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INTEGRAL SPINOR NORMS IN DYADIC LOCAL FIELDS I

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The spinor norms of integral rotations on some quadratic forms over an arbitrary dyadic local field are determined. As an application, the results obtained in Bon Durant's paper are improved.

1. Introduction. Because of the absence of a local-global principle in the equivalence of integral quadratic forms over a global field, Eichler and Kneser developed the theory of spinor genera, which strongly depend on spinor norms of local integral rotations. For nondyadic cases the spinor norms are well understood (see [**6**]), but they become difficult to be determined when 2 is no longer a unit. Hsia [**5**], the author [**10**] computed the spinor norms of local integral rotations on modular quadratic forms completely. Earnest and Hsia [**3**] determined the spinor norms of integral rotations on arbitrary quadratic forms over a dyadic local field in which 2 is prime, and Bon Durant [**1**] recently considered the spinor norms of integral rotations over a local field which is a quadratic ramified extension field of Q_2. In the present article we first determine the spinor norms of all binary quadratic forms over an arbitrary dyadic local field, and then use these results to obtain the spinor norms of some quadratic forms over an arbitrary dyadic local field which Jordan components are one dimension. As an application, we consider the spinor norms of quadratic forms over the dyadic local fields which ramification index of 2 is 2 and improve the sufficient condition for the class number of an indefinite quadratic form over the ring of integers of a number field to be a divisor of the class number of the field.

Notation and terminology used here is that of [**8**]. In particular, F denotes a dyadic local field, ϑ the ring of integers in F, p the maximal ideal of ϑ, U the group of units in ϑ, $e = \operatorname{ord} 2$ the ramification index of 2 in F, π a fixed prime element in F, $D(\, , \,)$ the quadratic defect function, Δ a fixed unit of quadratic defect 4ϑ, V a regular quadratic space over F associated symmetric bilinear form $B(x, y)$, L a lattice on V, $O^+(V)$ the group of rotations on V, $O^+(L)$ the corresponding subgroup of units of L, and $\theta(\, , \,)$ the spinor norm function. We use the symbol $\langle a, b, c, \dots\rangle$ for lattices, and $[a, b, c, \dots]$ for spaces.

2. Binary cases. In this section we determine the spinor norms of local integral rotations on binary lattices completely. All the results are expressed in convenient closed forms. Some cases have already been treated in [**4**], but we may repeat them for completeness.

Let L be a binary lattice. If L is modular then $\theta(O^+(L))$ has been determined in [**5**] and [**10**]. So we only consider the nonmodular cases. Since the spinor norm is not affected by scaling, we can put $L = \vartheta x \perp \vartheta y$ with $Q(x) = 1$ and $Q(y) = \varepsilon\pi^r$ where $r \geq 1$ and $\varepsilon \in U$.

PROPOSITION 2.1. *If $r > 4e$, then $\theta(O^+(L)) = \dot{F}^2 \cup \varepsilon\pi^r\dot{F}^2$.*

Proof. Take a symmetry τ_z in $O(L)$ where $z = ax + by$ with $a, b \in \vartheta$ and one of them is in U. By [**4**, Prop. 3.2], we get ord $a \leq e$; or ord $a \geq r-e$, ord $b = 0$, so $Q(x)$ is in $\dot{F}^2$ or $\varepsilon\pi^r\dot{F}^2$ respectively by the Local Square Theorem, and $\theta(O^+(L)) = \dot{F}^2 \cup \varepsilon\pi^r\dot{F}^2$ by [**7**, 1.1].

PROPOSITION 2.2. *Suppose $2e < r \leq 4e$.*

(i) *When r is odd, $\theta(O^+(L)) = ((1+p^{r-2e})\dot{F}^2 \cup \varepsilon\pi(1+p^{r-2e})\dot{F}^2) \cap Q(\dot{F}L)$.*

(ii) *When r is even, $D(\varepsilon) = p^d$ with $1 \leq d \leq 2e-r/2$, $\theta(O^+(L)) = ((1+p^{r-2e+d})\dot{F}^2 \cup \varepsilon(1+p^{r-2e+d})\dot{F}^2) \cap Q(\dot{F}L)$.*

(iii) *When r is even, $D(\varepsilon) = p^d$ with $d > 2e-r/2$, or 0; $\theta(O^+(L)) = (1+p^{r/2})\dot{F}^2 \cup \varepsilon(1+p^{r/2})\dot{F}^2$.*

Proof. Take a symmetry τ_z in $O(L)$ where $z = ax + by$ with $a, b \in \vartheta$ and one of them is in U. By [**4**, Prop. 3.2], we obtain ord $a = 0$, ord $b \geq 0$; or $0 <$ ord $a \leq e$, ord $b = 0$; or ord $a \geq r-e$, ord $b = 0$.

Case (i). If ord $a \leq e$, then

$$\operatorname{ord}(a^{-2}b^2\varepsilon\pi^r) \geq r - 2\operatorname{ord} a \geq r - 2e.$$

So

$$Q(z) = a^2(1 + a^{-2}b^2\varepsilon\pi^r)$$

is in $(1+p^{r-2e})\dot{F}^2 \cap Q(\dot{F}L)$.

If $\operatorname{ord} a \geq r-e$, $\operatorname{ord} b = 0$; then

$$\operatorname{ord}(\varepsilon^{-1}\pi^{-r}(b^{-1}a)^2) = 2\operatorname{ord} a - r \geq r-2e.$$

So

$$Q(z) = \varepsilon\pi^r b^2(1+\varepsilon^{-1}\pi^{-r}(b^{-1}a)^2)$$

is in $\varepsilon\pi(1+\mathfrak{p}^{r-2e})\dot{F}^2 \cap Q(\dot{F}L)$. Therefore

$$\theta(O^+(L)) \subseteq ((1+\mathfrak{p}^{r-2e})\dot{F}^2 \cup \varepsilon\pi(1+\mathfrak{p}^{r-2e})\dot{F}^2) \cap Q(\dot{F}L)$$

by [**7**, 1.1].

Conversely, take an element h in $(1+\mathfrak{p}^{r-2e})\dot{F}^2 \cap Q(\dot{F}L)$, so there exists z in FL such that $Q(z) = h$. Without loss of generality, we assume $z = ax+by$ where $a, b \in \vartheta$ and one of them is a unit.

If $e < \operatorname{ord} a < r/2$, then $\operatorname{ord} b = 0$. Put

$$k = \operatorname{ord}(\varepsilon\pi^r(ba^{-1})^2) = r - 2\operatorname{ord} a < r-2e \leq 2e.$$

Note that k is a positive odd integer. Since h is in $(1+\mathfrak{p}^{r-2e})\dot{F}^2$, we obtain

$$(1+c\pi^{r-2e})f^2 = h = a^2+\varepsilon\pi^r b^2 = a^2(1+(a^{-1}b)^2\varepsilon\pi^r)$$

where $f \in \dot{F}^2$, $c \in \vartheta$. Let $\eta = af^{-1}$, so η is in U by the above equation, and

$$\eta^2(1+\varepsilon\pi^r(ba^{-1})^2) = (1+c\pi^{r-2e}).$$

But

$$D(\eta^2(1+\varepsilon\pi^r(ba^{-1})^2) = \mathfrak{p}^k \supset \mathfrak{p}^{r-2e} \supseteq D(1+c\pi^{r-2e})$$

by [**8**, 63:5], a contradiction.

If $r-e > \operatorname{ord} a > r/2$, then $\operatorname{ord} b = 0$ again,

$$\operatorname{ord} h = \operatorname{ord}(a^2+\varepsilon\pi^r b^2) = r$$

is odd. But h is in $(1+\mathfrak{p}^{r-2e})\dot{F}^2$, so that $\operatorname{ord} h$ is even. This is a contradiction. Now there remains $\operatorname{ord} a = 0$; or $0 < \operatorname{ord} a \leq e$, $\operatorname{ord} b = 0$; or $\operatorname{ord} a \geq r-e$, $\operatorname{ord} b = 0$, so τ_z is in $O(L)$ by [**4**, Prop. 3.2]. Therefore

$$\theta(O^+(L)) = ((1+\mathfrak{p}^{r-2e}) \cup \varepsilon\pi(1+\mathfrak{p}^{r-2e})\dot{F}^2) \cap Q(\dot{F}L).$$

Case (ii). Since $D(\varepsilon) = \mathfrak{p}^d$, we can assume $\varepsilon = 1+\sigma\pi^d$ for convenience, where $\sigma \in U$ and d is odd. Note $1 \leq d \leq 2e - r/2 < 2e$.

If $\operatorname{ord} a = 0$, then

$$\operatorname{ord}(\varepsilon\pi^r(a^{-1}b)^2) = r + 2\operatorname{ord} b \geq r > r-2e+d.$$

So

$$Q(z) = a^2(1 + \varepsilon\pi^r(a^{-1}b)^2)$$

is in $(1+p^{r-2e})\dot{F}^2 \cap Q(\dot{F}L)$. If $0 < \operatorname{ord} a \leq e < r/2$, $\operatorname{ord} b = 0$ then

$$\operatorname{ord}(\sigma\pi^{d+r}(a^{-1}b)^2) = r + d - 2\operatorname{ord} a \geq r + d - 2e$$

and

$$\operatorname{ord}(-2\pi^{r/2}(a^{-1}b)) = e + r/2 - \operatorname{ord} a \geq r/2 \geq r + d - 2e.$$

So

$$\begin{aligned} Q(z) &= a^2(1 + \varepsilon\pi^r(a^{-1}b)^2) = a^2(1 + (1 + \sigma\pi^d)\pi^r(a^{-1}b)^2) \\ &= a^2(1 + \pi^r(a^{-1}b)^2 + \sigma\pi^{r+d}(a^{-1}b)^2) \\ &= a^2((1 + \pi^{r/2})^2 - 2\pi^{r/2}(a^{-1}b) + \sigma\pi^{d+r}(a^{-1}b)^2) \end{aligned}$$

is in $(1 + p^{r-2e+d})\dot{F}^2 \cap Q(\dot{F}L)$.

