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ANY KNOT COMPLEMENT COVERS AT MOST ONE KNOT COMPLEMENT

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It follows from Culler, Gordon, Luecke and Shalen's Cyclic Surgery Theorem that any knot complement is covered by at most two knot complements. Gonzales-Acuna and Whitten proved a result on the other direction: A given knot complement can cover at most finitely many knot complements. This paper is to show that the best possible result in this direction holds: A given knot complement can nontrivially cover at most one knot complement. Moreover, if the knot is not a torus knot, then the covering map is unique up to equivalence.

Given a 3-manifold M, there are generically infinitely many manifolds which cover M. However, if we are restricted to the category of knot complements, the situation is quite different. It can be shown (see Lemma 1 and below) that if the complement $E(K)$ of a knot K is n-fold covered by some knot complement, then the covering is cyclic, and K admits a cyclic surgery, i.e. a Dehn surgery such that the fundamental group of the resulting manifold is a cyclic group $\mathbf{Z}_n$. It follows from the Cyclic Surgery Theorem of **[CGLS]** that if K is not a torus knot, then there are at most two such coverings. The situation is also clear if K is a torus knot: By a theorem of Moser **[M]**, a Dehn surgery on a (p, q) torus knot $T(p, q)$ is a cyclic surgery if and only if the surgery coefficient is $(kpq \pm 1)/k$ for some k. Now the $kpq \pm 1$ fold cyclic covering of the complement $E(K)$ of K is always homeomorphic to $E(K)$ itself, with possibly an orientation reversing homeomorphism. So $E(K)$ is only covered by one knot complement, although there are infinitely many different covering maps.

In this paper we will study a closely related problem: How many knot complements are nontrivially covered by a given knot complement $E(K)$? The problem was studied by Gonzales-Acuna and Whitten in **[GW]**, where they proved that a knot complement covers at most finitely many knot complements up to homeomorphism. The main result of this paper is

THEOREM 1. *The complement $E(K)$ of any knot K can nontrivially cover at most one knot complement $E(K')$ up to homeomorphism.*

REMARK 1. We say two coverings $p_1, p_2: M \to N$ are equivalent if there is a homeomorphism $h: M \to M$ such that $p_1 = p_2 \circ h$. If K is a torus knot, then as we have seen above, there are infinitely many nonequivalent covering maps $p: E(K) \to E(K)$. In the proof of Theorem 1 we will actually show that if K is not a torus knot, then the degree of the covering map $p: E(K) \to E(K')$ is unique. It follows that p is unique up to equivalence, because a cyclic covering is determined by its degree if the first homology group of N is cyclic.

REMARK 2. A generalization of Gonzales-Acuna and Whitten's Theorem was proved in [**W3**]: Any aspherical 3-manifold with nonempty boundary covers only finitely many 3-manifolds.

All notations not defined in the paper are standard, see [**G**], [**H**], [**BM**], [**S**] and [**T**].

DEFINITIONS. For a knot K in S^3, $N(K)$ denotes the regular neighborhood of K; $E(K)$ denotes the knot complement $S^3 - \operatorname{Int} N(K)$; $(K, n/r)$ denotes the 3-manifold obtained by surgery on K with coefficient n/r, where n and r are coprime integers; $L(n)$ denotes a lens space $L(n, *)$. A strong inversion on K is a π-rotation of S^3 which leaves K invariant and has axis meeting K at exactly two points. Finally, in this paper the terms cover and covering always mean nontrivial ones (i.e., of degrees greater than 1).

The first lemma reduces coverings between knot complements to cyclic coverings.

LEMMA 1. *If $p: E(K) \to E(K_1)$ is a covering between knot complements, then p is a cyclic covering.*

Proof. This is Theorem 1.1 of [**GW**]. □

LEMMA 2. *Suppose f is a nontrivial (i.e. not the identity) periodic map on a torus T. Then f is isotopic to the identity if and only if f is fixed point free and orientation preserving.*

Proof. Let A_f be the induced matrix of f on $H_1(T)$. By 12.4 of [**H**], under a suitable basis of $H_1(T)$, A_f is one of the seven matrices on p. 123 of [**H**]. If f is fixed point free and orientation preserving, then we have $\det A_f = 1$, and the Lefschetz number $L(f) = 2 - \operatorname{tr} A_f = 0$. These conditions force A_f to be the identity matrix. Hence f is isotopic to the identity.

