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L^n SOLUTIONS OF THE STATIONARY AND NONSTATIONARY NAVIER-STOKES EQUATIONS IN R^n

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It is shown that the Navier-Stokes equations in the whole space R^n $(n \geq 3)$ admit a unique small stationary solution which may be formed as a limit of a nonstationary solution as $t \to \infty$ in L^n-norms.

0. Introduction. As is well known, the existence of solutions to the exterior stationary Navier-Stokes equations was studied by Finn [**2**, **3**], and small solutions from Finn [**2**, **3**] may be formed as limits of nonstationary solutions as time $t \to \infty$ in local or global L^2-norms (cf. Heywood [**9**, **10**], Galdi and Rionero [**6**], Miyakawa and Sohr [**16**], Borchers and Miyakawa [**1**]) and in the norms of other function spaces (cf. Heywood [**11**], Musuda [**14**]). However, it is still unknown even in the case of whole spaces whether or not

$$\|v(t) - w\|_n + t^{1/2}\|Dv(t) - Dw\|_n + t^{1/2}\|v(t) - w\|_\infty \to 0 \quad \text{as } t \to \infty, \tag{0.1}$$

provided that w and v are, respectively, the solutions to the stationary Navier-Stokes equations

$$-\Delta w + (w \cdot D)w + d\overline{p} = f, \quad D \cdot w = 0 \quad \text{in } R^n \tag{0.2}$$

and the nonstationary Navier-Stokes equations

$$\begin{aligned} v_t - \Delta v + (v \cdot D)v + D\overline{\overline{p}} = f, \quad D \cdot v = 0 \quad \text{in } R^n \times (0, \infty), \\ v(0) = v_0 \quad \text{in } R^n. \end{aligned} \tag{0.3}$$

Here and in what follows, $n \geq 3$ denotes the space dimension, $\overline{p}$ and $\overline{\overline{p}}$ represent the pressures associated with w and v, respectively, $D =$ the gradient, $f = f(x)$ is a prescribed function, the dot $\cdot$ denotes the scalar product in R^n, and $\|\cdot\|_r$ denotes the norm of the Lebesgue space $L^r = L^r(R^n; R^n)$.

The purpose of the paper is to show that (0.2) and (0.3) admit small regular solutions w and $v(t)$ in L^n, respectively, such that (0.1) is valid. The problem above is, as usual, said to be a stability problem for w, which has been studied by Kozono and Ozawa [13] in the case of bounded domains. From our view point, the global existence results of Kato [12] may be regarded as the stability theorems around the rest flow $w \equiv 0$.

In this paper we shall use the following spaces.

$$\begin{aligned}
C_0^\infty &= \text{the set of compactly supported solenoidal } u \in C^\infty(R^n; R^n), \\
J^r &= \text{the completion of } C_0^\infty \text{ in } L^r \text{ for } 1 < r < \infty, \\
W^{k,r} &= \text{the Sobolev space } W^{k,r}(R^n; R^n) \text{ for } 1 < r < \infty \text{ and } k = 1, 2, \\
\widehat{W}^{1,r} &= \{u \in L^{nr/(n-r)};\ Du \in L^r(R^n; R^{n^2})\} \text{ for } 1 < r < n, \\
\widehat{W}^{2,r} &= \{u \in W^{1,nr/(n-r)};\ D^2u \in L^r(R^n; R^{n^3})\} \text{ for } 1 < r < n/2,
\end{aligned}$$

where $D^2 =$ the Hessian matrix $[D_iD_j]_{n\times n}$ with $D_k = \partial/\partial x_k$. Moreover, we denote by P the linear bounded projection from L^r onto J^r for $1 < r < \infty$ (cf. [15] for details), by A the Stokes operator $-P\Delta$ associated with the domain $W^{2,r} \cap J^r$ for $1 < r < \infty$, by $(\cdot, \cdot)$ the duality pairing between L^r and $(L^r)^*$ for $1 \le r < \infty$, and we set

$$\|u\|_{-1,r} = \sup\{|(u, v)|;\ v \in C_0^\infty,\ \|Dv\|_{r/(r-1)} = 1\} \quad \text{for } 1 < r < \infty.$$

Our main results read as follows.