If $\operatorname{ord} a \geq r - e$, $\operatorname{ord} b = 0$; then

$$\operatorname{ord}(-2(\varepsilon^{-1}b^{-1})a\pi^{-r/2}) = e + \operatorname{ord} a - r/2 \geq r - 2e + d$$

and

$$\operatorname{ord}(\sigma(\varepsilon^{-1}b^{-1})^2a^2\pi^{-r+d}) = 2 \operatorname{ord} a - r + d \geq r - 2e + d.$$

So

$$\begin{aligned} Q(z) &= \varepsilon\pi^r b^2(1 + \varepsilon(\varepsilon^{-1}b^{-1})^2a^2\pi^{-r}) \\ &= \varepsilon\pi^r b^2(1 + (1 + \sigma\pi^d)(\varepsilon^{-1}b^{-1})^2a^2\pi^{-r}) \\ &= \varepsilon\pi^r b^2(1 + (\varepsilon^{-1}b^{-1})^2a^2\pi^{-r} + \sigma(\varepsilon^{-1}b^{-1})^2a^2\pi^{-r+d}) \\ &= \varepsilon\pi^r b^2((1 + \varepsilon^{-1}b^{-1}a\pi^{-r/2})^2 - 2\varepsilon^{-1}b^{-1}a\pi^{-r/2} \\ &\qquad + \sigma(\varepsilon^{-1}b^{-1})^2a^2\pi^{-r+d}) \end{aligned}$$

is in $\varepsilon(1 + p^{r-2e+d})\dot{F}^2 \cap Q(\dot{F}L)$. Therefore

$$\theta(O^+(L)) \subseteq ((1 + p^{r-2e+d})\dot{F}^2 \cup \varepsilon(1 + p^{r-2e+d})\dot{F}^2) \cap Q(\dot{F}L).$$

Conversely, take an element h in $(1 + p^{r-2e+d})\dot{F}^2 \cap Q(\dot{F}L)$, so there exists z in FL such that $Q(z) = h$. Without loss of generality, we assume $z = ax + by$ where $a, b \in \vartheta$ and one of them is a unit.

If $e < \operatorname{ord} a < r/2$, then $\operatorname{ord} b = 0$. Put

$$\begin{aligned} k &= \operatorname{ord}(\sigma(a^{-1}b)^2\pi^{r+d}) = r + d - 2\operatorname{ord} a, \\ &= (e + r/2 - \operatorname{ord} a) + (r/2 + d - e - \operatorname{ord} a) \\ &< \min\{(e + r/2 - \operatorname{ord} a) + (r/2 + d - 2e), r + d - 2e\} \\ &\leq \min\{(e + r/2 - \operatorname{ord} a), r + d - 2e\} \\ &= \min\{\operatorname{ord}(-2a^{-1}b\pi^{r/2}), r + d - 2e\} \end{aligned}$$

and k is a positive odd integer. Since h is in $(1+\mathfrak{p}^{r-2e+d})\dot{F}^2$, we obtain

$$f^2(1+c\pi^{r-2e+d}) = h = (a^2+\varepsilon\pi^r b^2) = a^2(1+(a^{-1}b)^2\varepsilon\pi^r)$$

where $f \in \dot{F}$, $c \in \vartheta$. Let $\eta = af^{-1}$, so η is in U by the above equation, and

$$\begin{aligned} 1+c\pi^{r-2e+d} &= \eta^2(1+\varepsilon\pi^r(a^{-1}b)^2) \\ &= \eta^2(1+(1+\sigma\pi^d)\pi^r(a^{-1}b)^2) \\ &= \eta^2(1+\pi^r(a^{-1}b)^2+\sigma\pi^{r+d}(a^{-1}b)^2) \\ &= \eta^2((1+(a^{-1}b)\pi^{r/2})^2-2(a^{-1}b)\pi^{r/2}+\sigma\pi^{r+d}(a^{-1}b)^2). \end{aligned}$$

But

$$\begin{aligned} &D(\eta^2((1+a^{-1}b\pi^{r/2})^2-2(a^{-1}b)\pi^{r/2}+\sigma\pi^{r+d}(a^{-1}b)^2)) \\ &\quad = \mathfrak{p}^k \supset \mathfrak{p}^{r+d-2e} \supseteq D(1+c\pi^{r-2e+d}) \end{aligned}$$

by [**8**, 63:5], a contradiction.

If $\operatorname{ord} a = r/2$, then $\operatorname{ord} b = 0$. Let $\xi = a^{-1}\pi^{r/2}b$, so ξ is in U. Suppose

$$\begin{aligned} h &= a^2(1+\varepsilon\pi^r(a^{-1}b)^2) = a^2(1+\varepsilon\xi^2) = a^2(1+(1+\sigma\pi^d)\xi^2) \\ &= a^2(1+\xi^2+\sigma\xi^2\pi^d) = a^2((1+\xi)^2-2\xi+\sigma\xi^2\pi^d) \end{aligned}$$

is in $(1+\mathfrak{p}^{r-2e+d})\dot{F}^2$. Note $d \leq 2e-r/2 < e$, and d is odd, so $2\operatorname{ord}(1+\xi) = \operatorname{ord}(1+\xi)^2 < d < e$.

Writing $1+\xi = \delta\pi^s$ with $\delta \in U$, $s \geq 0$, we obtain

$$\eta^2(1-\delta^{-2}\xi 2\pi^{-2s}+\sigma\delta^{-2}\xi^2\pi^{d-2s}) = (1+c\pi^{r-2e+d})$$

where $\eta \in U$, $c \in \vartheta$. But

$$\begin{aligned} &D(\eta^2(1-\delta^{-2}\xi 2\pi^{-2s}+\sigma\delta^{-2}\xi^2\pi^{d-2s})) \\ &\quad = \mathfrak{p}^{d-2s} \supseteq \mathfrak{p}^d \supset \mathfrak{p}^{r-2e+d} \supseteq D(1+c\pi^{r-2e+d}) \end{aligned}$$

by [**8**, 63:5], a contradiction

If $r/2 < \operatorname{ord} a < r-e$, then $\operatorname{ord} b = 0$. We have

$$\operatorname{ord}(2b^{-1}a\pi^{-r/2}) = e + \operatorname{ord} a - r/2 > e > 2e-r/2 \geq d \geq 1,$$

and

$$\begin{aligned} h = Q(z) &= b^2\pi^r(\varepsilon+(b^{-1}a\pi^{-r/2})^2) = b^2\pi^r(1+\sigma\pi^d+(b^{-1}a\pi^{-r/2})^2) \\ &= b^2\pi^r((1+b^{-1}a\pi^{-r/2})^2-2b^{-1}a\pi^{-r/2}+\sigma\pi^d) \end{aligned}$$

is in $(1+\mathfrak{p}^{r-2e+d})\dot{F}^2$ by hypothesis. So

$$\eta^2((1+b^{-1}a\pi^{-r/2})^2-2b^{-1}a\pi^{-r/2}+\sigma\pi^d) = (1+c\pi^{r-2e+d})$$

where $\eta \in U$ and $c \in \vartheta$. But

$$D(\eta^2((1 + b^{-1}a(\pi^{-r/2})^2 - 2b^{-1}a\pi^{-r/2} + \sigma\pi^d))$$
$$= p^d \supset p^{r-2e+d} \supseteq D(1 + c\pi^{r-2e+d})$$

by [**8**, 63:5], a contradiction.

Now there remains ord $a = 0$ or $0 < \text{ord}\, a \leq e$, ord $b = 0$; or ord $a \geq r - e$, ord $b = 0$, so τ_z is in $O(L)$ by [**4**, Prop. 3.2]. Therefore

$$\theta(O^+(L)) = ((1 + p^{r-2e+d})\dot{F}^2 \cup \varepsilon(1 + p^{r-2e+d})\dot{F}^2) \cap Q(\dot{F}L).$$

Case (iii). Write $\varepsilon = 1 + \sigma\pi^d$ with $d > 2e - r/2$, $\sigma \in U$.

If ord $a = 0$, then

$$\text{ord}(\varepsilon\pi^r(a^{-1}b)^2) = r + 2\,\text{ord}\, b \geq r \geq r/2,$$

so

$$Q(z) = a^2(1 + \varepsilon\pi^r(a^{-1}b)^2)$$

is in $(1 + p^{r/2})\dot{F}^2$.

If $0 < \text{ord}\, a \leq e < r/2$, ord $b = 0$; then

$$\begin{aligned} r/2 \leq \text{ord}(-2(a^{-1}b)\pi^{r/2}) &= e - \text{ord}\, a + r/2 \\ &= (r + d - 2\,\text{ord}\, a) + (\text{ord}\, a - r/2 - d + e) \\ &\leq (r + d - 2\,\text{ord}\, a) + (2e - r/2 - d) \\ &< r + d - 2\text{ ord}\, a = \text{ord}(\sigma\pi^{r+d}(a^{-1}b)^2). \end{aligned}$$

So

$$\begin{aligned} Q(z) = a^2(1 + \varepsilon\pi^r(a^{-1}b)^2) &= a^2(1 + (1 + \sigma\pi^d)\pi^r(a^{-1}b)^2) \\ &= a^2(1 + (a^{-1}b)^2\pi^r + \sigma(a^{-1}b)^2\pi^{r+d}) \\ &= a^2(1 + a^{-1}b\pi^{r/2})^2 - 2a^{-1}b\pi^{r/2} + \sigma(a^{-1}b)^2\pi^{r+d}) \end{aligned}$$

is in $(1 + p^{r/2})\dot{F}^2$.

