})$ and $\mathscr{E}(\mathscr{L}'_{\mathfrak{p}}) = \mathscr{E}(\mathscr{L}')_{\mathfrak{p}}$, it follows that

$$\mathscr{E}(\mathscr{L}_{\mathfrak{p}}) \cong_0 \mathscr{E}(\mathscr{L}'_{\mathfrak{p}}) \quad \text{for } \mathfrak{p} \in S(\Phi). \tag{7.8}$$

Next let $A = \operatorname{Aut}(\Omega \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{L}})^0$. It is well known that A is an almost simple adjoint algebraic group of type D_4. We give A the structure of an algebraic group defined over Φ using the Φ-form $\mathscr{L}$ of $\widetilde{\mathscr{L}}$. Then, $A(\widetilde{\Phi}) = \operatorname{Aut}(\widetilde{\mathscr{L}})^0$ and $A(\Phi_{\mathfrak{p}}^{\sim}) = \operatorname{Aut}(\Phi_{\mathfrak{p}}^{\sim} \otimes_{\widetilde{\Phi}} \widetilde{\mathscr{L}})^0$ for $\mathfrak{p} \in S(\Phi)$.

Now let $(\alpha_s)_{s\in G}$ and $(\alpha'_s)_{s\in G}$ be the Galois precocycles determined by $\mathscr{L}$ and $\mathscr{L}'$ respectively. Then, as in proof of Proposition 3.3, α_s (resp. α'_s) extends to an s-linear automorphism β_s (resp. β'_s) of $\widetilde{\mathscr{E}}$ which maps E_i to $E_{p(\alpha_s)^{-1}i}$ (resp. $E_{p(\alpha'_s)^{-1}i}$) and fixes the elements of $\mathscr{Z}(\mathscr{L})$ (resp. $\mathscr{Z}(\mathscr{L}')$). But $\mathscr{Z}(\mathscr{L}) = \mathscr{Z}(\mathscr{L}')$ since $\psi(\mathscr{Z}(\mathscr{L})) = \mathscr{Z}(\mathscr{L}')$ and $\psi|\widetilde{\mathscr{Z}} = I$. Thus, $\beta'_s(\beta_s)^{-1}$ is a linear automorphism of $\widetilde{\mathscr{E}}$ which fixes the elements of $\mathscr{Z}(\mathscr{L})$ and hence the elements of $\widetilde{\mathscr{Z}}$. Hence, $p(\alpha_s) = p(\alpha'_s)$. Thus, putting $\zeta_s = \alpha'_s\alpha_s^{-1}$, we have $p(\zeta_s) = (1)$ and hence

$$\zeta_s \in A(\widetilde{\Phi}) \tag{7.9}$$

for $s \in G$. Therefore, $(\zeta_s)_{s\in G}$ is a continuous 1-cocycle with values in $A(\widetilde{\Phi})$ which therefore represents an element $\zeta \in H^1(\Phi, A)$. (This is the standard assignment of a cohomology class to a Φ-form relative to $\mathscr{L}$. Under this assignment $\mathscr{L}' \to \zeta$ and $\mathscr{L} \to 1$. (7.9) says that $\mathscr{L}'$ is an *inner twist* of $\mathscr{L}$.) But then $\zeta = 1$ if and only if $\mathscr{L} \cong_0 \mathscr{L}'$. Thus, the injectivity theorem tells us that

$$\mathscr{L}_{\mathfrak{p}} \cong_0 \mathscr{L}'_{\mathfrak{p}} \quad \text{for all } \mathfrak{p} \in S(\Phi) \Rightarrow \mathscr{L} \cong_0 \mathscr{L}'. \tag{7.10}$$

So it suffices to verify that $\mathscr{L}_{\mathfrak{p}} \cong_0 \mathscr{L}'_{\mathfrak{p}}$ for all $\mathfrak{p} \in S(\Phi)$. But then by (7.8) and Proposition 7.2, it is enough to show that $\mathscr{L}_{\mathfrak{p}} \cong \mathscr{L}'_{\mathfrak{p}}$ for all $\mathfrak{p} \in S(\Phi)$. If $\mathfrak{p}$ is real this is being assumed, if $\mathfrak{p}$ is complex it is trivial, and if $\mathfrak{p}$ is finite it follows from (5.5) and (7.8). □

8. Construction of D_4's over number fields. This section contains the main results of the paper. If Φ is a number field, we show that the construction in §1 is *complete* in the sense that it yields all Lie algebras of type D_4 over Φ. We also give necessary and sufficient conditions for isomorphism of the Lie algebras obtained from the construction.

If Φ is a number field and $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, we may identify $\Phi_{\mathfrak{p}}$ and $\mathbb{R}$ by means of $\sigma_{\mathfrak{p}}$. If $\mathscr{L}$ is a Lie algebra of type D_4 and $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, we may then refer to the signature $\mathrm{sig}(\mathscr{L}_{\mathfrak{p}})$ of $\mathscr{L}_{\mathfrak{p}}$.

LEMMA 8.1. *Suppose $\mathscr{L}$ is a Lie algebra of type D_4 over a number field Φ. Suppose $\mathscr{B}$ is as in* §1 *and μ is a totally negative scalar from $\Phi^\times$ so that $\mathscr{E}(\mathscr{L}) \cong M_4(\mathscr{Q})$, where $\mathscr{Q}$ is the quaternion algebra determined by $\mathscr{B}$ and μ. Then, there exists $\gamma_2 \in \Phi^\times$ so that $\mathscr{L} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$, where $\gamma = \mathrm{diag}(1, \gamma_2, 1)$.*

Proof. By Proposition 4.1, we have $\mathscr{E}(\mathscr{L}) \cong \mathscr{E}(\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma))$ for all choices of γ as in §1. Thus, by the D_4-isomorphism theorem, it suffices to show that we can choose $\gamma_2 \in \Phi^\times$ so that $\mathscr{L}_{\mathfrak{p}} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)_{\mathfrak{p}}$ for all $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, where $\gamma = \mathrm{diag}(1, \gamma_2, 1)$. So let

$$S := \{\mathfrak{p} \in S_{\mathbb{R}}(\Phi) : \mathrm{sig}(\mathscr{L}_{\mathfrak{p}}) = -14 \text{ or } -28\}.$$

Choose, by the approximation theorem, $\gamma_2 \in \Phi^\times$ so that

$$\begin{aligned} &\gamma_2 >_{\mathfrak{p}} 0 \quad \text{for all } \mathfrak{p} \in S, \quad \text{and} \\ &\gamma_2 <_{\mathfrak{p}} 0 \quad \text{for all } \mathfrak{p} \in S_{\mathbb{R}}(\Phi) - S. \end{aligned} \tag{8.2}$$

Put $\gamma = \mathrm{diag}(1, \gamma_2, 1)$ and $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$.

