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FREE BANACH-LIE ALGEBRAS, COUNIVERSAL BANACH-LIE GROUPS, AND MORE

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The construction of free Banach-Lie algebra over a normed space enables us to build a connected separable Banach-Lie group of which any other connected separable Banach-Lie group is a quotient. New proofs are given to the result on representability of any Banach-Lie algebra as a quotient of an enlargable Banach-Lie algebra (due to van Est and Świerczkowski) and to the result on representability of any topological group as a quotient of a group with no small subgroups (due to successive efforts of Morris and Thompson, the author, and Sipacheva and Uspenskiĭ).

1. Introduction. Over the last 50 years a number of constructions of "universal arrows" (see, e.g., [**Go**]) to the categories of topological algebraic systems have been studied. Important contributions are those by Markov [**M**], Graev [**Gr**], and Arhangel'skiĭ [**A2**] on free topological groups, Mal'cev [**Mc**] on free topological algebras, Arens and Eells [**AE**], Raĭkov [**R**], and Uspenskiĭ [**U**] on free Banach spaces and free locally convex spaces. By virtue of these constructions a first ever example of a non-normal Hausdorff topological group was obtained [**M**], and the representability of any topological group as a quotient group of a zero-dimensional group was proved [**A1**]. Here we apply the concept of a free complete normed Lie algebra to theory of topological and Lie groups. Our construction is an extension of the well-known construction of Arens-Eells [**AE**] to the case of normed Lie algebras. Our main result is that there exists a couniversal separable connected Banach-Lie group, that is, such a separable connected Banach-Lie group that any other such Banach-Lie group is its quotient Lie group. This follows from observation that any free Banach-Lie algebra is enlargable, that is, comes from an approriate Banach-Lie group. Also we give entirely new and rather transparent proofs of two earlier known results.

Cohomological technique has enabled van Est and independently Świerczkowski [**Ś2**] to prove that any Banach-Lie algebra is a quotient algebra of an enlargable Banach-Lie algebra. Here we deduce the result from enlargability of free Banach-Lie algebras.

In his book [**Ka**] Kaplansky asked whether a quotient group of a topological group with no small subgroups (NSS group) is again an NSS group. Morris [**Mo**] answered in negative, and later he and Thompson [**MT**] have presented the following

THEOREM A. *Let X be a submetrizable Tychonoff topological space (that is, a Tychonoff space admitting a continuous metric). Then the Markov free topological group $F(X)$ over X is an NSS group.* □

It was asked in [**MT**] whether the following result is true.

THEOREM B. *Each topological group is a quotient group of an NSS group.* □

The author [**Pe1**, **Pe2**] has deduced Theorem B from Theorem A. It was discovered, however, by Sipacheva and Uspenskiĭ [**SU**] that both the original proof of Theorem A by Morris and Thompson [**MT**] and the later proof proposed by Thompson [**T**] are not free of certain deficiencies. In the same work [**SU**] a correct proof of Theorem A was given. Thus, Theorem B—and its proof from [**Pe1**, **Pe2**]—still remain valid. The proof of Theorem A by Sipacheva and Uspenskiĭ is "hard"—it relies on combinatorial technique of words in free groups. The concept of free Banach-Lie algebra enables us to provide an entirely different proof of Theorem A which is purely Lie-theoretic and certainly "soft".

2. Free Banach-Lie algebras. A norm $\|\cdot\|$ on an algebra A is called *submultiplicative* if $\|x \star y\| \leq \|x\| \cdot \|y\|$ whenever $x, y \in A$, where $\star$ stands for the binary algebra operation. By a *normed algebra* we mean an algebra endowed with a submultiplicative norm. We will loosely refer to *complete normed* algebras as merely *Banach* algebras. A mapping $f\colon X \to Y$ between two metric spaces is *contracting*, or *non-expanding*, if $\rho_Y(fx, fy) \leq \rho_X(x, y)$ whenever $x, y \in X$. If X and Y are normed spaces and f is linear, this is equivalent to the condition $\|f\| \leq 1$.