If ord $a \geq r - e > r/2$, ord $b = 0$; then

$$\begin{aligned} r/2 \leq \text{ord}(-2\varepsilon^{-1}b^{-1}a\pi^{-r/2}) &= e + \text{ord}\, a - r/2 \\ &= (2\text{ ord}\, a + d - r) + (r/2 - \text{ord}\, a + e - d) \\ &\leq (2\text{ ord}\, a + d - r) + (2e - r/2 - d) \\ &< (2\text{ ord}\, a + d - r) = \text{ord}(\sigma(\varepsilon^{-1}b^{-1}a)^2\pi^{d-r}). \end{aligned}$$

So

$$\begin{aligned}Q(z) &= \varepsilon\pi^r b^2(1+\varepsilon(\varepsilon^{-1}b^{-1}a)^2\pi^{-r}) \\ &= \varepsilon\pi^r b^2(1+(1+\sigma\pi^d)(\varepsilon^{-1}b^{-1}a)^2\pi^{-r}) \\ &= \varepsilon\pi^r b^2(1+(\varepsilon^{-1}b^{-1}a)^2\pi^{-r}+\sigma(\varepsilon^{-1}b^{-1}a)^2\pi^{d-r}) \\ &= \varepsilon\pi^r b^2((1+\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2 - 2\varepsilon^{-1}b^{-1}a\pi^{-r/2} \\ &\qquad + \sigma(\varepsilon^{-1}b^{-1}a)^2\pi^{d-r})\end{aligned}$$

is in $\varepsilon(1+\mathfrak{p}^{r/2})\dot{F}^2$.

Therefore

$$\theta(O^+(L)) \subseteq (1+\mathfrak{p}^{r/2})\dot{F}^2 \cup \varepsilon(1+\mathfrak{p}^{r/2})\dot{F}^2$$

by [7, 1.1].

Conversely, take an element h in $(1+\mathfrak{p}^{r/2})\dot{F}^2$, by the Local Square Theorem, we assume $h = 1 + c\pi^{r/2}$ with $0 \leq \operatorname{ord} c < e$. Write $c = \lambda\pi^k$ with $\lambda \in U$. By Hensel's lemma, there exists η in U such that

$$(\pi^{r-2e+2k} - \sigma c^{-1}\pi^{d+r/2-2e+2k})\eta^2(2c^{-1}\pi^{k-e} + 2\pi^{r/2-e+k})\eta + 1 = 0.$$

Put $z = \pi^{e-k}x + \eta y$, so τ_z is in $O(L)$ by [4, Prop. 3.2], and

$$\begin{aligned}Q(z) &= \pi^{2e-2k} + \varepsilon\pi^r\eta^2 = \pi^{2e-2k} + (1+\sigma\pi^d)\pi^r\eta^2 \\ &= \pi^{2e-2k}(1+\pi^{r+2k-2e}\eta^2 + \sigma\pi^{d+r+2k-2e}\eta^2) \\ &= \pi^{2e-2k}((1+\pi^{r/2+k-e}\eta)^2 - 2\pi^{r/2+k-e}\eta + \sigma\pi^{d+r+2k-2e}\eta^2) \\ &= \pi^{2e-2k}((1+\pi^{r/2+k-e}\eta)^2 \\ &\qquad + (-2c^{-1}\pi^{k-e}\eta + \sigma c^{-1}\pi^{d+r/2+2k-2e}\eta^2)c\pi^{r/2}) \\ &= \pi^{2e-2k}((1+\pi^{r/2+k-e}\eta)^2 + (1+\pi^{r/2-e+k}\eta)^2 c\pi^{r/2}) \\ &= \pi^{2e-2k}(1+\pi^{r/2+k-e}\eta)^2(1+c\pi^{r/2}) = \pi^{2e-2k}(1+\pi^{r/2+k-e}\eta)^2 h.\end{aligned}$$

Therefore

$$\theta(O^+(L)) = (1+\mathfrak{p}^{r/2})\dot{F}^2 \cup \varepsilon(1+\mathfrak{p}^{r/2})\dot{F}^2.$$

PROPOSITION 2.3. *Suppose* $0 < r \leq 2e$.

(i) *When* r *is odd, or* r *is even and* $D(-\varepsilon) = \mathfrak{p}^d$ *with* $1 \leq d \leq e - r/2$; $\theta(O^+(L)) = Q(\dot{F}L)$.

(ii) *When* r *is even and* $D(-\varepsilon) = \mathfrak{p}^d$ *with* $(3e - r/2)/2 \geq d > e - r/2$; $\theta(O^+(L)) = ((1+\mathfrak{p}^{d-e+r/2})\dot{F}^2) \cap Q(\dot{F}L)$.

(iii) *When* r *is even and* $D(-\varepsilon) = \mathfrak{p}^d$ *with* $d > (3e - r/2)/2$, *or* 0; $\theta(O^+(L)) = (1+\mathfrak{p}^{e-[e/2-r/4]})\dot{F}^2$.

Proof. Take a maximal anisotorpic vector $z = ax+by$ in L, where $a, b \in \vartheta$ and one of them is a unit, we know τ_z is in $O(L)$ if and only if $2B(z, L) \subseteq Q(z)\vartheta$. This is equivalent to

$$\operatorname{ord}(a^2 + \varepsilon\pi^r b^2) \leq \min\{\operatorname{ord} a, r + \operatorname{ord} b\} + e.$$

In fact, this inequality is the same as $\operatorname{ord} a \neq r/2$; or $\operatorname{ord} a = r/2$, $\operatorname{ord} b = 0$, $0 \leq \operatorname{ord}((a\pi^{-r/2}b^{-1})^2 + \varepsilon) \leq e - r/2$.

Case (i). When r is odd, we always have $\operatorname{ord} a \neq r/2$; when r is even and $\operatorname{ord} a = r/2$, we have $0 \leq \operatorname{ord}(\varepsilon + (\pi^{-r/2}b^{-1}a)^2) \leq d \leq e - r/2$ since $D(-\varepsilon) = p^d$ with $1 \leq d \leq e - r/2$. Note FL is anisotropic in this case, so $\theta(O^+(L)) = Q(\dot{F}L)$.

Case (ii). Write $-\varepsilon = 1+\sigma\pi^d$ with $\sigma \in U$ and $2e > (3e-r/2)/2 \geq d > e - r/2$.

If $\operatorname{ord} a = 0$, then

$$\begin{aligned}\operatorname{ord}(\sigma(\pi^{r/2}ba^{-1})^2\pi^d) &= d + r + 2\operatorname{ord} b\\ &= (e + r/2 + \operatorname{ord} b) + \operatorname{ord} b + (d + r/2 - e)\\ &> e + r/2 + \operatorname{ord} b = \operatorname{ord}(-2\pi^{r/2}(ba^{-1})(1 + a^{-1}b\pi^{r/2}))\\ &\geq e + r/2 > d - e + r/2.\end{aligned}$$

So

$$\begin{aligned}Q(z) &= a^2(1 + \varepsilon\pi^r(a^{-1}b)^2) = a^2(1 - (1 + \sigma\pi^d)(a^{-1}b)^2\pi^r)\\ &= a^2((1 + a^{-1}b\pi^{r/2})^2\\ &\qquad - 2(a^{-1}b\pi^{r/2})(1 + a^{-1}b\pi^{r/2}) - \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\end{aligned}$$

is in $(1 + p^{d-e+r/2})\dot{F}^2 \cap Q(\dot{F}L)$.

If $0 < \operatorname{ord} a < r/2$, $\operatorname{ord} b = 0$; then

$$\begin{aligned}\operatorname{ord}(-2a^{-1}b\pi^{r/2}(1 + a^{-1}b\pi^{r/2})) &= e + r/2 - \operatorname{ord} a\\ &> e = (e + e/2 - r/4) + r/4 - e/2 \geq d + (r/2 - e)/2\\ &\geq d - e + r/2\end{aligned}$$

and

$$\operatorname{ord}(-\sigma(a^{-1}b\pi^{r/2})^2\pi^d) = d + r - 2\operatorname{ord} a \geq d \geq d - e + r/2.$$

So

$$\begin{aligned}Q(z) = a^2((1 + a^{-1}b\pi^{r/2})^2\\ - 2(a^{-1}b\pi^{r/2})(1 + a^{-1}b\pi^{r/2}) - \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\end{aligned}$$

is in $(1 + p^{d-e+r/2})\dot{F}^2 \cap Q(\dot{F}L)$.

If $\operatorname{ord} a > r/2$, $\operatorname{ord} b = 0$; then

$$\begin{aligned}\operatorname{ord}(-2\varepsilon^{-1}b^{-1}a\pi^{-r/2}&(1+\varepsilon^{-1}b^{-1}a\pi^{-r/2}))\\ &= e + \operatorname{ord} a - r/2 > e \geq d - e + r/2\end{aligned}$$

and

$$\operatorname{ord}(-\sigma(\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2\pi^d) = -r + 2\operatorname{ord} a + d > d \geq d - e + r/2.$$

So

$$\begin{aligned}Q(z) &= \varepsilon\pi^r b^2(1+\varepsilon(\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2)\\ &= \varepsilon\pi^r b^2(1-(1+\sigma\pi^d)(\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2)\\ &= \varepsilon\pi^r b^2((1+\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2\\ &\qquad - 2\varepsilon^{-1}b^{-1}a\pi^{-r/2}(1+\varepsilon^{-1}b^{-1}a\pi^{-r/2})\\ &\qquad\qquad - \sigma(\varepsilon^{-1}b^{-1}a\pi^{-r/2})^2\pi^d)\end{aligned}$$

is in $\varepsilon(1+\mathfrak{p}^{d-e+r/2})\dot{F}^2 \cap Q(\dot{F}L)$.