Now let $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$. We want to show that $\mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) = \mathrm{sig}(\mathscr{L}_{\mathfrak{p}})$. As noted in the proof of the D_4-isomorphism theorem, we have $\mathscr{E}(\mathscr{L}_{\mathfrak{p}}) \cong \mathscr{E}(\mathscr{L})_{\mathfrak{p}}$. Thus, $\mathscr{E}(\mathscr{K}_{\mathfrak{p}}) \cong \mathscr{E}(\mathscr{L}_{\mathfrak{p}})$. Hence, if $\mathrm{sig}(\mathscr{L}_{\mathfrak{p}}) = -4$, we have $\mathscr{E}(\mathscr{K}_{\mathfrak{p}}) \cong \mathscr{E}(\mathscr{L}_{\mathfrak{p}}) \cong M_8(\mathbb{R}) \oplus M_4(\mathbb{H})^{(2)}$ and hence $\mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) \cong -4$, using Proposition 5.1. Suppose next that $\mathrm{sig}(\mathscr{L}_{\mathfrak{p}}) = 2$ or -14. Then, arguing as above using Proposition 5.1, we see that $\mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) = 2$ or -14. But in that case, $\mathrm{sig}(\mathscr{L}_{\mathfrak{p}}) = -14 \Leftrightarrow \mathfrak{p} \in S \Leftrightarrow \gamma_2 >_{\mathfrak{p}} 0$ (by (8.2)) $\Leftrightarrow \mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) = -14$ (since $\mu <_{\mathfrak{p}} 0$). The argument when $\mathrm{sig}(\mathscr{L}_{\mathfrak{p}}) = 4$ or -28 is the same as for 2 or -14. □

THEOREM 8.3 (*Completeness theorem for the construction*). *Let Φ be a number field and suppose $\mathscr{L}$ is a Lie algebra of type D_4 over Φ. Then, there exist $\mathscr{B}$, μ, γ as in* §1 *so that*

$$\mathscr{L} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma).$$

Moreover, μ and γ can be chosen with the additional properties that

$$\mu \text{ is totally negative and } \gamma = \mathrm{diag}(1, \gamma_2, 1), \quad \text{where } \gamma_2 \neq 0 \in \Phi.$$

Proof. By Theorem 6.9, $\mathscr{E}(\mathscr{L}) \cong M_4(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ over Φ and $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$. Then, by Proposition 6.1 $\mathscr{D} \cong (\nu, \mu/\mathscr{Z})$ for some generator ν of $\mathscr{Z}$ so that $n_{\mathscr{Z}}(\nu) \in \Phi^{\times 2}$ and some totally negative $\mu \in \Phi^\times$. By the $\mathscr{B}$-construction lemma, we may choose $\mathscr{B}$ so that $\mathscr{D}$ is isomorphic to the quaternion algebra $\mathscr{Q}$ determined by $\mathscr{B}$ and μ. Thus, $\mathscr{E}(\mathscr{L}) \cong M_4(\mathscr{Q})$ and the theorem follows from Lemma 8.1. □

If $\mathscr{B}$ is as in §1, we say that $\mathscr{B}$ *has a* 1-*dimensional summand* if $\mathscr{B}$ has a 1-dimensional simple ideal. We say that $\mathscr{B}$ is *split* if $\mathscr{B} \cong \Phi^{(4)}$.

If $\mathscr{X}$ is a Φ-form of $M_n(\tilde{\Phi})^{(m)}$ for some m, n, we say that $\mathscr{X}$ is a *full matrix algebra over its centre* if $\mathscr{X} \cong M_n(\mathscr{Z})$, where $\mathscr{Z}$ is the centre of $\mathscr{X}$. Clearly, $\mathscr{X}$ is a full matrix algebra over its centre if and only if each of the simple summands of $\mathscr{X}$ are full matrix algebras over their centres. We say that $\mathscr{X}$ is *split* if $\mathscr{X} \cong M_n(\Phi)^{(m)}$.

The following lemma follows immediately from Proposition 5.1:

LEMMA 8.4. *Suppose Φ is a number field, $\mathscr{B}$, μ, γ are as in* §1, *and μ is totally negative. Let $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$. If $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, then*

(8.5) $\mathscr{B}_{\mathfrak{p}}$ *has a* 1-*dimensional summand* $\Leftrightarrow \mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) \neq -4$
$\Leftrightarrow \mathscr{E}(\mathscr{K})_{\mathfrak{p}}$ *is a full matrix algebra over its centre*

Also,

(8.6) $\mathscr{B}_{\mathfrak{p}}$ *is split* $\Leftrightarrow \mathrm{sig}(\mathscr{K}_{\mathfrak{p}}) = 4$ *or* $-28 \Leftrightarrow \mathscr{E}(\mathscr{K})_{\mathfrak{p}}$ *is split*.

THEOREM 8.7 (*Isomorphism theorem for the construction*). *Let Φ be a number field and $\mathscr{B}$, μ, γ and $\mathscr{B}'$, μ', γ' are as in* §1. *Suppose further that μ, μ' are totally negative and $\gamma_2 = \mathrm{diag}(1, \gamma_2, 1)$, $\gamma' = \mathrm{diag}(1, \gamma_2', 1)$ where $\gamma_2, \gamma_2' \neq 0 \in \Phi$. Let $\mathscr{Q}$ (resp. $\mathscr{Q}'$) be the quaternion algebra determined by $\mathscr{B}$ and μ (resp. $\mathscr{B}'$ and μ'). Then, $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma) \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}', \mu'), \gamma')$ if and only if the following conditions both hold:*

(a) $\mathscr{Q} \cong \mathscr{Q}'$.
(b) *For each $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$ such that $\mathscr{B}_{\mathfrak{p}}$ has a* 1-*dimensional summand, we have $\gamma_2\gamma_2' >_{\mathfrak{p}} 0$.*

Proof. Observe that if we assume (a), then the real primes $\mathfrak{p}$ for which $\mathscr{B}_\mathfrak{p}$ has a 1-dimensional summand are the same as those for which $\mathscr{B}'_\mathfrak{p}$ has a 1-dimensional summand (by (8.5)).