THEOREM 2.1. *Let E be a normed space. There exist a complete normed Lie algebra $\mathscr{FL}(E)$ and a contracting linear operator $i_E\colon E \to \mathscr{FL}(E)$ with the following properties:*

(1) *$i_E(E)$ topologically generates $\mathscr{FL}(E)$, that is, the least Lie subalgebra containing $i_E(E)$ is dense in $\mathscr{FL}(E)$.*

(2) *For an arbitrary complete normed Lie algebra $\mathscr{L}$ and any contracting linear operator $f\colon E \to \mathscr{L}$, there exists a contracting Lie algebra homomorphism $\hat{f}\colon \mathscr{FL}(E) \to \mathscr{L}$ such that $\hat{f} \circ i_E = f$.*

The pair $(\mathcal{FL}(E), i_E)$ *with the properties* (1) *and* (2) *is essentially unique. The operator* i_E *is an isometrical embedding* $E \hookrightarrow \mathcal{FL}(E)$. *If* $\dim E > 2$ *then* $\mathcal{FL}(E)$ *is centerless.*

Proof. Denote by $\mathbf{F}$ the class of (classes of isomorphisms of) all pairs (L, j) where L is a complete normed Lie algebra and $j : E \to L$ is a contracting linear operator such that the image $j(E)$ topologically generates L. $\mathbf{F}$ is a set. Let i_E stand for the diagonal product $\Delta\{j : (L, j) \in \mathbf{F}\}$, viewed as a mapping from E to the l_∞-type sum $\mathbf{L} = l_\infty - \bigoplus_{(L, j)\in\mathbf{F}} L$. Denote by $\mathcal{FL}(E)$ the least closed Lie subalgebra of the Lie algebra $\mathbf{L}$ containing the image $i_E(E)$. The properties (1) and (2) of the pair $(\mathcal{FL}(E), i_E(E))$ are checked immediately.

The proof of uniqueness is standard (cf. **[Go, Gr, M, R]**).

Since the pair (E, id_E) is in $\mathbf{F}$, where E is treated as a commutative normed Lie algebra, then for any element $x \in E$ one has $\|x\|_E \geq \|i_E(x)\|_{\mathcal{FL}(E)} \geq \|\mathrm{id}_E(x)\|_E = \|x\|_E$, that is, i_E is an isometrical embedding.

Now let $x \in \mathcal{FL}(E)$. One may assume that $\|x\| = 1$. There exists a Lie polynomial l of degree $n \in \mathbb{N}$ such that for some elements $x_1, \dots, x_m \in E$ one has $\|l(x_1, \dots, x_m) - x\| \leq \frac{1}{3}$. There is a projection, π, from E to the subspace V spanned by $x_1, \dots, x_m$. The free degree k, $k \geq n$ nilpotent Lie algebra $\mathbf{N}_k(V)$ over V is finite-dimensional and therefore it is a normed space. By rescaling a norm on $\mathbf{N}_k(V)$, one can assume that it is submultiplicative. Let $C > 0$ be the norm of π calculated with respect to a new norm on $V \subset \mathbf{N}_k(V)$; the operator $C^{-1}\pi : E \to \mathbf{N}_k(V)$ is contracting and it is clear that the element $\widehat{C^{-1}\pi}(l(x_1, \dots, x_m)) = l(C^{-1}x_1, \dots, C^{-1}x_m)$ is non-zero in $\mathbf{N}_k(V)$. If k has been chosen sufficiently large, then $[\widehat{C^{-1}\pi}(x), y] \neq 0$ for some $y \in \mathbf{N}_k(V)$; this means that $[x, z] \neq 0$ for an arbitrary $z \in (\widehat{C^{-1}\pi})^{-1}(y)$. □

THEOREM 2.2. *Let* $X = (X, \rho, \star)$ *be a pointed metric space. There exist a complete normed Lie algebra* $\mathcal{FL}_X$ *and a contracting mapping* $i_X : X \to \mathcal{FL}_X$ *with the following properties*:

(1) $i_X(\star) = 0_{\mathcal{FL}_X}$.

(2) *The Lie algebra* $\mathcal{FL}_X$ *is topologically generated by the set* $i_X(X)$.

(3) *For an arbitrary complete normed Lie algebra* $\mathcal{L}$ *and any contracting mapping* $f : X \to \mathcal{L}$ *which sends* $\star$ *to* $0_{\mathcal{L}}$, *there exists a contracting Lie algebra homomorphism* $\hat{f} : \mathcal{FL}_X \to \mathcal{L}$.