If $\operatorname{ord} a = r/2$, $\operatorname{ord} b = 0$, and $\operatorname{ord}(\varepsilon + (\pi^{-r/2}ab^{-1})^2) \leq e - r/2$, note

$$\operatorname{ord}(2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2})) = e + \operatorname{ord}(1+a^{-1}b\pi^{r/2}) \geq e > e - r/2$$

and

$$\operatorname{ord}(\sigma(a^{-1}b\pi^{r/2})^2\pi^d) = d > e - r/2.$$

Then

$$\begin{aligned}&\operatorname{ord}(1+a^{-1}b\pi^{r/2})^2\\ &\quad= \operatorname{ord}((1+a^{-1}b\pi^{r/2})^2\\ &\qquad -2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2})^2 - \sigma(a^{-1}b\pi^{r/2})^2\pi^d\\ &\qquad + 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) + \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\\ &\quad= \operatorname{ord}(1-(a^{-1}b\pi^{r/2})^2 - \sigma(a^{-1}b\pi^{r/2})^2\pi^d\\ &\qquad\qquad + 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) + \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\\ &\quad= \operatorname{ord}(1+(a^{-1}b\pi^{r/2})^2(-1-\sigma\pi^d)\\ &\qquad\qquad + 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) + \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\\ &\quad= \operatorname{ord}((1+\varepsilon(a^{-1}b\pi^{r/2})^2)\\ &\qquad\qquad + 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) + \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\\ &\quad= \operatorname{ord}(1+\varepsilon(a^{-1}b\pi^{r/2})^2) \leq e - r/2.\end{aligned}$$

So

$$\operatorname{ord}(\sigma\pi^d(a^{-1}b\pi^{r/2})^2(1+a^{-1}b\pi^{r/2})^{-2}) \\ = d - 2\operatorname{ord}(1+a^{-1}b\pi^{r/2}) \geq d-e+r/2$$

and

$$\operatorname{ord}((-2a^{-1}b\pi^{r/2})(1+a^{-1}b\pi^{r/2})^{-1}) = e - \operatorname{ord}(1+a^{-1}b\pi^{r/2}) \\ \geq e-(e-r/2)/2 \geq d-e+r/2.$$

Thus

$$\begin{aligned} Q(z) &= a^2(1+(a^{-1}b\pi^{r/2})\varepsilon) = a^2(1-(1+\sigma\pi^d)(a^{-1}b\pi^{r/2})^2) \\ &= a^2((1+a^{-1}b\pi^{r/2})^2 \\ &\qquad - 2(1+a^{-1}b\pi^{r/2})a^{-1}b\pi^{r/2} - \sigma(a^{-1}b\pi^{r/2})^2\pi^d) \\ &= a^2(1+a^{-1}b\pi^{r/2})^2(1-2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2})^{-1} \\ &\qquad - \sigma\pi^d(a^{-1}b\pi^{r/2})^2(1+a^{-1}b\pi^{r/2})^{-2}) \end{aligned}$$

is in $(1+\mathfrak{p}^{d-e+r/2})\dot{F}^2 \cap Q(\dot{F}L)$. Therefore

$$\theta(O^+(L)) \subseteq ((1+\mathfrak{p}^{d-e+r/2})\dot{F}^2 \cap \varepsilon(1+\mathfrak{p}^{d-e+r/2})\dot{F}^2) \cap Q(\dot{F}L).$$

Conversely, take an element h in $(1+\mathfrak{p}^{d-e+r/2})\dot{F}^2 \cap Q(\dot{F}L)$, so there exists z in $\dot{F}L$ such that $h = Q(z)$. We can assume $z = ax+by$ with $a, b \in \vartheta$ and one of them is in U. We claim that τ_z is in $O(L)$. In fact, if $\operatorname{ord} a = r/2$, $\operatorname{ord} b = 0$, and $\operatorname{ord}((\pi^{-r/2}ab^{-1})^2+\varepsilon) > e-r/2$; note

$$\begin{aligned} h = Q(z) &= a^2(1+\varepsilon(a^{-1}b\pi^{r/2})^2) = a^2(1+(-1-\sigma\pi^d)(a^{-1}b\pi^{r/2})^2) \\ &= a^2((1+a^{-1}b\pi^{r/2})^2 - 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) - \sigma(a^{-1}b\pi^{r/2})^2\pi^d) \end{aligned}$$

is in $(1+\mathfrak{p}^{d-e+r/2})\dot{F}^2$, and d is odd, $d < 2e$; then

$$\begin{aligned} &\operatorname{ord}(1+a^{-1}b\pi^{r/2})^2 \\ &\quad = \operatorname{ord}((1+a^{-1}b\pi^{r/2})^2 \\ &\qquad - 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2}) - \sigma(a^{-1}b\pi^{r/2})^2\pi^d) \\ &\quad = \operatorname{ord}(1+\varepsilon(a^{-1}b\pi^{r/2})^2) > e-r/2. \end{aligned}$$

Write $1+a^{-1}b\pi^{r/2} = \lambda\pi^s$ with $\lambda \in U$, $s \geq 0$. So $d > 2s > e-r/2$, and

$$\eta^2(1-(\lambda^{-1}a^{-1}b\pi^{r/2})(2\pi^{-s}) - \sigma(\lambda^{-1}a^{-1}b\pi^{r/2})^2\pi^{d-2s}) = 1+c\pi^{d-e+r/2}$$

where $\eta \in U$, $c \in \vartheta$. Since $d \leq e+(e-r/2)/2 < e+s$, that is $0 < d-2s < e-s$, and $d-2s$ is odd, we have

$$D(\eta^2(1-(\lambda^{-1}a^{-1}b\pi^{r/2})(2\pi^{-s}) - \sigma(\lambda^{-1}a^{-1}b\pi^{r/2})^2\pi^{d-2s})) = \mathfrak{p}^{d-2s} \\ \supset \mathfrak{p}^{d-e+r/2} \supseteq D(1+c\pi^{d-e+r/2}).$$

This leads to a contradiction. So τ_z is in $O(L)$, and

$$\theta(O^+(L)) = ((1+p^{d-e+r/2})\dot{F}^2 \cup \varepsilon(1+p^{d-e+r/2})\dot{F}^2) \cap Q(\dot{F}L).$$

Case (iii). Write $-\varepsilon = 1+\sigma\pi^d$ with $\sigma \in U$ and $d > (3e-r/2)/2$.

If $\operatorname{ord} a = 0$, then

$$\begin{aligned}\operatorname{ord}(-2(1+a^{-1}b\pi^{r/2})a^{-1}b\pi^{r/2}) &= e + \operatorname{ord} b + r/2 \\ &\geq e + r/2 > e - [e/2 - r/4]\end{aligned}$$

and

$$\begin{aligned}\operatorname{ord}(-\sigma(a^{-1}b\pi^{r/2})^2\pi^d) &= 2\operatorname{ord} b + r + d \\ &\geq r + d > d > e + [e/2 - r/4] \geq e - [e/2 - r/4].\end{aligned}$$

So

$$\begin{aligned}Q(z) &= a^2(1 - (a^{-1}b\pi^{r/2})^2(1+\sigma\pi^d)) \\ &= a^2((1+a^{-1}b\pi^{r/2})^2 \\ &\qquad - 2(1+a^{-1}b\pi^{r/2})a^{-1}b\pi^{r/2} - \sigma(a^{-1}b\pi^{r/2})^2\pi^d)\end{aligned}$$

is in $(1+p^{e-[e/2-r/4]})\dot{F}^2$.

If $\operatorname{ord} a > r/2$, $\operatorname{ord} b = 0$; then

$$\begin{aligned}&\operatorname{ord}(-2(\varepsilon^{-1}b^{-1}\pi^{-r/2}a)(1+\varepsilon^{-1}b^{-1}\pi^{-r/2}a)) \\ &\qquad = e - r/2 + \operatorname{ord} a > e \geq e - [e/2 - r/4]\end{aligned}$$

and

$$\begin{aligned}\operatorname{ord}(-\sigma(\varepsilon^{-1}b^{-1}\pi^{-r/2}a)^2\pi^d) &= d + 2\operatorname{ord} a - r \\ &> d > e + [e/2 - r/4] \geq e - [e/2 - r/4].\end{aligned}$$

So

$$\begin{aligned}Q(z) &= \varepsilon\pi^r b^2(1 + \varepsilon(\varepsilon^{-1}b^{-1}\pi^{-r/2}a)^2) \\ &= \varepsilon\pi^r b^2(1 - (1+\sigma\pi^d)(\varepsilon^{-1}b^{-1}\pi^{-r/2}a)^2) \\ &= \varepsilon\pi^r b^2((1+\varepsilon^{-1}b^{-1}\pi^{-r/2}a)^2 \\ &\qquad - 2(1+\varepsilon^{-1}b^{-1}\pi^{-r/2}a)(\varepsilon^{-1}b^{-1}\pi^{-r/2}a) \\ &\qquad\qquad - \sigma(\varepsilon^{-1}b^{-1}\pi^{-r/2}a)^2\pi^d)\end{aligned}$$

is in $\varepsilon(1+p^{e-[e/2-r/4]})\dot{F}^2$.