Now put $\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ and $\mathscr{K}' = \mathscr{K}(\mathrm{CD}(\mathscr{B}', \mu'), \gamma')$. By Proposition 4.1, we may assume that (a) holds. Thus, if $\mathfrak{p} \in S_\mathbb{R}(\Phi)$ is such that $\mathscr{B}_\mathfrak{p}$ has no 1-dimensional summand, then $\mathscr{K}_\mathfrak{p} \cong \mathscr{K}'_\mathfrak{p}$ automatically (by (8.5)). Thus, by the D_4-isomorphism theorem, it suffices to show that for $\mathfrak{p} \in S_\mathbb{R}(\Phi)$ such that $\mathscr{B}_\mathfrak{p}$ has a 1-dimensional summand, we have

$$\mathscr{K}_\mathfrak{p} \cong \mathscr{K}'_\mathfrak{p} \Leftrightarrow \gamma_2\gamma'_2 >_\mathfrak{p} 0.$$

Since $\mathscr{B}'_\mathfrak{p}$ also has a 1-dimensional summand and $\mathscr{E}(\mathscr{K}_\mathfrak{p}) \cong \mathscr{E}(\mathscr{K}'_\mathfrak{p})$ and μ, μ' are negative at $\mathfrak{p}$, this follows from Proposition 5.1. □

9. Anisotropic D_4's over number fields. In this section, we identify the anisotropic D_4's over a number field Φ. The first lemma holds over any field Φ of characteristic zero.

LEMMA 9.1. *Suppose $\mathscr{B}$, μ, γ are as in* §1. *Let*

$$\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma).$$

Then,

$$\mathscr{K} \text{ is strongly isotropic} \Leftrightarrow \mathscr{K} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma_0).$$

Proof. "$\Leftarrow$" follows from Proposition 4.7. For "$\Rightarrow$", suppose $\mathscr{K}$ is strongly isotropic. By Proposition 4.7, $\mathscr{K} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}', \mu'), \gamma_0)$ for some $\mathscr{B}'$, μ'. But then by Proposition 4.1, we have $\mathscr{Q} \cong \mathscr{Q}'$, where $\mathscr{Q}$ (resp. $\mathscr{Q}'$) is the quaternion algebra determined by $\mathscr{B}$, μ (resp. $\mathscr{B}'$, μ'). Thus, by Proposition 4.7, $\mathscr{K}(\mathrm{CD}(\mathscr{B}', \mu'), \gamma_0) \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma_0)$ and so $\mathscr{K} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma_0)$. □

THEOREM 9.2. *Let Φ be a number field. Suppose*

$$\mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma),$$

where $\mathscr{B}$, μ, γ are as in §1, *μ is totally negative and $\gamma = \mathrm{diag}(1, \gamma_2, 1)$ with $\gamma_2 \in \Phi^\times$.*

(a) *$\mathscr{K}$ is strongly isotropic if and only if for each $\mathfrak{p} \in S_\mathbb{R}(\Phi)$ such that $\mathscr{B}_\mathfrak{p}$ has a 1-dimensional summand we have $\gamma_2 <_\mathfrak{p} 0$.*

(b) *If $\mathscr{K}$ is orthogonal, then $\mathscr{K}$ is isotropic if and only if for each $\mathfrak{p} \in S_\mathbb{R}(\Phi)$ so that $\mathscr{B}_\mathfrak{p}$ is split we have $\gamma_2 <_\mathfrak{p} 0$.*

(c) *If $\mathscr{K}$ is not orthogonal, then $\mathscr{K}$ is isotropic if and only if for each $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$ so that $\mathscr{B}_{\mathfrak{p}}$ has a 1-dimensional summand we have $\gamma_2 <_{\mathfrak{p}} 0$.*

Proof. (a) follows from Lemma 9.1 and the isomorphism theorem for the construction, and (c) follows from (a). For (b), suppose $\mathscr{K} \cong \mathfrak{o}(q)$ for some 8-dimensional nondegenerate quadratic form q. Now it is well known that $\mathfrak{o}(q)$ is isotropic if and only if q is isotropic (over any Φ). (See for example [**T1**, 2.4].) Thus, by the local global principle for isotropic quadratic forms [**Lam**, Corollary 3.5. p. 169], $\mathscr{K}$ is isotropic if and only if $\mathscr{K}_{\mathfrak{p}}$ is isotropic for all $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$. But if $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, $\mathscr{K}_{\mathfrak{p}}$ is isotropic if and only if $\mathscr{B}_{\mathfrak{p}}$ is not split or $\gamma_2 <_{\mathfrak{p}} 0$ (by Proposition 5.1). □

REMARK 9.3. The completeness theorem together with Theorem 9.2 (b) and (c) describes all anisotropic Lie algebras of type D_4 over a number field Φ. Given $\mathscr{B}$, μ, γ as in Theorem 9.2, one can use Corollary 4.5 to determine which part of Theorem 9.2 ((b) or (c)) to apply to test for anisotropicity.

As a consequence of Theorem 9.2, we obtain the following *local global principles*:

COROLLARY 9.4. *Suppose $\mathscr{L}$ is a Lie algebra of type D_4 over a number field Φ.*

(a) *$\mathscr{L}$ is strongly isotropic if and only if $\mathscr{L}_{\mathfrak{p}}$ is strongly isotropic for all (real) primes $\mathfrak{p}$ of Φ.*

(b) *If $\mathscr{L}$ is orthogonal or has type $D_{4\mathrm{I}}$ for $D_{4\mathrm{III}}$, then $\mathscr{L}$ is isotropic if and only if $\mathscr{L}_{\mathfrak{p}}$ is isotropic for all (real) primes $\mathfrak{p}$ of Φ.*

Proof. By the completeness theorem, we may assume that $\mathscr{L} = \mathscr{K} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$, with $\mathscr{B}$, μ, γ as in Theorem 9.2.

(a) If $\mathfrak{p} \in S(\Phi)$, then $\mathscr{L}_{\mathfrak{p}}$ is strongly isotropic if and only if $\mathfrak{p}$ is finite, $\mathfrak{p}$ is complex or $\mathfrak{p}$ is real and $\mathscr{L}_{\mathfrak{p}}$ has signature -4, 2 or 4 (see §5). Thus, (a) follows From Theorem 9.2 (a) and Proposition 5.1.