THEOREM 0.1. *For $n \ge 3$ there is a small $0 < d < 1$ such that* (0.1) *admits a unique solution*

$$w \in J^n \cap \widehat{W}^{1,2n/3} \cap \widehat{W}^{1,2n/5} \quad \text{with } \|Dw\|_{n/2} \le d$$

satisfying

$$\begin{aligned}
&\|Dw\|_{n/2} + \|w\|_n \le C\|f\|_{-1,n/2}, \\
&\|Dw\|_{2n/5} + \|Dw\|_{2n/3} + \|w\|_{2n} + \|w\|_{2n/3} \\
&\qquad \le C(\|f\|_{-1,2n/5} + \|f\|_{-1,2n/3})
\end{aligned}$$

with C independent of f and w, provided that

$$f \in C_0^\infty \quad \text{and} \quad \|f\|_{-1,n/2} \le d^2.$$

THEOREM 0.2. *Let $n \ge 3$, $f \in C_0^\infty$, $v_0 \in J^n$, and let $\|v_0\|_n$ and $\|f\|_{-1,2n/5} + \|f\|_{-1,2n/3}$ be sufficiently small. Then* (0.3) *admits a unique solution*

$$v \in BC([0,\infty); J^n) \quad \text{and} \quad t^{1/2}D(v(t) - w) \in BC([0,\infty); L^n(R^n; R^{n^2}))$$

such that (0.1) *is valid, where* w *is the solution of* (0.2) *from Theorem* 0.1 *and* BC *denotes the class of bounded and continuous functions.*

Since there is no boundary to worry about in the whole space, our proof largely depends on the fact that P commutes with D, and also based on the theory of analytic semigroups in various L^r spaces. Such an approach is developed from Fujita and Kato **[5]** and Kato **[12]**.

In §1 we prove Theorem 0.1. In §2 we obtain resolvent estimates for the perturbed operator $Au + P(u \cdot D)w + P(w \cdot D)u$ and therefore deduce decay estimates for the analytic semigroups generated by the perturbed operator. Theorem 0.2 is proved in §3.

1. Proof of Theorem 0.1. From the Sobolev inequality

$$(1.1) \qquad C^{-1}\|u\|_{nr/(n-2r)} \le \|Du\|_{nr/(n-r)} \le C\|D^2u\|_r \quad \text{for } 1 < r < n/2,$$

the Calderon-Zygmund inequality (cf. **[7]**)

$$\|D^2u\|_r \le C\|\Delta u\|_r \quad \text{for } 1 < r < \infty,$$

the density of $\{Au;\, u \in C_0^\infty\}$ in J^r for $1 < r < n/2$, and the fact that P commutes with Δ, it follows that the Stokes operator A can be extended to a bounded and invertible operator from $J^{nr/(n-2r)} \cap \widehat{W}^{2,r}$ onto J^r for $1 < r < n/2$. Consequently, we set the operator

$$T\colon J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3} \to \widehat{W}^{2,r} \quad \text{for } n/3 < r < n/2$$

such that

$$Tw = T_f w = A^{-1}(f - P(w \cdot D)w).$$

It is easy to see that to seek solutions of (0.2) means to seek fixed points of T.