If $\operatorname{ord} a < r/2$, $\operatorname{ord} b = 0$; then

$$\begin{aligned}&\operatorname{ord}(-2(1+a^{-1}b\pi^{r/2})a^{-1}b\pi^{r/2}) \\ &\qquad = e - \operatorname{ord} a + r/2 > e \geq e - [e/2 - r/4]\end{aligned}$$

and

$$\operatorname{ord}(-\sigma(a^{-1}b\pi^{r/2})^2\pi^d) = r + d - 2\operatorname{ord} a$$
$$> d > e + [e/2 - r/4] \geq e - [e/2 - r/4].$$

So

$Q(z) = a^2((1+a^{-1}b\pi^{r/2})^2 - 2(1+a^{-1}b\pi^{r/2})a^{-1}b\pi^{r/2} - \sigma(a^{-1}b\pi^{r/2})^2\pi^d)$ is in $(1+p^{e-[e/2-r/4]})\dot{F}^2$. If $\operatorname{ord} a = r/2$, $\operatorname{ord} b = 0$, and

$$\operatorname{ord}(\varepsilon + (b^{-1}a\pi^{-r/2})^2) \leq e - r/2,$$

note

$$\operatorname{ord}(\sigma(a^{-1}b\pi^{r/2})^2\pi^d) = d > e + (e/2 - r/4) \geq e > e - r/2$$

and

$$\operatorname{ord}(2(1+a^{-1}b\pi^{r/2})(a^{-1}b\pi^{r/2})) = e + \operatorname{ord}(1+a^{-1}b\pi^{r/2}) \geq e > e - r/2$$

Then

$$\begin{aligned}
&\operatorname{ord}(1+a^{-1}b\pi^{r/2})^2 \\
&\quad = \operatorname{ord}((1+a^{-1}b\pi^{r/2})^2 - 2(1+a^{-1}b\pi^{r/2})(a^{-1}b\pi^{r/2}) \\
&\qquad - \sigma(a^{-1}b\pi^{r/2})^2\pi^d \\
&\qquad + 2(1+a^{-1}b\pi^{r/2})(a^{-1}b\pi^{r/2}) + \sigma(a^{-1}b\pi^{r/2})^2\pi^d) \\
&\quad = \operatorname{ord}((1+\varepsilon(a^{-1}b\pi^{r/2})^2) + 2(1+a^{-1}b\pi^{r/2})(a^{-1}b\pi^{r/2}) \\
&\qquad + \sigma(a^{-1}b\pi^{r/2})^2\pi^d) \\
&\quad = \operatorname{ord}(1+\varepsilon(a^{-1}b\pi^{r/2})^2) \leq e - r/2.
\end{aligned}$$

So

$$\begin{aligned}
&\operatorname{ord}(-2(1+a^{-1}b\pi^{r/2})^{-1}(a^{-1}b\pi^{r/2})) \\
&\quad = e - \operatorname{ord}(1+a^{-1}b\pi^{r/2}) \geq e - [e/2 - r/4]
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{ord}(-\sigma(a^{-1}b\pi^{r/2})^2\pi^d(1+a^{-1}b\pi^{r/2})^{-2}) \\
&\quad = d - 2\operatorname{ord}(1+a^{-1}b\pi^{r/2}) \geq d - e + r/2 \geq e - [e/2 - r/4].
\end{aligned}$$

Consider

$$\begin{aligned}
Q(z) = a^2(1+a^{-1}b\pi^{r/2})^2(1 &- 2a^{-1}b\pi^{r/2}(1+a^{-1}b\pi^{r/2})^{-1} \\
&- \sigma(a^{-1}b\pi^{r/2})^2\pi^d(1+a^{-1}b\pi^{r/2})^{-2})
\end{aligned}$$

is in $(1+p^{e-[e/2-r/4]})\dot{F}^2$. Therefore

$$\theta(O^+(L)) \subseteq (1+p^{e-[e/2-r/4]})\dot{F}^2 \cup \varepsilon(1+p^{e-[e/2-r/4]})\dot{F}^2.$$

Conversely, take an element $h=1+c\pi^{e-[e/2-r/4]}$ in $(1+p^{e-[e/2-r/4]})\dot{F}^2$, and write $c=\lambda\pi^k$ with $\lambda\in U$, $k\geq 0$. If $0\leq k\leq [e/2-r/4]$, there exists η in U such that

$$\sigma\pi^{d-e-[e/2-r/4]+k}\eta^2+2\pi^{-e}\eta+\lambda=0$$

by Hensel's lemma. Put $z=(-1+\eta^{-1}\pi^{[e/2-r/4]-k})\pi^{r/2}x+y$ when $-1+\eta^{-1}\pi^{[e/2-r/4]-k}$ is in U, note

$$\operatorname{ord}(-\sigma\pi^d)=d>e+[e/2-r/4]>2[e/2-r/4]\geq 2[e/2-r/4]-2k$$

and

$$\operatorname{ord}(-2\eta^{-1}\pi^{[e/2-r/4]-k})=e+[e/2-r/4]-k>2[e/2-r/4]-2k$$

So

$$\begin{aligned}
&\operatorname{ord}((-1+\eta^{-1}\pi^{[e/2-r/4]-k})^2+\varepsilon)\\
&\quad=\operatorname{ord}(1+\eta^{-2}\pi^{2[e/2-r/4]-2k}-2\eta^{-1}\pi^{[e/2-r/4]-k}-1-\sigma\pi^d)\\
&\quad=\operatorname{ord}(\eta^{-2}\pi^{2[e/2-r/4]-2k}-2\eta^{-1}\pi^{[e/2-r/4]-k}-\sigma\pi^d)\\
&\quad=\operatorname{ord}(\eta^{-2}\pi^{2[e/2-r/4]-2k})\\
&\quad=2[e/2-r/4]-2k\leq 2[e/2-r/4]\leq e-r/2.
\end{aligned}$$

Thus, τ_z is in $O(L)$, and

$$\begin{aligned}
Q(z)&=\pi^r((-1+\eta^{-1}\pi^{[e/2-r/4]-k})^2+\varepsilon)\\
&=\pi^r(\eta^{-2}\pi^{2[e/2-r/4]-2k}-2\eta^{-1}\pi^{[e/2-r/4]-k}-\sigma\pi^d)\\
&=\pi^{r+2[e/2-r/4]-2k}\eta^{-2}(1-2\eta\pi^{k-[e/2-r/4]}-\sigma\eta^2\pi^{d+2k-[e/2-r/4]})\\
&=\eta^{-2}\pi^{r+2[e/2-r/4]-2k}\\
&\qquad\times(1+\pi^{k-[e/2-r/4]+e}(-2\pi^{-e}\eta-\sigma\eta^2\pi^{d+k-[e/2-r/4]-e}))\\
&=\eta^{-2}\pi^{r+2[e/2-r/4]-2k}(1+\lambda\pi^{k-[e/2-r/4]+e})\\
&=\eta^{-2}\pi^{r+2[e/2-r/4]-2k}(1+c\pi^{e-[e/2-r/4]})\\
&=\eta^{-2}\pi^{r+2[e/2-r/4]-2k}h.
\end{aligned}$$

If $k>[e/2-r/4]$, there exists η in U such that

$$\begin{aligned}
&\lambda\eta^2+(2\pi^{-e}+2\lambda\pi^{k-[e/2-r/4]})\eta+\lambda\pi^{2k-2[e/2-r/4]}\\
&\qquad+2\pi^{-e+k-[e/2-r/4]}+\sigma\pi^{d-e+k-[e/2-r/4]}=0
\end{aligned}$$

by Hensel's lemma. Put $z = \eta\pi^{r/2+[e/2-r/4]-k}x + y$, note $r/2 + [e/2-r/4]-k < r/2$, so τ_z is in $O(L)$, and

$$\begin{aligned}
Q(z) &= \eta^2\pi^{r+2[e/2-r/4]-2k} + \varepsilon\pi^r \\
&= \pi^{r+2[e/2-r/4]-2k}(\eta^2 - \pi^{2k-2[e/2-r/4]} - \sigma\pi^{d-2[e/2-r/4]+2k}) \\
&= \pi^{r+2[e/2-r/4]-2k}((\eta + \pi^{k-[e/2-r/4]})^2 + \pi^{e+k-[e/2-r/4]} \\
&\qquad \times (-2\eta\pi^{-e}2\pi^{-e+k-[e/2-r/4]} - \sigma\pi^{d-e+k-[e/2-r/4]})) \\
&= \pi^{r+2[e/2-r/4]-2k}((\eta + \pi^{k-[e/2-r/4]})^2 + \pi^{e+k-[e/2-r/4]} \\
&\qquad \times (\lambda\eta^2 + 2\lambda\pi^{k-[e/2-r/4]}\eta + \lambda\pi^{2k-2[e/2-r/4]})) \\
&= \pi^{r+2[e/2-r/4]-2k}(\eta + \pi^{k-[e/2-r/4]})^2(1 + \lambda\pi^k\pi^{e-[e/2-r/4]}) \\
&= \pi^{r+2[e/2-r/4]-2k}(\eta + \pi^{k-[e/2-r/4]})^2 h.
\end{aligned}$$

Therefore

$$\theta(O^+(L)) = (1 + p^{e-[e/2-r/4]})\dot{F}^2 \cup \varepsilon(1 + p^{e-[e/2-r/4]})\dot{F}^2$$

REMARK 1. Suppose two units ε, η with $D(\varepsilon) = p^s$ and $D(\eta) = p^t$, where s, t are odd integers, and $s + t > 2e$; then the Hilbert symbol $(\varepsilon, \eta)_p = 1$. This fact can be proved by an argument similar to that of the case (iii) of Proposition 2.2 or [**5**, Lemma 4]. On the other hand, consider a unit ε with $D(\varepsilon) = p^s$ where s is an odd integer. For any positive odd integer t, if $s = t \leq 2e$, then there exists a unit η with $D(\eta) = p^t$ and the Hilbert symbol $(\varepsilon, \eta)_p = -1$. This result can be obtained from [**5**, Lemma 3].