(b) If $\mathscr{L}$ is orthogonal, the claim follows from the argument in the proof of Theorem 9.2(b). Suppose $\mathscr{L}$ is not orthogonal and $\mathscr{L}$ had type $D_{4\mathrm{I}}$ or $D_{4\mathrm{III}}$. Then, $\mathscr{L}$ is isotropic iff $\mathscr{L}$ is strongly isotropic. Also, $\mathscr{Z}(\mathscr{L}) \cong \Phi \oplus \Phi \oplus \Phi$ or $\mathscr{Z}(\mathscr{L})$ is a $\mathbb{Z}/(3)$-cubic. Thus, if $\mathfrak{p} \in S_{\mathbb{R}}(\Phi)$, $\mathscr{Z}(\mathscr{L})_{\mathfrak{p}} \cong \Phi_{\mathfrak{p}} \oplus \Phi_{\mathfrak{p}} \oplus \Phi_{\mathfrak{p}}$. Hence, by Proposition 5.1, $\mathscr{L}_{\mathfrak{p}}$ is isotropic if and only if $\mathscr{L}_{\mathfrak{p}}$ is strongly isotropic. Thus, our claim follows from (a). □

REMARK 9.5. Kneser's local global principle for isotropic quaternion skew-hermitian forms [**Sch**, Theorem 4.1, p. 366] in the rank 4 case is closely related to Proposition 9.3 in the case when $\mathscr{L}$ has type $D_{4\mathrm{I}}$ or $D_{4\mathrm{II}}$.

REMARK 9.6. Part (b) of Corollary 9.4 is false for nonorthogonal $D_{4\mathrm{II}}$'s and for $D_{4\mathrm{VI}}$'s. Indeed, Example 11.8 will describe an anisotropic Lie algebra $\mathscr{L}$ of type $D_{4\mathrm{VI}}$ over the field $\mathbb{Q}$ of rational numbers so that $\mathscr{L}_{\mathbb{R}}$ is isotropic. An example of type $D_{4\mathrm{II}}$ is obtained by taking $\mathscr{L} = \mathscr{K}(\mathrm{CD}(\mathscr{B}, -3), I)$ over $\mathbb{Q}$, where $\mathscr{B} = \Phi[b_0]$ and b_0 has minimal polynomial $x^4 - 2$. (See also Remark 9.5.)

10. Jordan D_4's over number fields. Recall that a Lie algebra $\mathscr{L}$ of type D_4 over Φ is called a *Jordan* D_4 if $\mathscr{L} \cong \mathrm{Der}(\mathscr{J}/\mathscr{Z}) := \{D \in \mathrm{Der}\,\mathscr{J} : D\mathscr{Z} = \{0\}\}$ for some 27-dimensional exceptional central simple Jordan algebra $\mathscr{J}$ and some 3-dimensional separable associative subalgebra $\mathscr{Z}$. Allen has shown that

(10.1) $\mathscr{L}$ is Jordan $\Leftrightarrow \mathscr{E}(\mathscr{L})$ is a full matrix algebra over its centre

[**All1**, Theorem I]. As an application of our results and (10.1), we can give a simple description of the Jordan D_4's over a number field. We first need a lemma that holds over any field of characteristic 0.

LEMMA 10.2. *Suppose $\mathscr{B} = \Phi \oplus \mathscr{Z}$, where $\mathscr{Z}$ is a 3-dimensional separable associative commutative algebra over Φ, and $\mu \in \Phi^\times$. Then, the quaternion algebra $\mathscr{Q}$ determined by $\mathscr{B}$ and μ is isomorphic to $M_2(\mathscr{Z})$. Hence, for any γ as in §1, $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$ is a Jordan D_4 with Allen invariant isomorphic to $M_8(\mathscr{Z})$.*

Proof. We argue as in [**A3**, Corollary 6.6]. Let ζ be an invertible generator of $\mathscr{Z}$ of trace 0 and let $(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$ be its minimum polynomial over Φ, where $\lambda_1, \lambda_2, \lambda_3 \in \tilde{\Phi}$. Put $b_0 = (0, \zeta) \in \mathscr{B}$. Then, b_0 is a generator of $\mathscr{B}$ with minimum polynomial $f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)x$. So the polynomial $h(x)$ defined by (4.3) is $(x - \lambda_1^2)(x - \lambda_2^2)(x - \lambda_3^2)$. This is the minimum polynomial of ζ^2 over Φ and ζ^2 is therefore a generator of $\mathscr{Z}$. Thus, by Proposition 4.4, $\mathscr{Q} \cong (\zeta^2, \mu/\mathscr{Z}) \cong M_2(\mathscr{Z})$. □

REMARK 10.3. If $\mathscr{B} \cong \Phi \oplus \mathscr{Z}$ and μ are as in Lemma 10.2 and $\gamma_0 = \mathrm{diag}(1, -1, 1)$, then $\mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma_0)$ is the quasi-split (or Steinberg) D_4 with Allen invariant $M_8(\mathscr{Z})$. (See [**A2**, Proposition 9.1].)

THEOREM 10.4. *Suppose Φ is a number field. If $\mathscr{L}$ is a Jordan D_4 over Φ, then there exists a 3-dimensional separable commutative associative algebra $\mathscr{Z}$ and $\gamma_2 \neq 0 \in \Phi$ so that $\mathscr{L} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, -1), \gamma)$, where $\mathscr{B} = \Phi \oplus \mathscr{Z}$ and $\gamma = \mathrm{diag}(1, \gamma_2, 1)$. Moreover, if $\mathscr{B} = \Phi \oplus \mathscr{Z}$, $\mathscr{B}' = \Phi \oplus \mathscr{Z}'$, $\gamma = \mathrm{diag}(1, \gamma_2, 1)$ and $\gamma' = \mathrm{diag}(1, \gamma_2', 1)$, then*

$$\mathscr{K}(\mathrm{CD}(\mathscr{B}, -1), \gamma) \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}', -1), \gamma')$$
$$\Leftrightarrow \mathscr{Z} \cong \mathscr{Z}' \text{ and } \gamma_2\gamma_2' >_{\mathfrak{p}} 0 \quad \text{for all } \mathfrak{p} \in S_{\mathbb{R}}(\Phi).$$

Proof. By (10.1), $\mathscr{E}(\mathscr{L}) \cong M_8(\mathscr{Z})$, where $\mathscr{Z} = \mathscr{Z}(\mathscr{L})$. Let $\mathscr{B} = \Phi \oplus \mathscr{Z}$ and $\mu = -1$. By Lemma 10.2, $\mathscr{Q} \cong M_2(\mathscr{Z})$ and so $\mathscr{E}(\mathscr{L}) \cong M_4(\mathscr{Q})$. Thus, by Lemma 8.1, we have the first statement. The final statement follows from Lemma 10.2 and the isomorphism theorem for the construction. □

REMARK 10.5. Although we haven't checked this, a related description of the Jordan D_4's over a number field can likely also be obtained using the work of Allen in **[All1]** and the Albert-Jacobson classification of 27-dimensional exceptional central simple Jordan algebras over a number field **[A&J]**.