Let $2n/5 \le r \le 2n/3$, $w \in J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3}$, $v \in C_0^\infty$. Then by the divergence condition $D \cdot w = 0$, we have

$$\begin{aligned}(DTw\,,\, Dv) &= (f\,,\, v) - ((w \cdot D)w\,,\, v) \\ &= (f\,,\, v) + (w\,,\, (w \cdot D)v) \\ &\le (f\,,\, v) + \|w\|_n\|w\|_{nr/(n-r)}\|Dv\|_{r/(r-1)}.\end{aligned}$$

Combining this with the inequality (cf. **[17, 18]**)

$$\|DTw\|_r \le C\sup\{|(DTw\,,\, Dv)|;\, v \in C_0^\infty,\ \|Dv\|_{r/(r-1)} = 1\}$$

with $C = C(n)$, we have, by (1.1),

$$\|DTw\|_r \le C(n)(\|f\|_{-1,r} + \|Dw\|_{n/2}\|Dw\|_r),$$

and, similarly, for u, $w \in J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3}$

$$\|DTw - DTu\|_r \le C(n)(\|Dw\|_{n/2} + \|Du\|_{n/2})\|Dw - Du\|_r .$$

Consequently, there is a small positive d such that T is a contraction mapping from the complete metric space

$$\{w \in J^n \cap \widehat{W}^{1,2n/5} \cap \widehat{W}^{1,2n/3};\ \|Dw\|_{n/2} \le d\}$$

into itself provided that $f \in C_0^\infty$ with $\|f\|_{-1,n/2} < d^2$. We thus obtain the desired assertion by making use of the contraction mapping principle and (1.1). The proof is complete.

2. $L^p - L^q$ estimates. In the remainder of the paper we denote by w the solution of (0.2) given in Theorem 0.1, and by C the various constants which are always independent of the quantities u, v, w, f, a, t, and z. Moreover we set

$$S = \{z \in \mathbb{C};\ -3\pi/4 < \arg z < 3\pi/4\},$$

$$Lu = Au + Bu; \quad Bu = P(u\cdot D)w + P(w\cdot D)u,$$

$$L^*u = Au + B^*u; \quad B^*u = -P(w\cdot D)u + \sum_{i=1}^{n} Pu^i Dw^i$$

for $u = (u^1, \dots, u^n)$ and $w = (w^1, \dots, w^n)$.

In arriving at $L^p - L^q$ estimates, we begin with the resolvent estimates for L and L^*.

LEMMA 2.1. *Let $z \in S$ and $u \in C_0^\infty$. Then we have*

$$|z|\,\|(L+z)^{-1}u\|_r \le C\|u\|_r \quad \text{for } 1 < r < \infty, \tag{2.1}$$

$$|z|\,\|(L^*+z)^{-1}u\|_r \le C\|u\|_r \quad \text{for } 1 < r < \infty, \tag{2.2}$$

$$|z|^{1/2}\|D(L+z)^{-1}u\|_r \le C\|u\|_r \quad \text{for } 1 < r < n, \tag{2.3}$$

$$|z|^{1/2}\|D(L^*+z)^{-1}u\|_r \le C\|u\|_r \quad \text{for } 1 < r < \infty, \tag{2.4}$$

provided that $\|Dw\|_{n/2}$ is sufficiently small;

$$|z|^{3/4}\|(L+z)^{-1}u\|_\infty \le C\|u\|_{2n}, \tag{2.5}$$

$$|z|^{1/2}\|D(L+z)^{-1}u\|_n \le C(\|u\|_n + |z|^{-1/4}\|u\|_{2n}), \tag{2.6}$$

provided that $\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}$ is sufficiently small.

Proof. Let us recall the well-known resolvent estimates for the Stokes operator (cf. **[15]**)

$$|z|\,\|(A+z)^{-1}u\|_r + |z|^{1/2}\|D(A+z)^{-1}u\|_r + \|D^2(A+z)^{-1}u\|_r \le C\|u\|_r \tag{2.7}$$

for $z \in S$, $1 < r < \infty$ and $u \in J^r$, and the Gagliardo-Nirenberg inequality (cf. [4])

$$\|u\|_q \leq C\|u\|_r^{1-h}\|Du\|_p^h \tag{2.8}$$

for $1 < r$, $p \leq q \leq \infty$, $0 \leq h < 1$, $-n/q = h(1-n/p) - (1-h)n/r$, $u \in C_0^\infty$. Let us suppose $z \in S$ and $u \in J^r \cap W^{1,r}$ for $1 < r < \infty$.