REMARK 2. Suppose E is a finite extension of F and $\overline{L}$ is the lifting of L to E, then we can check $N_{E/F}(\theta(O^+(\overline{L}))) \subseteq \theta(O^+(L))$ by the results obtained in [**5**], [**10**] and this section. These can give an alternative proof about spinor genus extension of binary lattices (see [**2**], [**4**]).

3. Main results. In this section let L be a lattice with $\dim FL \geq 3$ and Jordan splitting $L = \vartheta x_1 \perp \vartheta x_2 \perp \cdots \perp \vartheta x_n$ where $Q(x_1) = 1$, $Q(x_i) = \varepsilon_i\pi^{r_i}$, $r_i \in Z$, $\varepsilon_i \in U$, $i = 2, 3, \ldots, n$; and $0 < r_i < r_{i+1}$, $i = 2, 3, \ldots, n-1$. We assume $r_1 = 0$, $\varepsilon_1 = 1$. First we generalize [**3**, Th. 2.2] to an arbitrary dyadic local field.

THEOREM 3.1. *Suppose there is at least one k with $1 \leq k \leq n-1$ for which $r_{k+1} - r_k \leq 2e+1$ and $r_{k+1} - r_k$ is odd. Then if $0 < r_s - r_t \leq 4e$ and $r_s - r_t$ is even for any $s, t = 1, \ldots, n$, we have $\theta(O^+(L)) = \dot{F}$.*

Proof. Put $L_{k+1,k} = \vartheta x_k \perp \vartheta x_{k+1}$. When $r_{k+1} - r_k = 2e+1$, then

$$\begin{aligned}\theta(O^+(L)) &= ((1+p)\dot{F}^2 \cup \varepsilon\pi(1+p)\dot{F}^2) \cap Q([1, \varepsilon_k\dot{\varepsilon}_{k+1}\pi]) \\ &= (U\dot{F}^2 \cup \pi U\dot{F}^2) \cap Q([1, \varepsilon_k\dot{\varepsilon}_{k+1}\pi]) = Q([1, \varepsilon_k\dot{\varepsilon}_{k+1}\pi])\end{aligned}$$

by case (i) of Proposition 2.2. When $r_{k+1} - r_k < 2e$, we also obtain $\theta(O^+(L_{k+1,k})) = Q([1, \varepsilon_k\dot{\varepsilon}_{k+1}\pi])$ by case (i) of Proposition 2.3. Note $Q([1, \varepsilon_k\dot{\varepsilon}_{k+1}\pi])$ is a subgroup of $\dot{F}$ with index 2 which does not contain Δ.

Put $L_{s,t} = \vartheta x_s \perp \vartheta x_t$. By cases (ii), (iii) of Proposition 2.2 and Proposition 2.3, we check Δ in $\theta(O^+(L_{s,t}))$. Hence $\theta(O^+(L)) = \dot{F}$.

Let $S(L)$ be the group generated by symmetries in $O(L)$ (see [7]). We need the following lemma.

LEMMA 3.2. *If* $r_{i+1} - r_i \geq e$, *for all* $i = 1, 2, \dots, n-1$, *with at most one exception, then* $O(L) = S(L)$.

Proof. First we assume $r_2 \geq e$, otherwise we scale the dual lattice of L by $\varepsilon_n\pi^{r_n}$ and obtain the lattice L' which satisfies this assumption, and we know $O(L) = O(L')$.

Take σ in $O(L)$. Suppose $\sigma x_1 = \sum_{i=1}^n a_i x_i$, $a_i \in \vartheta$. So

$$1 = Q(\sigma x_1) = Q\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n \varepsilon_i \pi^{r_i} a_i^2 .$$

If $\operatorname{ord}(a_1 - 1) \leq e$, then

$$\operatorname{ord} Q(\sigma x_1 - x_1) = \operatorname{ord}\left((a_1-1)^2 + \sum_{i=2}^n a_i^2\varepsilon_i\pi^{r_i}\right) = \operatorname{ord} 2(a_1 - 1)$$

and

$$2B(\sigma x_1 - x_1, L)\vartheta = 2(a_1 - 1)\vartheta = Q(\sigma x_1 - x_1)\vartheta .$$

Thus, $\tau_{\sigma x_1 - x_1}$ is in $O(L)$.

If $\operatorname{ord}(a_1 - 1) > e$, then $\operatorname{ord}(a_1 + 1) = \operatorname{ord}(2 + (a_1 - 1)) = e$, and

$$\operatorname{ord} Q(\sigma x_1 + x_1) = \operatorname{ord}\left((a_1+1)^2 + \sum_{i=2}^n a_i^2\varepsilon_i\pi^{r_i}\right) = \operatorname{ord} 2(a_1 + 1)$$

and

$$2B(\sigma x_1 + x_1, L)\vartheta = 2(a_1 + 1)\vartheta = Q(\sigma x_1 + x_1)\vartheta .$$

Thus, $\tau_{\sigma x_1 + x_1}$ is in $O(L)$.

Since $\tau_{\sigma x_1 - x_1}\sigma$ and $\tau_{x_1}\tau_{\sigma x_1 + x_1}\sigma$ can be regarded as elements in $O(K)$ where $K = \vartheta x_2 \perp \cdots \perp \vartheta x_n$, by induction on n, we obtain $O(L) = S(L)$.

Now we deduce the Kneser-type result about spinor norms.

THEOREM 3.3. *If* $r_{i+1} - r_i > 4e$, *for all* $i = 1, 2, \ldots, n-1$; *then*

$$\theta(O^+(L)) = \bigcup_{k \text{ even}} \varepsilon_{i_1} \cdots \varepsilon_{i_k} \pi^{r_{i_1} + \cdots + r_{i_k}} \dot{F}^2 .$$

Proof. Take a symmetry τ_z in $O(L)$, where z is a maximal anisotropic vector in L. Then $2B(z, L)\vartheta \subseteq Q(z)\vartheta$. Write $z = \sum_{i=1}^n a_i x_i$ with $a_i \in \vartheta$, so

$$\operatorname{ord} Q(z) = \operatorname{ord}\left(\sum_{i=1}^n a_i^2 \varepsilon_i \pi^{r_i}\right) \leq \min_{1 \leq i \leq n} \{r_i + \operatorname{ord} a_i\} + e .$$

Choose k so that $\operatorname{ord}(a_k^2 \varepsilon_k \pi^{r_k}) = \min_{1 \leq i \leq n}\{\operatorname{ord}(a_i^2 \varepsilon_i \pi^{r_i})\}$. Then

$$2 \operatorname{ord} a_k + r_k \leq \operatorname{ord} Q(z) \leq \min_{1 \leq i \leq n} \{r_i + \operatorname{ord} a_i\} + e \leq r_j + \operatorname{ord} a_j + e$$

for $j = 1, 2, \ldots, n$. When we take $j = k$, $\operatorname{ord} a_k \leq e$ is obtained. When $j < k$, we have

$$\begin{aligned} \operatorname{ord}((\varepsilon_j \varepsilon_k^{-1})(a_j a_k^{-1})^2 \pi^{r_j - r_k}) &= 2 \operatorname{ord} a_j - 2 \operatorname{ord} a_k + r_j - r_k \\ &= 2(\operatorname{ord} a_j - 2 \operatorname{ord} a_k - r_k + r_j + e) + 2 \operatorname{ord} a_k + (r_k - r_j) - 2e \\ &\geq (r_k - r_j) - 2e > 2e . \end{aligned}$$

When $j > k$, consider

$$\begin{aligned} \operatorname{ord}((\varepsilon_j \varepsilon_k^{-1})(a_j a_k^{-1})^2 \pi^{r_j - r_k}) &= 2 \operatorname{ord} a_j - 2 \operatorname{ord} a_k + r_j - r_k \\ &\geq r_j - r_k - 2 \operatorname{ord} a_k \geq r_j - r_k - 2e > 2e . \end{aligned}$$

So

$$Q(z) = \varepsilon_k \pi^{r_k} a_k^2 \left(1 + \sum_{i=1}^{k-1} (\varepsilon_k^{-1} \varepsilon_i)(a_k^{-1} a_i)^2 \pi^{r_i - r_k} + \sum_{i=k+1}^{n} (\varepsilon_k^{-1} \varepsilon_i)(a_k^{-1} a_i)^2 \pi^{r_i - r_k}\right)$$

is in $\varepsilon_k \pi^{r_k} \dot{F}^2$ by the Local Square Theorem. By Lemma 3.2, we obtain $\theta(O^+(L)) = \bigcup_{k \text{ even}} \varepsilon_{i_1} \cdots \varepsilon_{i_k} \pi^{r_{i_1} + \cdots + r_{i_k}} \dot{F}^2$.

4. Application. In this section we will show that the restriction in **[1]**, that F be a ramified quadratic extension of Q_2, is unnecessary. In fact we have the following theorem.

THEOREM 4.1. *Suppose ϑ is the ring of integers of dyadic local field F with $e = \operatorname{ord} 2 = 2$. Let L be a regular ϑ-lattice with $sL \subseteq \vartheta$ and* $\operatorname{rank} L \geq 3$. *If* $\operatorname{ord}(dL) \leq 3$ *then* $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Proof. Let $L = L_1 \perp \cdots \perp L_t$ be a Jordan splitting of L. We can assume $t \geq 2$ and $\dim FL_i \leq 2$, $i = 1, \dots, t$, by **[8**, 93:20], and $2sL_i \subset nL_i$ by **[5**, Lemma 1], $i = 1, \dots, t$. Since $\operatorname{ord}(dL) \leq 3$, $t \leq 3$.