REMARK 10.6. Suppose $\mathscr{Z}$ is a 3-dimensional separable associative commutative algebra over a number field Φ. By the approximation theorem and Theorem 10.4, there are exactly 2^n Jordan D_4's (up to isomorphism) with Allen invariant isomorphic to $M_8(\mathscr{Z})$, where n is the number of real primes of Φ. By Theorem 9.2, if $\mathscr{Z}$ is a field or $\mathscr{Z} \cong \Phi\oplus\Phi\oplus\Phi$, then exactly one of these Jordan $\mathscr{D}_4$'s is isotropic (the quasi-split one with $\gamma = \gamma_0$). If $\mathscr{Z} \cong \Phi\oplus\Gamma$, where Γ/Φ is a quadratic extension, then exactly 2^{n-l} of these Jordan D_4's are isotropic, where l is the number of real primes $\mathfrak{p}$ so that $\Gamma_{\mathfrak{p}}$ is split. We will see a more general result of this type in the next section.

11. The classification problem for D_4's over a number field. *Suppose in this section that Φ is a number field*. We show how to construct the distinct (isomorphism classes) of D_4's over Φ with a specified Allen invariant $\mathscr{E}$. We begin by describing the construction.

Construction 11.1. Suppose $\mathscr{E} = M_4(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ over Φ so that $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$. (This is a necessary assumption by Theorem 6.12.) Choose a generator ν of $\mathscr{Z}$ and $\mu \in \Phi^\times$ so that

$$(11.2) \qquad \mathscr{D} \cong \left(\frac{\nu, \mu}{\mathscr{Z}}\right), \quad n_{\mathscr{Z}}(\nu) \in \Phi^{\times 2} \text{ and } \mu \text{ is totally negative}.$$

(See Proposition 6.1.) Let

$$h(x) = x^3 + \alpha_2 x^2 + \alpha_1 x - \eta^2$$

be the minimum polynomial of ν over Φ, where $\alpha_1, \alpha_2 \in \Phi$, $\eta \in \Phi^\times$ and $\eta^2 = n_{\mathscr{Z}}(\nu)$. Put

$$f(x) = x^4 + \frac{1}{2}\alpha_2 x^2 + \eta x + \frac{1}{16}(\alpha_2^2 - 4\alpha_1) \quad \text{and} \quad \mathscr{B} = \Phi[b_0],$$

where b_0 has minimum polynomial $f(x)$ over Φ. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the distinct real primes of Φ such that

(11.3) $\mathscr{D}_{\mathfrak{p}_i}$ is a full matrix algebra over its centre, $\quad i = 1, \dots, k$.

Choose $\gamma_2^{(1)}, \dots, \gamma_2^{(2^k)} \in \Phi^\times$ (by the approximation theorem) so that

(11.4) every sign configuration at the real primes $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ is achieved by some $\gamma_2^{(i)}$.

Put $\gamma^{(i)} := \operatorname{diag}(1, \gamma_2^{(i)}, 1)$ and

$$\mathscr{K}^{(i)} := \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma^{(i)}), \qquad i = 1, \dots, 2^k.$$

THEOREM 11.5. *Suppose Φ is a number field and $\mathscr{E} = M_4(\mathscr{D})$, where $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ over Φ so that $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$. Let k be the number of real primes $\mathfrak{p}$ so that $\mathscr{D}_{\mathfrak{p}}$ is a full matrix algebra over its centre. Then, the Lie algebras $\mathscr{K}^{(i)}$, $i = 1, \dots, 2^k$, described in Construction* 11.1 *are the distinct Lie algebras of type D_4 up to isomorphism whose Allen invariants are isomorphic to $\mathscr{E}$.*

Proof. By the $\mathscr{B}$-construction lemma $\mathscr{E}(\mathscr{K}^{(i)}) \cong \mathscr{E}$, $i = 1, \dots, 2^k$. Also $\mathscr{K}^{(i)}$ is not isomorphic to $\mathscr{K}^{(j)}$ for $i \neq j$, by the isomorphism theorem for the construction and (8.5). Suppose finally that $\mathscr{L}$ is a Lie algebra of type D_4 so that $\mathscr{E}(\mathscr{L}) \cong \mathscr{E}$. By the $\mathscr{B}$-construction lemma and Lemma 8.1, there exists $\gamma_2 \in \Phi^\times$ so that $\mathscr{L} \cong \mathscr{K}(\mathrm{CD}(\mathscr{B}, \mu), \gamma)$, where $\gamma = \operatorname{diag}(1, \gamma_2, 1)$. But $\gamma_2 \gamma_2^{(i)} >_{\mathfrak{p}_j} 0$ for $j = 1, \dots, k$ and some $i \in \{1, \dots, 2^k\}$, by (11.4). Thus, by the isomorphism theorem for the construction, (11.3) and (8.5), we have $\mathscr{L} \cong \mathscr{K}^{(i)}$. □

COROLLARY 11.6. *Assume the hypotheses of Theorem* 11.5.

(a) *If $\mathscr{D}$ has a simple summand isomorphic to $M_2(\Phi)$, then the Lie algebras $\mathscr{K}^{(i)}$, $i = 1, \dots, 2^k$, are orthogonal and exactly 2^{k-l} of* *these Lie algebras are isotropic, where l is the number of real primes $\mathfrak{p}$ so that $\mathscr{D}_\mathfrak{p}$ is split.*

(b) *If $\mathscr{D}$ has no simple summand isomorphic to $M_2(\Phi)$, then the Lie algebras $\mathscr{K}^{(i)}$, $i = 1, \dots, 2^k$, are not orthogonal and exactly one of these algebras is isotropic.*

Proof. The statements about orthogonality follow from Remark 3.20. We need to prove the statements about the number of isotropic $\mathscr{K}^{(i)}$'s. If $\mathscr{D}$ has a simple summand isomorphic to $M_2(\Phi)$, we may number the $\mathfrak{p}_j$'s so that $\mathscr{D}_{\mathfrak{p}_j}$ is split if and only if $j \le l$, in which case $\mathscr{K}^{(i)}$ is isotropic if and only if $\gamma_2^{(i)} <_{\mathfrak{p}_j} 0$ for $j = 1, \dots, l$ (by Theorem 9.2(b) and (8.6)). If $\mathscr{D}$ has no simple summand isomorphic to $M_2(\Phi)$, then $\mathscr{K}^{(i)}$ is isotropic if and only if $\gamma_2^{(i)} <_{\mathfrak{p}_j} 0$ for $j = 1, \dots, k$ (by Theorem 9.2(c) and (8.5)). □

REMARK 11.7. Theorem 11.5 reduces the classification problem for Lie algebras of type D_4 over a given number field Φ to two associative problems:

(1) Classifying all associative algebras $\mathscr{D}$ up to isomorphism so that $\mathscr{D}$ is a quaternion algebra over a 3-dimensional separable algebra $\mathscr{Z}$ over Φ and $c_{\mathscr{Z}/\Phi}(\mathscr{D}) \sim \Phi$.