Conversely, if f is isotopic to the identity, then f is orientation preserving and $L(f) = 0$. Then it is an elementary fact that f is the identity or is fixed point free. □

The next lemma is quoted from **[W1]**. It builds up a connection between coverings of knot complements and cyclic surgeries on knots. The first part is also in **[GW]** with somewhat different proof.

LEMMA 3. *Let K be a nontrivial knot.*

(1) *If $E(K)$ is n-fold cyclic covered by $E(\tilde{K})$ for some knot $\tilde{K}$, then $\pi_1(K, n/r) = \mathbf{Z}_n$.*

(2) *If $(K, n/r) = L(n)$, then $E(K)$ is n-fold cyclic covered by $E(\tilde{K})$ for some knot $\tilde{K}$.*

(3) *If K is not a torus knot, then $r = 1$ in* (1) *and* (2).

Proof. (1) If $E(\tilde{K}) \to E(K)$ is an n-fold cyclic covering, then the action of deck transformation group $\Pi = (\tau | \tau^n = 1)$ on $E(\tilde{k})$ is orientation preserving and fixed point free. Let $T = \partial(N(\tilde{K}))$. Then $\tau|_T$, the restriction of τ on T, is an orientation preserving and fixed point free periodic map. By Lemma 2, $\tau|_T$ is isotopic to the identity. In particular it sends meridian to meridian. So we can use the standard disk extention to extend $\tau|_T$ over the solid torus $N(\tilde{k})$ in a periodic way, getting an action of Π on S^3 which has no fixed point in $S^3 - \tilde{K}$. As the action is cyclic and orientation preserving, the fixed point set of Π is either $\tilde{K}$ or the empty set. Now $\tilde{K}$ is knotted because K is, so by Smith Conjecture it cannot be the fixed point set of Π. Hence Π acts freely on $S^3 = E(\tilde{K}) \cup N(\tilde{K})$, and

$$S^3/\Pi = (E(\tilde{K})/\Pi) \cup (N(\tilde{K})/\Pi) = E(K) \cup (S^1 \times D^2) = (K, n/r)$$

where the surgery coefficient is of the form n/r because $\Pi = \mathbf{Z}_n$.

(2) If $(K, n/r) = L(n)$, then we have an n-fold cyclic covering $q: S^3 \to L(N)$. the meridian m of K represents a generator of $H_1((K, n/r)) = \pi_1((K, n/r)) = \mathbf{Z}_n$; so $q^{-1}(m)$ is connected. It implies that $q^{-1}(N(K))$ is a single solid torus. Let $\tilde{K}$ be the central curve of $q^{-1}(N(K))$. Then $q^{-1}(N(K))$ is a regular neighborhood of $\tilde{K}$, and q restricting to $E(\tilde{K}) = S^3 - q^{-1}(N(K))$ is a cyclic covering onto $E(K)$.

(3) follows from **[CGLS]**. □

By a hyperbolic structure (or metric) we mean one which is complete and has finite volume. A hyperbolic 3-manifold M is a manifold

whose interior admits such a hyperbolic structure. We use M_ρ to denote $\operatorname{Int} M$ together with a given hyperbolic metric ρ.

LEMMA 4. *Let h be a map on a hyperbolic 3-manifold M_ρ. Then the following two statements are equivalent:*

(1) *h is conjugate to an isometry on M_ρ;*

(2) *there exists a hyperbolic metric ρ_1 on M under which h is an isometry.*

Proof. Suppose g is a self-homeomorphism of M such that ghg^{-1} is an isometry on M_ρ. Let $\rho_1 = g^*\rho$ be defined by $\rho_1(x, y) = \rho(g(x), g(y))$ for all $x, y \in M$. Then we have

$$\begin{aligned}\rho_1(h(x), h(y)) &= \rho(gh(x)k, gh(y)) = \rho(ghg^{-1}g(x), ghg^{-1}g(y))\\ &= \rho(g(x), g(y)) = \rho_1(x, y);\end{aligned}$$

i.e., h is an isometry on M_{ρ_1}.

Conversely, suppose (2) is true. By Mostow's Rigidity Theorem [T, 5.7.2], id: $\pi_1(M_\rho) \to \pi_1(M_{\rho_1})$ can be realized by a unique isometry $g: M_\rho \to M_{\rho_1}$. So we have

$$\begin{aligned}\rho(x, y) &= \rho_1(g(x), g(y)) = \rho_1(hg(x), hg(y))\\ &= \rho(g^{-1}hg(x), g^{-1}hg(y));\end{aligned}$$

i.e., $g^{-1}hg$ is an isometry on M_ρ. □

The next two lemmas are used to prove Theorem 1 for hyperbolic knots.