(1) $L = L_1 \perp L_2$ with $\operatorname{rank} L_1 = \operatorname{rank} L_2 = 2$, so L_1 is unimodular and $sL_2 = p$. By **[8**, 93:17] we may write $L_i = \vartheta x_i + \vartheta y_i$ with $Q(x_i) = a_i$, $Q(y_i) = -\delta_i a_i^{-1}$, $B(x_i, y_i) = \pi^{i-1}$, and $a_i\vartheta = nL_i$, $\delta_i\vartheta = D(-dL_i)$, $i = 1, 2$.

If $nL_1 = psL_1 = p$, we only need to consider $D(-dL_1) = p^3$ by **[5**, Prop. C], and $\theta(O^+(L_1))$ contains Δ in this case by **[5**, Prop. B].

Suppose $nL_2 = psL_2 = p^2$. Put $K = \vartheta x_1 \perp \vartheta x_2$. Consider any maximal vector of K, say z, which is then also a maximal anisotropic vector of L. We can check that $2B(z, L)\vartheta \subseteq Q(z)\vartheta$, so τ_z is in $O(L)$, and $\theta(O^+(L)) \supseteq Q([1, \dot{a}_1 a_2])\dot{F}^2$ which is a subgroup of $\dot{F}$ with index 2 and does not contain Δ. Therefore $\theta(O^+(L)) = \dot{F}$.

Suppose $nL_2 = sL_2 = p$. Put $K = \vartheta y_1 \perp \vartheta x_2$, by taking the same argument as above, we obtain $\theta(O^+(L)) = \dot{F}$ again.

If $nL_1 = sL_1 = \vartheta$, then $\theta(O^+(L_1))$ contains Δ by the results obtained in **[5]** and **[10]**.

Suppose $nL_2 = psL_2 = p^2$. By **[5**, Prop. C], we only need to treat the case of $D(-dL_2) = p^5$. Put $K = \vartheta x_1 \perp \vartheta y_2$. Note any maximal vector of K, say z, which is also a maximal anisotropic vector of L, satisfies $2B(z, L)\vartheta \subseteq Q(z)\vartheta$. So τ_z is in $O(L)$, $\theta(O^+(L)) \supseteq Q([1, \dot{a}_1 b_2])$ which does not contain Δ. Therefore $\theta(O^+(L)) = \dot{F}$.

Suppose $nL_2 = sL_2 = p$. Put $K = \vartheta x_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$.

(2) $L = L_1 \perp L_2$ with $\operatorname{rank} L_1 = 2$, $\operatorname{rank} L_2 = 1$, so L_1 is unimodular.

Write $L_1 = \vartheta x_1 + \vartheta y_1$ with $Q(x_1) = a_1$, $Q(y_1) = -\delta_1 a_1^{-1}$, $B(x_1, y_1) = 1$, and $nL_1 = a_1\vartheta$, $D(-dL_1) = \delta_1\vartheta$ by **[8**, 93:17]. Let $L_2 = \vartheta x_2$ with $Q(x_2) = a_2 \in p$. If $nL_1 = psL_1 = p$, by **[5**, Prop. C], we only need to consider $D(-dL_1) = p^3$, and $\theta(O^+(L_1))$ contains Δ in this case by **[5**, Prop. B].

When $Q(x_2)\vartheta = a_2\vartheta = p$, put $K = \vartheta y_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$. When $Q(x_2)\vartheta = p^2$, put $K = \vartheta x_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$. When $Q(x_2)\vartheta = p^3$, put $K = \vartheta y_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$.

If $nL_1 = sL_1 = \vartheta$, [**5**, Prop. C], we only need to consider $\delta_1\vartheta = p$ or p^3, and we know that $\theta(O^+(L_1))$ contains Δ in these two cases by [**5**, Prop. B,E]. When $Q(x_2)\vartheta = p$, put $K = \vartheta x_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$. When $Q(x_2)\vartheta = p^3$, put $K = \vartheta x_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$. When $Q(x_2)\vartheta = p^2$, and $\delta_1\vartheta = p$, put $K = \vartheta y_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$. When $Q(x_2)\vartheta = a_2\vartheta = p^2$, and $\delta_1\vartheta = p^3$; we know $(1+p^2)\dot{F}^2 \subseteq \theta(O^+(L_1))$ by [**5**, Prop. E]. Write $a_2 = \sigma\pi^2$ with $\sigma \in U$, $-a_1^{-1}\delta_1 = \eta\pi^3$ with $\eta \in U$. Take an element $1+\lambda\pi$ in $1+p$ with $\lambda \in U$.

Suppose $\lambda\sigma\eta^{-1} = \xi^2 + \varepsilon\pi^d$ with ξ, $\varepsilon \in U$ and $d \geq 4$; then there exists t in ϑ such that $a_1t^2 + 2\pi^{-2}\xi t - \eta\varepsilon\pi^{d-1} = 0$ by Hensel's lemma. Put $z = \pi^2 t x_1 + \xi y_1 + x_2$, which is a maximal vector of L, and

$$\begin{aligned} Q(z) &= \pi^4 t^2 a_1 + 2\pi^2\xi t + \xi^2\eta\pi^3 + \sigma\pi^2 \\ &= \pi^4\eta\varepsilon\pi^{d-1} + \xi^2\eta\pi^3 + \sigma\pi^2 = \eta\pi^3(\xi^2 + \varepsilon\pi^d) + \sigma\pi^2 \\ &= \lambda\sigma\pi^3 + \sigma\pi^2 = \sigma\pi^2(1+\lambda\pi). \end{aligned}$$

So $2B(z, L)\vartheta \subseteq Q(z)\vartheta$, and τ_z is in $O(L)$. Then $(1+\lambda\pi)$ is in $\theta(O^+(L))$. Suppose $\lambda\sigma\eta^{-1} = \xi^2 + \varepsilon\pi^d$ with ξ, $\varepsilon \in U$ and $1 \leq d \leq 3$, and d is odd. Put $\sigma^{-1}\eta\varepsilon = \omega^2 + \delta\pi^h$ with ω, $\delta \in U$. Then there exists t in ϑ such that $\sigma^{-1}a_1t^2 + (2\pi^{-2})\sigma^{-1}\xi t - \delta\pi^{d+h-1} + (2\pi^{-2})\omega\pi^{(d+1)/2} = 0$ by Hensel's lemma. Put $z = \pi^2 t x_1 + \xi y_1 + (1+\omega\pi^{(d+1)/2})x_2$ which is a maximal vector of L, and

$$\begin{aligned} Q(z) &= \tau^4 t^2 a_1 + 2\pi^2 t\xi + \xi^2\pi^3\eta + (1+\omega\pi^{(d+1)/2})^2\sigma\pi^2 \\ &= \sigma\pi^2(\pi^2(\sigma^{-1}a_1t^2 + 2\pi^{-2}\sigma^{-1}\xi t) \\ &\qquad + \sigma^{-1}\xi^2\pi\eta + 1 + 2\omega\pi^{(d+1)/2} + \omega^2\pi^{d+1}) \\ &= \sigma\pi^2(\delta\pi^{d+h+1} - 2\omega\pi^{(d+1)/2} + \sigma^{-1}\xi^2\pi\eta + 1 \\ &\qquad + 2\omega\pi^{(d+1)/2} + \omega^2\pi^{d+1}) \\ &= \sigma\pi^2(1 + \sigma^{-1}\xi^2\eta\pi + \pi^{d+1}(\omega^2 + \delta\pi^h)) \\ &= \sigma\pi^2(1 + \sigma^{-1}\xi^2\eta\pi + \sigma^{-1}\eta\varepsilon\pi^{d+1}) \\ &= \sigma\pi^2(1 + \sigma^{-1}\eta\pi(\xi^2 + \varepsilon\pi^d)) \\ &= \sigma\pi^2(1 + \sigma^{-1}\eta\pi\lambda\sigma\eta^{-1}) = \sigma\pi^2(1+\lambda\pi). \end{aligned}$$

So $2B(z, L) \subseteq Q(z)\vartheta$, and τ_z is in $O(L)$, then $(1+\lambda\pi)$ is in $\theta(O^+(L))$. Therefore, $\theta(O^+(L)) \supseteq (1+p)\dot{F}^2 = U\dot{F}^2$.

(3) $L = L_1 \perp L_2$ with rank $L_1 = 1$, rank $L_2 = 2$. We scale the dual lattice of L by π and reduce to case (2).

(4) $L = L_1 \perp L_2 \perp L_3$ with rank $L_1 =$ rank $L_2 =$ rank $L_3 = 1$. Let $L_i = \vartheta x_i$, $i = 1, 2, 3$; so $Q(x_i)\vartheta = p^{i-1}$, $1 \leq i \leq 3$. Therefore $\theta(O^+(L)) = \dot{F}$ by Theorem 3.1.

(5) $L = L_1 \perp L_2 \perp L_3$ with rank $L_1 = 2$, rank $L_2 =$ rank $L_3 = 1$. So L_1 is unimodular, and $L_i = \vartheta x_i$ with $Q(x_i)\vartheta = p^{i-1}$, $i = 2.3$. Since Δ is in $\theta(O^+(L_1))$ by the results obtained in **[5]** and **[10]**, put $K = \vartheta x_2 \perp \vartheta x_3$, obtain $\theta(O^+(L)) = \dot{F}$.