(2) Given $\mathscr{D}$ as in (1), expressing $\mathscr{D}$ in the form (11.2)

The remaining parts of Construction 11.1 are the comparatively straightforward. In particular, choosing the real primes $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ so that (11.3) holds is equivalent to determining the real primes $\mathfrak{p}$ so that the polynomial $h(x)$ does not have a negative root in $\Phi_\mathfrak{p}$.

EXAMPLE 11.8. Suppose $\Phi = \mathbb{Q}$ and $\mathscr{D} = (\nu, -3/\Lambda)$, where $\Lambda = \Phi(\nu)$ and ν has minimum polynomial $h(x) = x^3 + 2x - 9$ over Φ. Λ is an S_3-cubic extension of Φ and, by Proposition 6.1, $c_{\Lambda/\Phi}(\mathscr{D}) \sim \Phi$. If we reduce mod 3, $h(x)$ factors as $x(x-1)(x+1)$. Thus, $h(x)$ has a root ν_0 in the 3-adic integers so that the image of ν_0 in $\mathbb{Z}/(3)$ under the residue class map is a nonsquare. Hence, $(\nu_0, -3/\Phi_{(3)})$ is a division algebra [**Lam**, Theorem 2.2, p. 149] and so $\mathscr{D}_{(3)}$ is not a full matrix algebra over its centre. Therefore, $\mathscr{D}$ is a division algebra. Finally, $h(x)$ has one positive real root and two conjugate nonreal roots. Thus, $\mathscr{D}_\mathbb{R} \cong M_2(\mathbb{R}) \oplus M_2(\mathbb{C})$. We now carry out Construction 11.1 starting with $\mathscr{D}$. Let $\mathscr{B} = \Phi[b_0]$, where b_0 has minimum polynomial $f(x) = x^4 + 3x - \frac{1}{2}$. Let $\gamma^{(1)} = \operatorname{diag}(1, -1, 1)$ and $\gamma^{(2)} = \operatorname{diag}(1, 1, 1)$. Then, $\mathscr{K}^{(i)} := \mathscr{K}(\operatorname{CD}(\mathscr{B}, -3), \gamma^{(i)})$, $i = 1, 2$, are the Lie algebras of type D_4 with Allen invariant isomorphic to

$M_4(\mathscr{D})$. These are non-Jordan Lie algebras of type $D_{4\mathrm{VI}}$. $\mathscr{K}^{(1)}$ is isotropic and $\mathscr{K}^{(2)}$ is anisotropic.

EXAMPLE 11.9. Suppose $\Phi = \mathbb{Q}$ and $\mathscr{D} = (\nu, -1/\Lambda)$, where $\Lambda = \Phi(\nu)$ and ν has minimum polynomial $h(x) = x^3 - 3x - 1$. Then, Λ is a $\mathbb{Z}/(3)$-cubic and $c_{\Lambda/\Phi}(\mathscr{D}) \sim \Phi$. $h(x)$ has 3 real roots exactly one of which is positive. Thus, $\mathscr{D}_{\mathbb{R}} \cong \mathbb{H} \oplus \mathbb{H} \oplus M_2(\mathbb{R})$. Hence, $\mathscr{D}$ is a division algebra. Applying Construction 11.1, we let $\mathscr{B} = \Phi[b_0]$, where b_0 has minimum polynomial $f(x) = x^4 + x - \frac{3}{4}$, and $\gamma^{(1)} = \mathrm{diag}(1, -1, 1)$. Then, $\mathscr{K}^{(1)} := \mathscr{K}(\mathrm{CD}(\mathscr{B}, -1), \gamma^{(1)})$ is the unique Lie algebra of type D_4 with Allen invariant isomorphic to $\mathscr{D}$. $\mathscr{K}^{(1)}$ is an isotropic non-Jordan Lie algebra of type $D_{4\mathrm{III}}$.

12. Appendix: Proof of the injectivity theorem. In this appendix, we give the proof, postponed from §7, of the injectivity theorem (Theorem 7.6). We assume throughout the section that A is an almost simple adjoint algebraic group of type D_4 over a number field Φ. Let B be the simply connected covering group defined over Φ of A [**T1**, §2.6] and let C be the centre of A.

We wish to prove that the map

$$H^1(\Phi, A) \longrightarrow \prod_{\mathfrak{p} \in S(\Phi)} H^1(\Phi_{\mathfrak{p}}, A) \text{ is injective.} \tag{12.1}$$

Now, by a theorem of Kneser, we have $H^1(\Phi_{\mathfrak{p}}, B) = \{1\}$ for all finite primes $\mathfrak{p}$ of Φ (see [**Kn1**, Satz 1] or [**B&T**, Proposition 7]). Also, by a theorem of Harder [**Ha**], the map $H^1(\Phi, B) \to \prod_{\mathfrak{p} \in S(\Phi)} H^1(\Phi_{\mathfrak{p}}, B)$ is injective. Using these two facts, a standard argument involving a twist of the Galois action and a diagram chase (see for example [**Kn2**, §5.1] or [**F1**, §2]) shows that, for the proof of (12.1), it suffices to show that the map

$$H^2(\Phi, C) \longrightarrow \prod_{\mathfrak{p} \in S(\Phi)} H^2(\Phi_{\mathfrak{p}}, C) \text{ is injective} \tag{12.2}$$

and that the map

$$H^1(\Phi, C) \longrightarrow \prod_{\mathfrak{p} \in S_{\mathbb{R}}(\Phi)} H^1(\Phi_{\mathfrak{p}}, C) \text{ is surjective.} \tag{12.3}$$

Now [**T1**, §1.5], $C(\widetilde{\Phi})$ is a Klein 4-group. Thus,

$$C(\widetilde{\Phi}) = \{1, c_1, c_2, c_3\},$$

with $c_i^2 = 1$ and $c_1 c_2 = c_3$. Then, there exists a homomorphism

$s \to q_s$ of G into S_3 so that the action of G on $C(\widetilde{\Phi})$ is given by

$$sc_i = c_{q_s i} \quad \text{for } s \in G, \ i = 1, 2, 3.$$

We put

$$H := \ker(s \to q_s) \quad \text{and} \quad \Gamma := \mathrm{Fix}(H).$$

As in the proof of the D_4-isomorphism theorem, it will be convenient to regard $\widetilde{\Phi}$, $\Phi_\mathfrak{p}$ and $\Phi_\mathfrak{p}^\sim$ for $\mathfrak{p} \in S(\Phi)$ as subfields of some large algebraically closed extension Ω/Φ. If $\mathfrak{p} \in S(\Phi)$, we put $G_\mathfrak{p} := \mathrm{Gal}(\Phi_\mathfrak{p}^\sim/\Phi_\mathfrak{p})$ and identify $G_\mathfrak{p}$ as a subgroup of G (by the restriction map). Also since $C(\Phi_\mathfrak{p}^\sim)$ has order 4, we may identify $C(\Phi_\mathfrak{p}^\sim) = C(\widetilde{\Phi}) = \{1, c_1, c_2, c_3\}$.