LEMMA 5. *Suppose $E = E(K)$ is hyperbolic, and $\Pi_1 = (\tau_1 | \tau_1^p = 1)$ and $\Pi_2 = (\tau_2 | \tau_2^q = 1)$ act freely on E. Then there is a hyperbolic metric ρ on $\operatorname{Int} E$ and a homeomorphism g on E such that $G = \langle \tau_1, g\tau_2 g^{-1} \rangle$ is a finite abelian group.*

Proof. Let $p_i: E \to E/\Pi_i$ be the covering map. Since E is hyperbolic, E/Π_i is homotopic to a hyperbolic manifold [T, 6.7.3]. Since E/Π_i is a P^2-irreducible 3-manifold (by [MSY, Theorem 3]) with a torus boundary component, it is a Haken manifold. So E/Π_i itself admits a hyperbolic structure. For any given hyperbolic metric $\overline{\rho}$ on E/Π_i, let $\rho_i = p_i^*\overline{\rho}$ be the pullback metric on E. Then τ_i is an isometry on E_{ρ_i}.

Now fix a metric ρ_1 on E under which τ_1 is an isometry. We have just shown that τ_2 is an isometry under certain hyperbolic metric.

So by Lemma 4 it is conjugate to an isometry with respect to ρ_1; i.e., there is a homeomorphism g such that $g\tau_2 g^{-1}$ is an isometry on E_{ρ_1}. Thus $G = \langle \tau_1, g\tau_2 g^{-1} \rangle$ is a group of isometries. By the Mostow-Thurston Theorem [T, 5.7.4], G is a finite group.

It remains to show that G is abelian. Let E_ε be the submanifold of $\operatorname{Int} E$ consisting of points contained in some hyperbolic ball of radius ε. Choose ε small enough so that E_ε is homeomorphic to E (see 5.10.2 and 5.11.1 of [T]). Since the elements of G are isometries on E, E_ε is invariant under the G action. Especially, $T = \partial E_\varepsilon$ is invariant under the G action. As $\tau_1|_T$ and $\tau_2|_T$ are fixed point free orientation preserving periodic maps on T, they both induce the identity map on $\pi_1(T)$ by Lemma 2. Thus every element in G induces identity on $\pi_1(T)$. Again by Lemma 2, except the identity, every element in $G|_T$ is a fixed point free orientation preserving periodic map. In other words, $G|_T$ is a free action. So we can consider T as a covering space over $T/(G|_T)$, and $G|_T$ as the covering transformation group. Since $T/(G|_T)$ is a torus, the quotient group $G|_T = \pi_1(T/(G|_T))/\pi_1(T)$ is abelian. □

LEMMA 6. *If a non-torus knot K admits a symmetry which is not a strong inversion, then $E(K)$ cannot be covered by a knot complement.*

Proof. In [**WZ**], Wang and Zhou proved that if a non-torus knot K admits a symmetry which is not a strong inversion, then there is no cyclic surgery on K. Lemma 6 then follows from Lemma 3(2). □

We use $K(p, q)$ to denote the (p, q)-torus knot, and $K(p, q; r, s)$ to denote the (r, s)-cable of $K(p, q)$. Without loss of generality we may assume $p > q > 0$ (otherwise change the orientation of S^3). The next two lemmas are used to prove the theorem for satellite knots. Recall that (K, n) denotes the manifold obtained from S^3 by Dehn surgery on K with slope n.

LEMMA 7. *Suppose K is a satellite knot. Then (K, n) is a lens space if and only if $K = K(p, q; 2, 2pq \pm 1)$ and $n = 4pq \pm 1$.*

Proof. This is a result of Bleiler-Litherland [**BL**] and Wu [**Wu**]. □

Let M be the complement of $K(p, q; 2, 2pq + 1)$, let N be the complement of $K(p, q; 2, 2pq - 1)$. Denote by $\widetilde{M}$ the $4pq+1$ fold cyclic covering of M, and by $\widetilde{N}$ the $4pq - 1$ fold covering of N.

LEMMA 8. *$\widetilde{M}$ is not homeomorphic to $\widetilde{N}$.*

Proof. Let T be a torus which divides M into two maximal Seifert pieces M_1 and M_2, where $M_1 = E(k(p, q))$, and M_2 is a cable space $C(2, 2pq+1)$ (see e.g. [G] for definition of cable spaces). Consider T as the boundary of the knot complement M_1. Choose a pair of simple closed curves m and l on T so that they represent the homology of the standard meridian-longitude pair of $K(p, q)$ (rather than that of $K(p, q; 2, 2pq+1)$).