The Lie algebra $\mathscr{FL}_X$ with the properties (1) *and* (2) *is essentially unique. For any metric space X the mapping i_X is an isometrical embedding. Free Banach-Lie algebras over the same metric space (X, ρ) with different distinguished points are isometrically isomorphic.*

Proof. It is known **[R, Pe3]** that for any pointed metric space $X = (X, \rho, \star)$ there exists an essentially unique Banach space $B(X, \star)$ (called the free Banach space over X) containing X as a metric subspace in such a way that $\star$ is identified with the zero element of $B(X, \star)$ and any contracting mapping f from X to a Banach space E, taking $\star$ to zero, extends to a unique contracting linear operator $\hat{f} : B(X, \star) \to E$. Now it suffices to put $\mathscr{FL}_X = \mathscr{FL}(B(X, \star))$ and use the above theorem together with known facts about free Banach spaces **[Pe3]**. □

Assertion 2.3. Let $f: E \to F$ be an open linear mapping onto between normed spaces. Then the normed Lie algebra morphism $\hat{f}: \mathscr{FL}(E) \to \mathscr{FL}(F)$ extending f is an open homomorphism onto.

Proof. Denote by A the Banach algebra quotient of $\mathscr{FL}(E)$ by a closed Lie ideal $\ker \hat{f}$. There is a natural continuous homomorphism $i: A \to \mathscr{FL}(F)$. On the other hand, since A contains F as a normed subspace, there is a contracting homomorphism $\widehat{\mathrm{id}}_E : \mathscr{FL}(F) \to A$. It is easy to see that i and $\widehat{\mathrm{id}}_E$ are mutually inverse maps. This proves that A and $\mathscr{FL}(F)$ are isomorphic and $\hat{f}$ is a quotient homomorphism between Banach algebras, as desired. □

3. Couniversal Banach-Lie groups.

THEOREM 3.1. *For any normed space E, the free Banach-Lie algebra $\mathscr{FL}(E)$ is enlargable.*

Proof. If $\dim E = 1$, it is trivial. Otherwise, use Theorem 2.1 and the following fact: any centerless Banach-Lie algebra is enlargable **[vEK]**. □

COROLLARY 3.2. *For any pointed metric space $(X, \rho, \star)$, the free Banach-Lie algebra $\mathscr{FL}_X$ is enlargable.* □

THEOREM 3.3 **[Ś2]**. *Every Banach-Lie algebra is a quotient algebra of an enlargable Banach-Lie algebra.*

Proof. Denote by $\mathbf{g}^+$ the Banach space of an arbitrary Banach-Lie algebra $\mathbf{g}$. The identity mapping $\mathrm{id}_{\mathbf{g}}$ extends to a quotient Banach-

Lie algebra homomorphism from $\mathscr{FL}(\mathbf{g}^+)$ onto $\mathbf{g}$ (Theorem 2.1 and Assertion 2.3). Finally, $\mathscr{FL}(\mathbf{g}^+)$ is enlargable. □

There exists still another proof of the above result, sketched in **[Pe5]**; it is based on nonstandard Lie theory **[Pe4]**.

THEOREM 3.4. *Let τ be a cardinal number. There exists a couniversal Banach-Lie algebra $\mathbf{g}$ of density τ. In other terms, $\mathbf{g}$ contains a dense subset of cardinality $\leq \tau$ and for every other Banach-Lie algebra $\mathbf{h}$ with the same property, there exists a quotient Lie algebra homomorphism onto, $\mathbf{g} \to \mathbf{h}$. In particular, there exits a couniversal separable Banach-Lie algebra.*

Proof. The desired Banach-Lie algebra is $\mathscr{FL}(l_1(\tau))$. One should take into account that a Banach space of density $\leq \tau$ is a quotient space of the Banach space $l_1(\tau)$ **[LT]** and use Theorems 2.1, 3.1 and Assertion 2.3. □

THEOREM 3.5. *Let τ be a cardinal number. Then there exists a couniversal connected Banach-Lie group G of density τ. In other terms, G contains a dense subset of cardinality $\leq \tau$ and any other connected Banach-Lie group with the same property is a quotient Lie group of G. In particular, there exists a couniversal separable Banach-Lie group.*

Proof. Take as G a connected simply connected Banach-Lie group corresponding to the Banach-Lie algebra $\mathscr{FL}(l_1(\tau))$ (use Theorem 3.1). Let H be an arbitrary connected Banach-Lie group of density $\leq \tau$. According to 3.4, the Lie algebra $\mathrm{Lie}(H)$ is a quotient Banach-Lie algebra of $\mathscr{FL}(l_1(\tau))$; let π denote the corresponding quotient homomorphism. It follows from Th. 3.6.2.1, Prop. 3.6.4.10(i), and Prop. 3.4.4.8 in **[Bou]** and the connectedness of H that there is a quotient Banach-Lie group morphism from G onto H. □

In particular, every connected finite dimensional Lie group is a quotient group of an arbitrary couniversal Banach-Lie group.