LEMMA 4.2. *Suppose ϑ is the ring of integers of a dyadic local field with $e =$ ord $2 = 2$. Let L be a regular ϑ-lattice with $sL \subseteq \vartheta$ and* rank $L \geq 4$. *If* $\text{ord}(dL) \leq 7$, *then* $\theta(O^+(L)) \supseteq U\dot{F}^2$.

Proof. Using the above theorem, considering components and dual lattices when necessary, there remain two cases to be treated.

(1) $L = L_1 \perp L_2$ where L_1 is binary unimodular, L_2 is binary p^2-modular. Write $L_1 = \vartheta x_1 \perp \vartheta y_1$ with $Q(x_1) = a_1$, $B(x_1, y_1) = 1$, $Q(y_1) = -\delta_1 a^{-1}$, and $nL_1 = a_1\vartheta$, $D(-dL_1) = \delta_1\vartheta$; and $L_2 = \vartheta x_2 + \vartheta y_2$ with $Q(x_2) = a_2$, $B(x_2, y_2) = \pi^2$, $Q(y_2) = -\delta_2 a_2^{-1}$, and $nL_2 = a_2\vartheta$. $D(-dL_2) = \delta_2\vartheta$ by **[8**, 93:17].

We know that L_i has an orthogonal base if and only if $nL_i = sL_i$ $(i = 1, 2)$. Using the above theorem, considering components and dual lattices when necessary, we assume $nL_1 = p(sL_1) = p$, and $nL_2 = p(sL_2) = p^3$. By **[5**, Prop. C], we only need to consider $D(-dL_1) = p^3$, and Δ is in $\theta(O^+(L_1))$ by **[5**, Prop. B] in this case. Put $K = \vartheta y_1 \perp \vartheta x_2$. Then $\theta(O^+(L)) = \dot{F}$.

(2) $L = L_1 \perp L_2$ where L_1 is binary unimodular and L_2 is binary p^3-modular. We know L_i has an orthogonal base if and only if $nL_i = sL_i$ $(i = 1, 2)$. Using the above theorem, considering components and dual lattices when necessary, we assume $nL_1 = p(sL_1) = p$ and $nL_2 = psL_2 = p^4$. Write $L_1 = \vartheta x_1 + \vartheta y_1$ with $Q(x_1)\vartheta = nL_1 = p$, and $L_2 = \vartheta x_2 + \vartheta y_2$ with $Q(x_2)\vartheta = nL_2 = p^4$, note Δ is in $\theta(O^+(L_1))$ by **[5]** and **[10]**, put $K = \vartheta x_1 \perp \vartheta x_2$, then $\theta(O^+(L)) = \dot{F}$.

By using Theorem 4.1, Lemma 4.2 and Theorem 3.1, we can also prove **[1**, Lemma 3.4] and **[1**, Theorem 3.5] when ϑ is the ring of integers of a dyadic local field with $e =$ ord $2 = 2$. Finally we consider an example.

EXAMPLE 4.3. The bound $\text{ord}(dL) \leq 7$ given in Lemma 4.2 cannot be unconditionally improved. Consider the lattice $L = L_1 \perp L_2$ over the ring of integers of a ramified quadratic extension of Q_2, where $L_1 = \vartheta x_1 + \vartheta y_1$ with $Q(x_1) = 1$, $Q(y_1) = \pi^3$, $B(x_1, y_1) = 1$, and $L_2 = \vartheta x_2 + \vartheta y_2$ with $Q(x_2) = \pi^4$, $Q(y_2) = \pi^7$, $B(x_2, y_2) = \pi^4$. We claim $\theta(O^+(L)) = (1+p^2)\dot{F}^2 \subset U\dot{F}^2$.

First we prove that $O(L)$ is generated by symmetries of L. Note L_i has an orthogonal base $(i = 1, 2)$, and write $L_1 = \vartheta z \perp \vartheta w$. Put $K = \vartheta w \perp L_2$. Then $O(K)$ is generated by symmetries of K by the proof of Lemma 3.2. Let $\sigma \in O(L)$, and $\sigma z = az + bw + u$ with $u \in L_2$, so $Q(z) = a^2Q(z) + b^2Q(w) + Q(u)$ and $Q(\sigma z - z) = 2(1-a)Q(z)$, $Q(\sigma z + z) = 2(1+a)Q(z)$.

If $\mathrm{ord}(1-a) = 0$, then $2B(\sigma z - z, L) = Q(\sigma z - z)\vartheta$, and $\tau_{\sigma z - z} \in O(L)$. If $\mathrm{ord}(1-a) = 1$, then $b^2Q(w) = (1-a)(1+a)Q(z) - Q(u) \in p$, and $\mathrm{ord}\, b \geq 1$. So $2B(\sigma z - z, L) = Q(\sigma z - z)\vartheta$, and $\tau_{\sigma z - z} \in O(L)$.

If $\mathrm{ord}(1-a) = 2$, then $\mathrm{ord}(1+a) = \mathrm{ord}(2-(1-a)) > 2$, and $b^2Q(w) = (1+a)(1-a)Q(z) - Q(u) \in p^4$, so $\mathrm{ord}\, b \geq 2$, and $2B(\sigma z - z, L) = Q(\sigma z - z)\vartheta$. Thus $\tau_{\sigma z - z} \in O(L)$.

If $\mathrm{ord}(1-a) > 2$, then $\mathrm{ord}(1+a) = \mathrm{ord}(2-(1-a)) = 2$, and $b^2Q(w) = (1+a)(1-a)Q(z) - Q(u) \in p^4$, so $\mathrm{ord}\, b \geq 2$, and $2B(\sigma z + z, L) \subseteq Q(\sigma z + z)\vartheta$.Thus $\tau_{\sigma z + z} \in O(L)$.

Note that $\tau_{\sigma z - z}\sigma$ and $\tau_z \tau_{\sigma z + z}\sigma$ can be regarded as elements in $O(K)$ which is generated by symmetries of K. Therefore $O(L)$ is generated by symmetries of L.

Now we calculate $\theta(O^+(L))$. Let $\tau_t \in O(L)$ where t is a maximal anisotropic vector of L, write $t = cx_1 + dy_1 + fx_2 + gy_2$ with $c, d, f, g \in \vartheta$.

When $\mathrm{ord}\, c \leq 1$, then $2c^{-1}d + c^{-2}d^2\pi^3$ is always in p^2 whenever c and d are units or not. So

$$Q(t) = c^2(1 + (2c^{-1}d + c^{-2}d^2\pi^3) + c^{-2}\pi^4(f^2 + 2fg + g^2\pi^3))$$

is in $(1+p^2)\dot{F}^2$.

When $\mathrm{ord}(d) = 0$, and $\mathrm{ord}(c) > 1$; then $\mathrm{ord}\, Q(t) = 3$, and $2B(t, L) = p^2 \supset Q(t)\vartheta$, contradicting the assumption that τ_t is in $O(L)$.

When $\mathrm{ord}(f) = 0$, $\mathrm{ord}(d) > 0$, and $\mathrm{ord}(c) > 1$; then $\mathrm{ord}(c) \geq 2$, and the inclusion $2B(t, L) \subseteq Q(t)\vartheta$ forces $\mathrm{ord}(d) \geq 2$.

Suppose $\mathrm{ord}(c) = 2$. Then $\mathrm{ord}(c^2 + f^2\pi^4) \geq 5$. So

$$\mathrm{ord}\, Q(t) = \mathrm{ord}((c^2 + f^2\pi^4) + 2cd + d^2\pi^3 + 2\pi^4 fg + g^2\pi^7) \geq 5$$

and $2B(t, L) = p^4 \supset Q(t)\vartheta$, a contradiction.

Therefore $\text{ord}(c) \geq 3$, and

$$Q(t) = f^2\pi^4(1 + f^{-2}c^2\pi^{-4} + 2f^{-2}cd\pi^{-4} + d^2f^{-2}\pi^{-1} + 2f^{-1}g + f^{-2}g^2\pi^3)$$

is in $(1+p^2)\dot{F}^2$.

When $\text{ord}(g) = 0$, $\text{ord}(f) > 0$, $\text{ord}(d) > 0$, and $\text{ord}(c) > 1$; then $\text{ord}(c) \geq 2$, and $\text{ord}(d) \geq 2$.

Suppose $\text{ord}(c) = 2$. Then

$$Q(t) = c^2(1 + 2c^{-1}d + (c^{-1}d)^2\pi^3 + c^{-2}\pi^4(f^2 + 2fg + g^2\pi^3))$$

is in $(1+p^2)\dot{F}^2$.

Suppose $\text{ord}(c) \geq 3$. Then $\text{ord}(c) \geq 4$ and $\text{ord}(d) \geq 4$.

If $\text{ord}(f) = 1$, then $2f^{-1}g + (f^{-1}g)^2\pi^3 \in p^2$, and

$$Q(t) = f^2\pi^4(1 + (f^{-1}c\pi^{-2})^2 + 2cdf^{-2}\pi^{-4} + d^2f^{-2}\pi^{-1} + (2f^{-1}g + (f^{-1}g)^2\pi^3))$$

is in $(1+p^2)\dot{F}^2$.

If $\text{ord}(f) \geq 2$, then $\text{ord}\, Q(t) = 7$, and $2B(t, L) = p^6 \supset Q(t)\vartheta$. But τ_t is in $O(L)$ and t is a maximal vector, a contradiction. In any case, we have $\theta(O^+(L)) \subseteq (1+p^2)\dot{F}^2$. On the other hand, $\theta(O^+(L)) \supseteq \theta(O^+(L_1)) \supseteq (1+p^2)\dot{F}^2$ by [5, Prop. E]. Thus, we conclude $\theta(O^+(L)) = (1+p^2)\dot{F}^2$.

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