Denote by $\tilde{m}$, $\tilde{l}$, $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{T}$ the lifting (preimage) of m, l, M_1, M_2, T, respectively. It is easy to show that if c is a simple closed curve in M representing an element k in $\mathbf{Z} = H_1 M$, then its lifting $\tilde{c}$ in an n-fold cyclic covering $\widetilde{M}$ of M has $\gcd(k, n)$ components. Since homologically m is 2 times a meridian of $K(p, q; 2, 2pq+1)$ and since 2 is coprime with $4pq+1$, $\tilde{m}$ is connected. Especially, $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{T}$ are all connected. Since l is null-homologous, $\tilde{l}$ consists of $4pq+1$ parallel simple closed curves. Let l' be one of these curves. Then $[\tilde{l}] = (4pq+1)[l']$. Note that $\tilde{m}$ intersects $\tilde{l}$ transversely at $4pq+1$ points (because m intersects l at one point), so $\tilde{m}$ intersects l' exactly once. Thus the simple closed curves $\tilde{m}$ and l' generate $H_1(\widetilde{T})$.

It is well known that Seifert fibrations of both M_1 and M_2 are unique up to isotopy (see Theorem 3.9 of [S] and its refinement). Let α_i be the simple closed curve on T representing a fiber of M_i. One can check that $[\alpha_1] = pq[m]+[l]$, and $[\alpha_2] = (2pq+1)[m]+2[l]$. Let $\tilde{\alpha}_i$ be the lift of α_i. Since both pq and $2pq+1$ are coprime with $4pq+1$, the $\tilde{\alpha}_i$'s are connected; so each $\tilde{\alpha}_i$ is a fiber of the lifted fibration of $\widetilde{M}_i$. We have

$$[\tilde{\alpha}_1] = pq[\tilde{m}] + [\tilde{l}] = pq[\tilde{m}] + (4pq+1)[\tilde{l}'],$$
$$[\tilde{\alpha}_2] = (2pq+1)[\tilde{m}] + 2[\tilde{l}] = (2pq+1)[\tilde{m}] + 2(4pq+1)[l'].$$

Since $[\tilde{m}]$, $[\tilde{l}']$ generates $H_1(T)$, we can compute the intersection number of $[\tilde{\alpha}_1]$ and $[\tilde{\alpha}_2]$ by

$$[\tilde{\alpha}_1][\tilde{\alpha}_2] = \left|\det \begin{pmatrix} pq & 4pq+1 \\ 2pq+1 & 8pq+2 \end{pmatrix}\right| = 4pq+1.$$

Construct in a similar way a decomposing torus $\widetilde{T}'$ which divides $\widetilde{N}$ into $\widetilde{N}_1$ and $\widetilde{N}_2$, and simple closed curves $\tilde{\beta}_1$, $\tilde{\beta}_2$ which represent fibers of $\tilde{n}_1$ and $\widetilde{N}_2$ respectively. By a similar computation we have $[\tilde{\beta}_1][\tilde{\beta}_2] = 4pq-1$.

If $\widetilde{M}$ and $\widetilde{N}$ are homeomorphic, by the Torus Decomposition Theorem of Jaco-Shalen-Johannson (see [J]) and the uniqueness of fibration of $\widetilde{M}_i$ (see e.g. [J, Theorem VI.18]), there is a homeomorphism $\varphi\colon \widetilde{M} \to \widetilde{N}$ such that $\varphi(\widetilde{T}) = \widetilde{T}'$ and $\varphi(\tilde{\alpha}_i) = \tilde{\beta}_i$, $i = 1, 2$, so $[\tilde{\alpha}_1][\tilde{\alpha}_2] = [\tilde{\beta}_1][\tilde{\beta}_2]$. But this contradicts the above computation. □

LEMMA 9. *Suppose $E(K_1) \to E(K_2)$ is a cyclic covering between knot complements. If one of the K_i is a torus knot $K(p, q)$, then both $E(K_1)$ and $E(K_2)$ are homeomorphic to $E(K(p, q))$, and the covering degree is $rpq \pm 1$ for some r.*

Proof. If K_i is a torus knot, then it is a Seifert manifold. Hence both $E(K_1)$ and $E(K_2)$ are Seifert manifolds. It implies that both K_1 and K_2 are torus knots (see [J, IX 22]). So we may assume $K_2 = K(p, q)$.