Step 1. We prove (2.1) and (2.2). From (2.7), (1.1), the Hölder inequality and the boundedness of P in L^r-spaces it follows that for $1 < r < n/2$, $p = nr/(n-r)$ and $q = nr/(n-2r)$,

$$\begin{aligned}\|B(A+z)^{-1}u\|_r &\leq C\|w\|_n\|D(A+z)^{-1}u\|_p \\ &\quad + C\|Dw\|_{n/2}\|(A+z)^{-1}u\|_q \\ &\leq C\|Dw\|_{n/2}\|D^2(A+z)^{-1}u\|_r \\ &\leq C\|Dw\|_{n/2}\|u\|_r \\ &\leq (1/2)\|u\|_r, \quad \text{by setting } C\|Dw\|_{n/2} < 1/2.\end{aligned}$$

This is together with (2.7) and the identity

$$L + z = (1 + B(A+z)^{-1})(A+z)$$

implies

$$|z|\,\|(L+z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < n/2.$$

Similarly, we have

$$|z|\,\|(L^*+z)^{-1}u\|_r \leq C\|u\|_r \quad \text{for } 1 < r < n/2.$$

This yields for $n < r < \infty$, $v \in L^{r'}$ with $r' = r/(r-1)$,

$$((L+z)^{-1}u, v) = (u, (L^*+z)^{-1}Pv) \leq C|z|^{-1}\|u\|_r\|v\|_{r'}$$

and hence the validity of (2.1) with $n < r < \infty$. Thus (2.1) with $n/2 \leq r \leq n$ follows immediately from the Marcinkiewicz interpolation theorem (cf. [7]). (2.2) is verified in the same way.

Step 2. We prove (2.3). Observing that $1 < r < n$ and applying the condition $D \cdot u = D \cdot w = 0$ and the fact that D commutes with P yields

$$(A+z)^{-1}Bu = \sum_{i=1}^{n} D_i(A+z)^{-1}P(u^i w + w^i u), \tag{2.9}$$

we have, by (2.7) and (1.1),

$$\begin{aligned}\|(A+z)^{-1}Bu\|_r &\leq C|z|^{-1/2}\|w\|_n\|u\|_{nr/(n-r)} \\ &\leq C\|Dw\|_{n/2}\|Du\|_r|z|^{-1/2} \\ &\leq 2^{-1}|z|^{-1/2}\|Du\|_r, \quad \text{by setting } C\|Dw\|_{n/2} < 1/2\end{aligned}$$

and

$$(2.10)\qquad \begin{aligned}\|D(A+z)^{-1}Bu\|_r &\le C\|w\|_n\|u\|_{nr/(n-r)}\\ &\le C\|Dw\|_{n/2}\|Du\|_r\\ &\le (1/2)\|Du\|_r, \quad \text{by setting } C\|Dw\|_{n/2} < 1/2.\end{aligned}$$

Consequently, we have

$$(2.11)\quad \|((A+z)^{-1}B)^k u\|_r \le 2^{-k}|z|^{-1/2}\|Du\|_r, \quad \text{for integer } k > 0,$$

and so

$$D(L+z)^{-1}u = \sum_{k=0}^{\infty} D((A+z)^{-1}B)^k(A+z)^{-1}u \quad \text{in } L^r.$$

Applying (2.10) to the preceding identity repeatedly and using (2.7), we have

$$\|D(L+z)^{-1}u\|_r \le 2\|D(A+z)^{-1}u\|_r \le C|z|^{-1/2}\|u\|_r$$

as required.