We now prove (12.2) using work of K. Hoechsmann **[Ho]**:

LEMMA 12.4. *The map* $H^2(\Phi, C) \to \prod_{\mathfrak{p}\in S(\Phi)} H^2(\Phi_\mathfrak{p}, C)$ *is injective.*

Proof. The character group $\mathrm{Hom}_\mathbb{Z}(C(\widetilde{\Phi}), \widetilde{\Phi}^\times)$ of $C(\widetilde{\Phi})$ is a G-module with fixing group H. Thus, if $\mathrm{Fix}(H)/\Phi$ is a cyclic Galois extension, the required injectivity is a consequence of **[Ho**, 6.1 and 6.3]. So we may assume that $[\Gamma : \Phi] = 6$. We now argue as in **[F2**, p. 205]. Let Λ/Φ be one of the degree 3 subextensions of Γ/Φ. Then, we have the commutative diagram:

$$\begin{array}{ccc} H^2(\Phi, C) & \longrightarrow & H^2(\Lambda, C) \\ \downarrow & & \downarrow \\ \prod_{\mathfrak{p}\in S(\Phi)} H^2(\Phi_\mathfrak{p}, C) & \longrightarrow & \prod_{\mathfrak{P}\in S(\Lambda)} H^2(\Lambda_\mathfrak{P}, C). \end{array}$$

The top row is injective since $[\Lambda : \Phi]$ is relatively prime to the order of $C(\widetilde{\Phi})$ **[Ser1**, I-11]. The vertical map on the right-hand side is injective by the case considered previously. Thus, the vertical map on the left-hand side is injective as required. □

So it remains to prove (12.3). We let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the distinct real primes of Φ labelled so that the primes of Γ lying above $\mathfrak{p}_i$ are all real if $1 \le i \le m$ and all complex if $m+1 \le i \le n$. (This is possible since Γ/Φ is Galois.)

LEMMA 12.5. *If* $1 \le i \le m$, *then* $G_{\mathfrak{p}_i}$ *acts trivially on* $C(\widetilde{\Phi})$. *If* $m+1 \le i \le n$, *then* $G_{\mathfrak{p}_i}$ *acts nontrivially on* $C(\widetilde{\Phi})$ *and* $H^1(\Phi_{\mathfrak{p}_i}, C) = \{1\}$.

Proof. First if $1 \le i \le n$, then $G_{\mathfrak{p}_i}$ acts trivially on $C(\widetilde{\Phi}) \Leftrightarrow G_{\mathfrak{p}_i} \subseteq H \Leftrightarrow \Gamma \subseteq \Phi_{\mathfrak{p}_i} \Leftrightarrow 1 \le i \le m$. So if $m+1 \le i \le n$, $G_{\mathfrak{p}_i}$ is a cyclic group of order 2 acting nontrivially on the Klein 4-group $C(\widetilde{\Phi})$, in which case an easy calculation shows that $H^1(G_{\mathfrak{p}_i}, C(\widetilde{\Phi})) = \{1\}$. □

LEMMA 12.6. *The map* $H^1(\Phi, C) \to \prod_{\mathfrak{p}\in S_{\mathbb{R}}(\Phi)} H^1(\Phi_{\mathfrak{p}}, C)$ *is surjective.*

Proof. In this proof, we identify $C(\widetilde{\Phi})$ with the multiplicative group $\{(\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_i = \pm 1,\ \varepsilon_1\varepsilon_2\varepsilon_3 = 1\}$ by means of the identification

$$c_1 = (1, -1, -1), \quad c_2 = (-1, 1, -1), \quad c_3 = (-1, -1, 1).$$

In that case the action of G on $C(\widetilde{\Phi})$ is given by

$$s(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (\varepsilon_{q_s^{-1}1}, \varepsilon_{q_s^{-1}2}, \varepsilon_{q_s^{-1}3}) \quad \text{for } s \in G. \tag{12.7}$$

Suppose next that $1 \le i \le m$. Then, $\Gamma \subseteq \Phi_{\mathfrak{p}_i}$ and we let $\mathfrak{P}_i$ be the real prime of Γ determined by the restriction of the absolute value on $\Phi_{\mathfrak{p}_i}$ to Γ. Thus, the completions $\Gamma_{\mathfrak{P}_i}$ and $\Phi_{\mathfrak{p}_i}$ are equal. Also, the distinct real primes of Γ lying above $\mathfrak{p}_i$ are the primes $s\mathfrak{P}_i$, $s \in \mathrm{Gal}(\Gamma/\Phi)$. Finally, $G_{\mathfrak{p}_i}$ acts trivially on $C(\widetilde{\Phi})$ and so

$$H^1(\Phi_{\mathfrak{p}_i}, C) = \{1, \chi_{i1}, \chi_{i2}, \chi_{i3}\},$$

where $\chi_{ij} : G_{\mathfrak{p}_i} \to C(\widetilde{\Phi})$ is the group homomorphism so that $\chi_{ij}(s_i) = c_j$, $j = 1, 2, 3$, and s_i denotes the generator (of order 2) of $G_{\mathfrak{p}_i}$.