By the work of Moser [M], $\pi_1(k(p, q), n/r)$ is cyclic if and only if $n = rpq \pm 1$. Thus $E(K_1)$ is an $rpq \pm 1$ fold cyclic covering of $E(k(p, q))$. But such a covering space is homeomorphic to $E(K(p, q))$, because $T(p, q)$ is a fibered knot with holonomy a period map of order pq (see e.g. [R]). □

With Lemma 9 in hand, we need only to prove Theorem 1 for hyperbolic knots and satellite knots.

Proof of Theorem 1 *for hyperbolic knots.* Suppose the complement $E(K)$ of a hyperbolic knot K covers two knot complements $E(K_1)$ and $E(K_2)$. By Lemma 1, these coverings are cyclic. Let $\Pi_1 = (\tau_1 | \tau_1^p = 1)$ and $\Pi_2 = (\tau_2 | \tau_2^q = 1)$ be the deck transformation groups. For any homeomorphism g of $E(K)$, $E(K)/\langle \tau_i \rangle$ is homeomorphic to $E(K)/\langle g\tau_i g^{-1} \rangle$. So by Lemma 5, after replacing τ_2 by some $g\tau_2 g^{-1}$ if necessary, we may assume that τ_1, τ_2 and the hyperbolic metric on $E(K)$ have been chosen so that the action of $G = \langle \tau_1, \tau_2 \rangle$ is the action of a finite abelian group.

Claim. $G = \langle \tau_1 \rangle = \langle \tau_2 \rangle$.

Otherwise, suppose $\langle \tau_1 \rangle = \langle \tau_2 \rangle$, say, is a proper subgroup of G. As G is abelian, $\langle \tau_1 \rangle$ is a normal subgroup; so the non-trivial group $G/\langle \tau_1 \rangle$ acts on $E(K)\langle \tau_1 \rangle = E(K_1)$. Since the restriction of G on $\partial E(K)$ is a free action, it follows from covering space theory that the restriction of $G/\langle \tau_1 \rangle$ on $\partial E(K_1)$ is a free action; so the action of

$G/\langle\tau_1\rangle$ on $E(K_1)$ is not a strong inversion. By Lemma 3 $E(K_1)$ admits a cyclic surgery, but by Lemma 6 it cannot admit such a surgery, a contradiction. So we have $G = \langle\tau_1\rangle = \langle\tau_2\rangle$.

Therefore, $E(K_1)$ and $E(K_2)$ are homeomorphic. □

Proof of Theorem 1 *for satellite knots.* Suppose the complement $E(K)$ of a satellite knot K covers two non-homeomorphic knot complements $E(K_1)$ and $E(K_2)$, with degree n_1 and n_2 respectively. By Lemma 3, $E(K_i)$ admits a cyclic surgery with coefficient n_i/r_i. By Lemma 7, we must have $K_i = K(p_i, q_i; 2, 2p_iq_i + \varepsilon_i)$ and $n_i = 4p_iq_i + \varepsilon_i$, where $\varepsilon_i = \pm 1$; so $E(K)$ is a $4p_iq_i + \varepsilon_i$ fold cyclic covering of $E(K_i)$. Use the notations in the proof of Lemma 8, and write $E(K_i) = M_1^{(i)} \cup M_1^{(i)} = E(K(p_i, q_i)) \cup C(2, 2p_iq_1 + \varepsilon_i)$. It was implied in the proofs of Lemmas 8 and 9 that the preimage $\widetilde{M}_1^{(i)}$ of $M_1^{(i)}$ in $E(K)$ is homeomorphic to $M_1^{(i)}$ itself, while by the uniqueness of torus decomposition of $E(K)$ we must have $\widetilde{M}_1^{(1)} \cong \widetilde{M}_1^{(2)}$. Therefore, $E(K(p_1, q_1)) \cong \widetilde{M}_1^{(1)} \cong \tilde{m}_1^{(2)} \cong E(K(p_2, q_2))$. It is well known that torus knots are determined by their complements, so we have $(p_1, q_1) = (p_2, q_2) = (p, q)$ for some p, q. Now if $K_1 = K(p, q; 2, 2pq + \varepsilon_1)$ and $K_2 = K(p, q; 2, 2pq + \varepsilon_2)$ are not the same, then $\varepsilon_1 = -\varepsilon_2$, and $E(K)$ is a $4pq + \varepsilon$ fold covering of $K(p, q; 2, 2pq + \varepsilon)$ for both $\varepsilon = \pm 1$, which is impossible by Lemma 8. □

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