Step 3. We prove (2.4). Observing that

$$\begin{aligned}(D_i(L^*+z)^{-1}u, v) &= -(u, (L+z)^{-1}D_iPv)\\ &\le \|u\|_r\|(L+z)^{-1}D_iPv\|_{r'}\end{aligned}$$

for $i = 1, \dots, n$, $1 < r < \infty$, $r' = r/(r-1)$ and $v \in W^{1,r'}$, we need only to show the estimate

$$(2.12)\qquad \|(L+z)^{-1}Du\|_r \le C|z|^{-1/2}\|u\|_r, \quad \text{for } 1 < r < \infty.$$

Indeed, taking (2.9), (1.1) and (2.7) into account, we have for $n \le r < \infty$,

$$\begin{aligned}\|(A+z)^{-1}Bu\|_r &\le C\sum_{i=1}^{n}\|DD_i(A+z)^{-1}P(uw^i + wu^i)\|_{nr/(n+r)}\\ &\le C\|Dw\|_{n/2}\|u\|_r \le (1/2)\|u\|_{r'}\end{aligned}$$

by setting $C\|Dw\|_{n/2} < 1/2$, and hence for $n \le r < \infty$,

$$\begin{aligned}\|(L+z)^{-1}Du\|_r &= \|(1+(A+z)^{-1}B)^{-1}D(A+z)^{-1}u\|_r\\ &\le 2\|D(A+z)^{-1}u\|_r\\ &\le C|z|^{-1/2}\|u\|_r\end{aligned}$$

which arrives at (2.12) for $n \le r < \infty$. Moreover (2.12) with $1 < r < n$ is verified as follows:

$$\begin{aligned}\|(L+z)^{-1}Du\|_r &= \|(1+(A+z)^{-1}B)^{-1}D(A+z)^{-1}u\|_r \\ &\le \|D(A+z)^{-1}u\|_r + |z|^{-1/2}\|D^2(A+z)^{-1}u\|_r \\ &\le C|z|^{-1/2}\|u\|_r,\end{aligned}$$

where we have used (2.11) and (2.7).

Step 4. We prove (2.5). By (2.8) and (2.7), we obtain

$$\begin{aligned}(2.13)\quad \|(A+z)^{-1}u\|_\infty &\le C\|(A+z)^{-1}u\|_{2n}^{1/2}\|D(A+z)^{-1}u\|_{2n}^{1/2} \\ &\le C|z|^{-3/4}\|u\|_{2n},\end{aligned}$$

and, by (2.7), (2.8), (1.1) and (2.9),

$$\begin{aligned}(2.14)\quad \|(A+z)^{-1}Bu\|_\infty &\le C\|(A+z)^{-1}Bu\|_{2n}^{1/2}\|D(A+z)^{-1}Bu\|_{2n}^{1/2} \\ &\le C\|D(A+z)^{-1}Bu\|_{2n/3}^{1/2}\|D(A+z)^{-1}Bu\|_{2n}^{1/2} \\ &\le C\sum_{i=1}^{n}\|u^i w + w^i u\|_{2n/3}^{1/2}\|u^i w + w^i u\|_{2n}^{1/2} \\ &\le C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_\infty \\ &\le (1/2)\|u\|_\infty, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.\end{aligned}$$

We thus obtain

$$\begin{aligned}\|(L+z)^{-1}u\|_\infty &= \|(1+(A+z)^{-1}B)^{-1}(A+z)^{-1}u\|_\infty \\ &\le 2\|(A+z)^{-1}u\|_\infty \le C|z|^{-3/4}\|u\|_{2n}\end{aligned}$$

and hence the validity of (2.5).

Step 5. We prove (2.6). By (1.1), (2.9) and (2.7),

$$\begin{aligned}\|(A+z)^{-1}Bu\|_n &\le C\|D(A+z)^{-1}Bu\|_{n/2} \\ &\le C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}\|u\|_n \\ &\le (1/2)\|u\|_n, \quad \text{by setting } C\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2} < 1/2\end{aligned}$$