4. On a question of Kaplansky on NSS groups. The author considers the following two results as a development of some ideas of Gelbaum **[Ge]**.

THEOREM 4.1. *Let $X = (X, \rho, \star)$ be a pointed metric space of diameter* diam $X \leq 1$. *Then the image $\exp_{\mathscr{FL}_X}(X \backslash \{\star\})$ of the set*

$X\backslash\{\star\}$ under the exponential mapping forms a free group basis for a subgroup generated by that set in the simply connected Banach-Lie group associated to $\mathcal{FL}_X$.

Proof. The group $\mathrm{SU}(2)$ contains a free group with an infinite number of generators **[DGD]**. By virtue of a theorem of Mycielski **[My]**, for any non-trivial irreducible word $w(x_1, \ldots, x_n)$ the identity $w = 0$ holds over no neighbourhood of zero in $\mathrm{SU}(2)$ (otherwise the same identity would be true over the whole of $\mathrm{SU}(2)$).

Let $x_1, \ldots, x_n$ be an arbitrary collection of distinct points in $X\backslash\{\star\}$ and let ε be the minimum of distances $\rho(x_i, x_j)$, $i \neq j$, and $\rho(x_i, \star)$. For any n and any irreducible word $w(x_1, \ldots, x_n)$ there are elements $u_1, \ldots, u_n$ in the Lie algebra $\mathbf{su}(2)$ such that $w(\exp(u_1), \ldots, \exp(u_n)) \neq e_{\mathrm{SU}(2)}$ and $\|u_i\| \leq \varepsilon$, where $\|\cdot\|$ is a fixed submultiplicative norm on $\mathbf{su}(2)$ (say, a doubled operator norm). The composition, f, of the mapping $x \mapsto (\rho(x, x_1), \ldots, \rho(x, x_n), \rho(x, \star))$ and a linear mapping from $\mathbf{R}^{n+1}$ to $\mathbf{su}(2)$ sending the images of x_i to u_i and the image of $\rho(x, \star)$ to 0, is a contracting map from X to $\mathbf{su}(2)$, sending x_i to u_i and $\star$ to 0. Therefore, it extends to a Banach-Lie algebra morphism $\hat{f}\colon \mathcal{FL}_X \to \mathbf{su}(2)$. Furthermore, there exists a Lie group morphism $f^\star$ from the simply connected Lie group associated with $\mathcal{FL}_X$ to $\mathrm{SU}(2)$ commuting with the corresponding exponential mappings. Now it is clear that

$$f^\star[w(\exp_{\mathcal{FL}_X}(x_1), \ldots, \exp_{\mathcal{FL}_X}(x_n))] = \exp_{\mathbf{su}(2)}[w(u_1, \ldots, u_n)] \neq e_{\mathrm{SU}(2)}. \qquad \square$$

COROLLARY 4.2. *An arbitrary metrizable topological space X can be homeomorphically embedded into a Banach-Lie group G as a free group basis for a subgroup $\mathrm{gp}_G(X)$ generated by X in G.*

Proof. Follows from the preceding theorem after an appropriate metrization of X. □

Now we will show that Theorem B (Introduction) admits a Lie-theoretic proof.

COROLLARY 4.3. *Let X be a submetrizable Tychonoff topological space. Then the free topological group $F(X)$ over X has no small subgroups.*

Proof. Pick a continuous one-to-one mapping f from X to a metrizable topological space Y. Let i_Y be a homeomorphic embedding of Y into a Banach-Lie group G as a free group basis for a subgroup $\mathrm{gp}_G(Y)$ generated by $i_Y(Y)$ in G. The composition $i_Y \circ f$ extends to a continuous homomorphism $\widehat{i_Y \circ f}: F(X) \to G$ by the very definition of a free topological group **[M, Gr, A2]**. Since any Banach-Lie group has no small subgroups (**[Bou]**, corol. 1 de Th. 3.4.2.2), then there is a neighbourhood U of unity in G that contains no small subgroups. This property is shared by a neighbourhood of unity $(\widehat{i_Y \circ f})^{-1}(U)$ in $F(X)$. □

THEOREM 4.4. **[Pe1, Pe2, SU]** *Every topological group is a topological quotient group of a group with no small subgroups.*

Proof. Any topological space—in particular, G—is an image of an appropriate submetrizable Tychonoff topological space X under a quotient mapping π **[J]**. Extend π to an open homomorphism $\hat{\pi}: F(X) \to G$ **[A2]** and apply Theorem 4.2. □

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This paper is dedicated to the author's topologist friends.