Now, by Lemma 12.5, we must show that the map

$$H^1(\Phi, C) \to \prod_{i=1}^{m} H^1(\Phi_{\mathfrak{p}_i}, C) \tag{12.8}$$

is surjective. Thus, with the above notation, it suffices to show that $(1, \dots, \chi_{i_0 j_0}, , \dots, 1)$ is in the image of this map for $1 \le i_0 \le m$, $1 \le j_0 \le 3$. So we fix $1 \le i_0 \le m$ and $1 \le j_0 \le 3$. Suppose for the moment that we have chosen $\alpha_1, \alpha_2, \alpha_3 \in \Gamma^\times$ so that:

$$s\alpha_j = \alpha_{q_s j} \quad \text{for } s \in G,\ j = 1, 2, 3, \tag{12.9}$$

$$\alpha_1\alpha_2\alpha_3 \in \Phi^{\times 2}, \tag{12.10}$$

and

$$\text{if } 1 \le i \le m \text{ and } 1 \le j \le 3 \text{ then } \alpha_j <_{\mathfrak{P}_i} 0 \Leftrightarrow i = i_0 \text{ and } j \ne j_0. \tag{12.11}$$

We then choose $\beta_1, \beta_2, \beta_3 \in \widetilde{\Phi}^\times$ so that $\beta_j^2 = \alpha_j$, $j = 1, 2, 3$. For $s \in G$, we define

$$\eta_s = ((s\beta_{q_s^{-1}1})\beta_1^{-1}, (s\beta_{q_s^{-1}2})\beta_2^{-1}, (s\beta_{q_s^{-1}3})\beta_3^{-1}). \tag{12.12}$$

From (12.9) and (12.10) it follows that $\eta_s \in C(\widetilde{\Phi})$ for $s \in G$. Also, using (12.7), one easily checks that $(\eta_s)_{s\in G}$ is a continuous 1-cocycle of G in $C(\widetilde{\Phi})$. Denote the corresponding element of $H^1(\Phi, C)$ by η. Observe that if $1 \le i \le m$, we have $q_{s_i} = (1)$ and so $\eta_{s_i} = ((s_i\beta_1)\beta_1^{-1}, (s_i\beta_2)\beta_2^{-1}, (s_i\beta_2)\beta_2^{-1})$. But for $1 \le j \le 3$, $(s_i\beta_j)\beta_j^{-1} = 1 \Leftrightarrow \beta_j \in \Phi_{\mathfrak{p}_i}^\times \Leftrightarrow \alpha_j \in \Phi_{\mathfrak{p}_i}^{\times 2} \Leftrightarrow \alpha_j \in \Gamma_{\mathfrak{P}_i}^{\times 2} \Leftrightarrow \alpha_j >_{\mathfrak{P}_i} 0 \Leftrightarrow i \ne i_0$ or $j = j_0$ (by (12.11)). Thus, under the map (12.8), η maps to $(1, \dots, \chi_{i_0 j_0}, \dots, 1)$ as required.

So it remains to show that given $1 \le i_0 \le m$ and $1 \le j_0 \le 3$, we may choose $\alpha_1, \alpha_2, \alpha_3 \in \Gamma^\times$ satisfying (12.9)–(12.11). For convenience, we may assume $i_0 = 1$, $j_0 = 1$. We consider cases for $[\Gamma : \Phi]$. Suppose first that $[\Gamma : \Phi] = 1$. Then, $q_s = (1)$ for all $s \in G$. Choose $\alpha_2 \in \Phi^\times$ so that $\alpha_2 <_{\mathfrak{P}_1} 0$ and $\alpha_2 >_{\mathfrak{P}_i} 0$ for $i = 2, \dots, m$. Put $\alpha_1 = 1$ and $\alpha_3 = \alpha_2$. Then, (12.9)–(12.11) hold. Suppose next that $[\Gamma : \Phi] = 2$. Choose $r \in G$ so that $q_r \ne (1)$. Then, $\mathrm{Gal}(\Gamma/\Phi) = \langle r|_\Gamma\rangle$ and so we may choose $\alpha \in \Gamma^\times$ so that $\alpha >_{\mathfrak{P}_1} 0$, $r\alpha <_{\mathfrak{P}_1} 0$, $\alpha >_{\mathfrak{P}_i} 0$, and $r\alpha >_{\mathfrak{P}_i} 0$ for $i = 2, \dots, m$. If $q_r = (12)$, we put $\alpha_1 = \alpha$, $\alpha_2 = r\alpha$ and $\alpha_3 = \alpha(r\alpha)$ in which case (12.9)–(12.11) hold. Similarly, if $q_r = (13)$, we put $\alpha_1 = \alpha$, $\alpha_2 = \alpha(r\alpha)$ and $\alpha_3 = r\alpha$. Finally, if $q_r = (23)$, we choose $\beta \in \Gamma^\times$ so that $\beta <_{\mathfrak{P}_1} 0$, $r\beta <_{\mathfrak{P}_1} 0$, $\beta >_{\mathfrak{P}_i} 0$ and $r\beta >_{\mathfrak{P}_i} 0$ for $i = 2, \dots, m$, and put $\alpha_1 = \beta(r\beta)$, $\alpha_2 = \beta$, $\alpha_3 = r\beta$. Suppose next that $[\Gamma : \Phi] = 3$. Choose $s \in G$ so that $q_s = (123)$. Then, $\mathrm{Gal}(\Gamma/\Phi) = \langle s|_\Gamma\rangle$ and so we may choose $\alpha \in \Gamma^\times$ so that $\alpha <_{\mathfrak{P}_1} 0$, $s\alpha <_{\mathfrak{P}_1} 0$, $s^2\alpha >_{\mathfrak{P}_1} 0$, and $s^j\alpha >_{\mathfrak{P}_i} 0$ for $i = 1, 2, \dots, m$, $j = 0, 1, 2$. We put $\alpha_1 = \alpha(s\alpha)$, $\alpha_2 = (s\alpha)(s^2\alpha)$, and $\alpha_3 = (s^2\alpha)\alpha$ and again we have (12.9)–(12.11). Suppose finally that $[\Gamma : \Phi] = 6$. Choose $r, s \in G$ so that $q_r = (13)$ and $q_s = (123)$. Then, $\mathrm{Gal}(\Gamma/\Phi)$ consists of the restrictions of 1, s, s^2, r, sr and s^2r to Γ. This time, we choose $\alpha \in \Gamma^\times$ so that $s^2\alpha <_{\mathfrak{P}_1} 0$, $t\alpha >_{\mathfrak{P}_1} 0$ for $t = 1, s, r, sr, s^2r$, and $t\alpha >_{\mathfrak{P}_i} 0$ for $i = 2, \dots, m$ and all $t \in G$. Then, we put $\alpha_1 = \alpha(s\alpha)(r\alpha)(sr\alpha)$, $\alpha_2 = (s\alpha)(s^2\alpha)(sr\alpha)(s^2r\alpha)$ and $\alpha_3 = (s^2\alpha)\alpha(s^2r\alpha)(r\alpha)$, and again (112.9)–(12.11) hold. □

By the remarks at the beginning of the section, Lemmas 12.4 and 12.6 complete the proof of the injectivity theorem.

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