and, by (2.9), (2.7) and (1.1),

$$\begin{aligned}\|D(A+z)^{-1}Bu\|_n &\le C\|w\|_n\|u\|_\infty \\ &\le C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2}\|u\|_\infty \\ &\le (1/2)\|u\|_\infty, \quad \text{by setting } C\|w\|_{2n/3}^{1/2}\|w\|_{2n}^{1/2} < 1/2.\end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned}\|D(L+z)^{-1}u\|_n &\le \sum_{k=0}^{\infty}\|D((A+z)^{-1}B)^k(A+z)^{-1}u\|_n \\ &\le \|D(A+z)^{-1}u\|_n + \sum_{k=0}^{\infty}\|((A+z)^{-1}B)^k(A+z)^{-1}u\|_\infty \\ &\le \|D(A+z)^{-1}u\|_n + \|(A+z)^{-1}u\|_\infty, \quad \text{by } (2.14), \\ &\le C(|z|^{1/2}\|u\|_n + |z|^{-3/4}\|u\|_{2n}), \quad \text{by } (2.7), (2.13).\end{aligned}$$

The proof is complete.

As an immediate consequence of (2.1) and (2.2), we conclude that L and L^* generate strongly continuous analytic semigroups e^{-tL} and e^{-tL^*} in J^r with $1 < r < \infty$, respectively, provided $\|Dw\|_{n/2}$ is sufficiently small. What is more, we can now proceed to the proof of the following $L^p - L^r$ estimates.

THEOREM 2.1. *Let* $t > 0$, $1 < q \le n$, $v \in J^q$ *and* $u \in C_0^\infty$. *Then we have*

$$\|e^{-tL}u\|_p \le Ct^{-(n/r-n/p)/2}\|u\|_r \quad \text{for } 1 < r \le p < \infty, \tag{2.15}$$

provided that $\|Dw\|_{n/2}$ *is sufficiently small*;

$$\|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n \le Ct^{-n/2r}\|u\|_r \quad \text{for } 1 < r \le n, \tag{2.16}$$

$$t^{n/2q}(t^{-1/2}\|e^{-tL}v\|_n + \|e^{-tL}v\|_\infty + \|De^{-tL}v\|_n) \to 0 \quad \text{as } t \to \infty, \tag{2.17}$$

provided that $\|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}$ *is sufficiently small.*

Proof. By making use of the semigroup property of e^{-tL}, Lemma 2.1, and the Dunford integral (cf. [8]) via a standard calculation, we have

$$\|e^{-tL^*}u\|_r + t^{1/2}\|De^{-tL^*}u\|_r \le C\|u\|_r \quad \text{for } 1 < r < \infty, \tag{2.18}$$

$$\begin{aligned}\|e^{-tL}u\|_\infty + \|De^{-tL}u\|_n \\ \le Ct^{-1/2}\|e^{-(t/2)L}u\|_n + Ct^{-1/4}\|e^{-(t/2)L}u\|_{2n}\end{aligned} \tag{2.19}$$

under the assumptions of Theorem 2.1.

It follows from (2.8) and (2.18) that

$$\|e^{-tL^*}u\|_p \le C\|e^{-tL^*}u\|_r^{1-n/r+n/p}\|De^{-tL^*}u\|_r^{n/r-n/p}$$
$$\le Ct^{-(n/r-n/p)/2}\|u\|_r$$

for $1 < r \le p < \infty$ and $n/r - n/p < 1$. Combining this with the semigroup property of e^{-tL^*}, we have for $1 < r \le p < \infty$ and $a \in L^{p/(p-1)}$,

$$\|e^{-tL^*}Pa\|_{r/(r-1)} \le Ct^{-(n/r-n/p)/2}\|a\|_{p/(p-1)},$$

and hence

$$(e^{-tL}u, a) = (u, e^{-tL^*}Pa) \le Ct^{-(n/r-n/p)}\|u\|_r\|a\|_{p/(p-1)}$$

This gives (2.15). (2.16) follows from (2.19) and (2.15).

To prove (2.17), we note for $a \in J^q \cap J^r$ with $1 < r < q$,

$$t^{n/2q}(t^{-1/2}\|e^{-tL}v\|_n + \|e^{-tL}v\|_\infty + \|De^{-tL}v\|_n)$$
$$\le C\|v - a\|_q + Ct^{-(n/r-n/q)/2}\|a\|_r,$$

where we have used (2.15) and (2.16). Hence the density of $J^q \cap J^r$ in J^q implies (2.17). The proof is complete.