REFERENCES

[AE] R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math., **6** (1956), 397–403.

[A1] A.V. Arhangel'skiĭ, *Any topological group is a quotient group of a zero-dimensional topological group*, Soviet Math. Dokl., **23** (1981), 615–618.

[A2] ——, *Classes of topological groups*, Russian Math. Surveys, **36** (1981), 151–174.

[Bou] N. Bourbaki, *Groupes et Algèbres de Lie,* ch. 2 et 3, Hermann, Paris.

[DGD] J. De Groot and T. Dekker, *Free subgroups of the orthogonal group*, Comp. Math., **12** (1954), 134–136.

[Ge] B. R. Gelbaum, *Free topological groups*, Proc. Amer. Math. Soc., **12** (1961), 737–743.

[Go] R. Goldblatt, *Topoi, the Categorial Analysis of Logic*, North-Holland, Amsterdam a.o., 1984.

[Gr] M. I. Graev, *Free topological groups*, Amer. Math. Soc. Transl., **35** (1951), 61 pp.

[J] H. J. K. Junnila, *Stratifiable pre-images of topological spaces.* Topology, Vol. II (Proc. Fourth Colloq., Budapest, 1978), pp. 689–703, Collog. Math. Soc. János Bolyai 23, North-Holland, Amsterdam-New York, 1980.

[Ka] I. Kaplansky, *Lie Algebras and Locally Compact Groups*, Chicago University Press, Chicago, 1971.

[Kc] V. Kac, *Constructing groups associated to infinite-dimensional Lie algebras*, Infinite-Dimensional Groups with Applications, (V. Kac, ed.), MSRI Publ. **4**, Springer-Verlag, Berlin and New York, 1985, pp. 167–216.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, Vol.* 1, *Sequence Spaces*, SpringerVerlag, Berlin a.o., 1977.

[Mc] A. I. Mal'cev, *Free topological algebras*, Amer. Math. Soc. Transl. (2), **17** (1961), 173–200.

[M] A. A. Markov, *Three papers on topological groups*, Amer. Math. Soc. Transl., no. 30 (1950), 120 pp.

[Mo] S. A. Morris, *Quotient groups of topological groups with no small subgroups*, Proc. Amer. Math. Soc., **31** (1972), 625–626.

[MT] S. A. Morris and H. B. Thompson, *Free topological groups with no small subgroups*, Proc. Amer. Math. Soc., **46** (1974), 431–437.

[My] J. Mycielski, *On the extension of equalities in connected topological groups*, Fund. Math., **44** (1957), 300–302.

[Pe1] V. G. Pestov, *On the structure and embeddings of topological groups*, Manuscript deposited at VINITI (Moscow) on April 13, 1981, no. 1495-81 Dep, 41 pp. (in Russian).

[Pe2] ——, *Topological groups and algebraic envelopes of topological spaces*, Ph.D. thesis, Moscow State University, Moscow, 1983, 78 pp., (in Russian).

[Pe3] ——, *Free Banach spaces and representations of topological groups*, Functional Anal. Appl., **20** (1986), 70–72.

[Pe4] ——, *Fermeture non standard des groupes et algèbres de Lie banachiques*, C. R. Acad. Sci. Paris, Ser. I, **306** (1988), 643–645.

[Pe5] ——, *An invitation to nonstandard superanalysis*, Seminar report 149/1990, Dipartimento di Matematica, Università di Genova, Nov. 1990, 21 pp.

[R] D. A. Raĭkov, *Free locally convex spaces for uniform spaces*, Mat. Sb. (N.S.), **63** (1964), 582–590 (in Russian).

[SU] O. V. Sipacheva and V. V. Uspenskiĭ, *Free topological groups with no small subgroups, and Graev metrics*, Moscow Univ. Math. Bull., **42** (1987), 24–29.

[Ś1] S. Świerczkowski, *Embedding theorems for local analytic groups*, Acta Math., **114** (1965), 207–235.

[Ś2] ——, *The path-functor on Banach-Lie algebras*, Indag. Math., **33** (1971), 235–239.

[T] H. B. Thompson, *A remark on free topological groups with no small subgroups*, J. Austral. Math. Soc., **18** (1974), 482–484.

[U] V. V. Uspenskiĭ, *On the topology of a free locally convex space*, Soviet Math. Dokl., **27** (1983), 771–775.

[vEK] W. T. van Est and T. J. Korthagen, *Non-enlargible Lie algebras*, Indag. Math., **26** (1964), 15–31.

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