3. Proof of Theorem 0.2. From Theorem 0.1 we can suppose that $\|Dw\|_{n/2} + \|w\|_{2n}^{1/2}\|w\|_{2n/3}^{1/2}$ is small such that (2.15)–(2.17) holds.

By using the projection P to (0.2)–(0.3), and setting $u(t) = v(t) - w$ and $a = v_0 - w$, then (0.2)–(0.3) leads to the evolution equation

$$(3.1) \qquad (d/dt)u + Lu = -P(u\cdot D)u \ (t > 0), \quad u(0) = a$$

in J^n. Hence, our goal now remains to show that (3.1) has a unique solution u belonging to the space

$$U \equiv \{u \in BC([0, \infty); J^n); t^{1/2}Du(t) \in BC([0, \infty); L^n(R^n; R^{n^2}))\}$$

such that

$$Hu(t) \equiv \|u(t)\|_n + t^{1/2}\|u(t)\|_\infty + t^{1/2}\|Du(t)\|_\infty \to 0 \quad \text{as } t \to \infty$$

provided that $a \in J^n$ with $\|a\|_n$ small enough.

Let us impose the following notation.

$$|||u||| = \sup_{t>0} Hu(t),$$
$$W = \{u \in U; |||u||| < \infty, \ Hu(t) \to 0 \text{ as } t \to \infty\},$$
$$Mu(t) = u_0(t) - \int_0^t e^{-(t-s)L}P(u\cdot D)u(s)\,ds; \quad u_0(t) = e^{-tL}a.$$

Observing that for $u \in C_0^\infty$,

$$t^{1/2}(\|e^{-tL}P(u\cdot D)u\|_\infty + \|De^{-tL}P(u\cdot D)u\|_n) + \|e^{-tL}P(u\cdot D)u\|_n$$
$$\le Ct^{-1/4}\|(u\cdot D)u\|_{2n/3}, \quad \text{by (2.15)–(2.16)},$$
$$\le Ct^{-1/4}\|u\|_{2n}\|Du\|_n,$$

and

$$\|u\|_{2n} \le \|u\|_n^{1/2}\|u\|_\infty^{1/2},$$

we have for $u \in W$,

$$\|Mu(t)\|_n + t^{1/2}\|Mu(t)\|_\infty + t^{1/2}\|DMu(t)\|_n \tag{3.2}$$
$$\le C\|a\|_n + C\int_0^t (t-s)^{-1/4}\|u(s)\|_{2n}\|Du(s)\|_n\,ds$$
$$+ C\int_0^t t^{1/2}(t-s)^{-3/4}\|u(s)\|_{2n}\|Du(s)\|_n\,ds,$$
$$\text{by (2.15)–(2.16)},$$
$$\le C\|a\|_n + C|||u|||^2,$$

and what is more, by using (2.17) and the property

$$Hu_0(t) + Hu(t) \to 0 \quad \text{as } t \to \infty$$

via a calculation similar to (3.2), we have

$$H(Mu)(t) \to 0 \quad \text{as } t \to \infty.$$

Moreover, by a standard calculation from **[19]** or **[12]**, we have $Mu \in U$ for $u \in W$, and so $M\colon W \to W$ and

$$|||Mu||| \le C\|a\|_n + C|||u|||^2.$$

Additionally, similar to (3.2), we obtain for $u_1, u_2 \in W$,

$$|||Mu_1 - Mu_2||| \le C(|||u_1||| + |||u_2|||)|||u_1 - u_2|||.$$

From contraction mapping principle it follows that M has a fixed point u in W provided $\|a\|_n$ is sufficiently small. As in **[12, 5]**, we find that the fixed point u is the desired solution which exists uniquely in U. The proof is complete.

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