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PRIMITIVE SUBALGEBRAS OF COMPLEX LIE ALGEBRAS. I. PRIMITIVE SUBALGEBRAS OF THE CLASSICAL COMPLEX LIE ALGEBRAS

I. V. CHEKALOV

This paper lists all the primitive reductive non-maximal subalgebras of simple complex classical Lie algebras.

The classification problem of primitive actions of connected Lie groups on manifolds, which was raised yet by Sophus Lie [**11**], and the problem of finding the maximal subgroups of connected Lie groups boil down to the classification problem of primitive subalgebras of Lie algebras (Lie groups and algebras are assumed to be finite-dimensional).

The classification of maximal connected subgroups of connected Lie groups reduces to the classification of maximal subalgebras of Lie algebras, which was obtained by the early fifties by V. V. Morozov [**13**], [**14**], F. I. Karpelevich [**9**], A. Borel and J. de Siebenthal [**7**], and E. B. Dynkin [**4**], [**5**].

Any maximal subalgebra of an algebra $\mathfrak{G}$, which contains no proper ideal of $\mathfrak{G}$, is primitive. In 1972, M. Golubitsky [**7**] found primitive non-maximal subalgebras among maximal-rank subalgebras of simple classical complex Lie algebras. Later M. Golubitsky and B. Rothschild [**8**] found primitive non-maximal subalgebras among maximal-rank subalgebras of simple exceptional complex Lie algebras.

M. Golubitsky [**7**] proved that primitive subalgebras of non-reductive primitive subalgebras of simple complex algebras are maximal. Thus the problem of finding primitive subalgebras of complex Lie algebras was reduced to the case of reductive subalgebras of simple Lie algebras.

This work provides the final solution of the question of primitive subalgebras of the classical complex Lie algebras; in fact, we list all the primitive reductive non-maximal subalgebras of simple complex classical Lie algebras.

This work's main results have been published in [**3**]. Note that three cases of primitive non-maximal subalgebras were left out in [**3**].

DEFINITION 1. A k-foliation of a differentiable manifold M is a collection $\{U_m\}$, $m \in M$, of connected k-dimensional imbedded submanifolds U_m such that for any $m, m' \in M$ the submanifold U_m has a denumerable base for its topology, $m \in U_m$, $U_m = U_{m'}$, or $U_m \cap U_{m'} = \varnothing$.

The foliation $\{U_m\}$ is called invariant with respect to the Lie group G action if $gU_m = U_{gm}$ for any $g \in G$, $m \in M$.

There are two trivial foliations of any manifold: the point-one and the foliation into connected components.

DEFINITION 2. A (transitive and effective) action of a Lie group G on a manifold M is called primitive if the only invariant foliations with respect to G are the trivial ones.

A closed proper subgroup P of a group G is called primitive if

> G acts on G/P in the primitive way, and P contains no proper normal divisor of G.

DEFINITION 3. A proper subalgebra $\mathfrak{P}$ of a Lie algebra $\mathfrak{G}$ is called primitive if

> $\mathfrak{P}$ contains no proper ideal of $\mathfrak{G}$,
> $\mathfrak{P}$ is the maximal invariant subalgebra with respect to the action of $\operatorname{Int}_{\mathfrak{P}} \mathfrak{G}$, where $\operatorname{Int}_{\mathfrak{P}} \mathfrak{G}$ is the subgroup of the group of inner automorphisms of the algebra $\mathfrak{G}$, which consists of those automorphisms which keep $\mathfrak{P}$ on its place.

Note that any maximal subalgebra of $\mathfrak{G}$ which contains no ideal of $\mathfrak{G}$ is primitive.

We shall be interested in the following problems.

1. List all the primitive actions of connected Lie groups on manifolds or, which is equivalent, list to within conjugation, all the primitive subgroups of connected complex Lie groups.

2. List all the maximal closed subgroups of connected complex Lie groups.

Problems 1 and 2 boil down to classification of primitive Lie subalgebras; indeed, the following theorem holds.

THEOREM A [7].

(1) *Let G be a connected Lie group, having the Lie algebra $\mathfrak{G}$; let $\mathfrak{P}$ be a primitive subalgebra of $\mathfrak{G}$; let P_0 be a connected closed subgroup*

of G, having the Lie algebra $\mathfrak{P}$. Then P_0 is a primitive subgroup of G and $\mathrm{Norm}_G P_0$ *is a maximal closed subgroup of G.*

(2) *If P is a primitive subgroup of a connected Lie group G, its Lie algebra is a primitive subalgebra of the G group's Lie algebra.*

(3) *Maximal closed subgroups of a connected Lie group are primitive.*

Classification of the maximal connected subgroups of connected complex Lie groups or, which is equivalent, of the maximal subalgebras of complex Lie algebras is provided by the works of V. V. Morozov **[13, 14]**, F. I. Karpelevich **[9]**, E. B. Dynkin **[5, 9]**, and A. Borel and J. de Siebenthal **[1]**. Therefore we shall be interested in primitive non-maximal subalgebras of complex Lie algebras.

THEOREM B [7]. *Let $\mathfrak{P}$ be a primitive subalgebra of a complex Lie algebra $\mathfrak{G}$. Then:*

(1) *If $\mathfrak{G}$ is not simple, $\mathfrak{P}$ is maximal in $\mathfrak{G}$;*

(2) *If $\mathfrak{G}$ is a simple algebra and $\mathfrak{P}$ is a non-reductive subalgebra, $\mathfrak{P}$ is maximal in $\mathfrak{G}$.*

Thus, to complete the classification of primitive and maximal subgroups of connected complex Lie groups, one has to find the primitive reductive non-maximal subalgebras of simple complex Lie algebras.

Let us present the definitions of the main classes of reductive subalgebras of semisimple complex Lie algebras **[5]**.

A subalgebra $\mathfrak{L}$ of a semisimple complex algebra $\mathfrak{G}$ is called regular if it has a basis which consists of elements of the Cartan subalgebra $\mathfrak{L}$ of $\mathfrak{G}$ and of root vectors of $\mathfrak{G}$ with respect to $\mathfrak{L}$.

A subalgebra $\mathfrak{L} \subset \mathfrak{G}$ is called a maximal-rank subalgebra if $\operatorname{rank} \mathfrak{L} = \operatorname{rank} \mathfrak{G}$. Maximal-rank subalgebras are regular ones.

A subalgebra $\mathfrak{L}$ is called an S-subalgebra if it does not belong to any proper regular subalgebra of $\mathfrak{G}$. A subalgebra $\mathfrak{L}$ is called an R-subalgebra if it belongs to some proper regular subalgebra of $\mathfrak{G}$. Classification of regular subalgebras of semisimple algebras is done in **[5]**. Some inaccuracies are corrected in **[10]**. If $\mathfrak{G} = \mathrm{sl}(n)$, $\mathrm{sp}(2n)$, $\mathrm{o}(2n+1)$, all the S-subalgebras are non-reducible subalgebras. If $\mathfrak{G} = \mathrm{o}(2n)$, all the S-subalgebras are non-reducible subalgebras and subalgebras of the form $\varphi_1 + \varphi_2$ where φ_i are non-reducible, orthogonal, odd-dimensional and $\varphi_1 \nsim \varphi_2$.

The problem of classification of maximal reductive subalgebras of simple complex Lie algebras is solved in **[4]**, **[5]**. Let us briefly formulate the results of this classification in the case of the classical algebras.

1. If $\mathfrak{P}$ is a simple non-reducible subalgebra of a simple classical algebra $\mathfrak{G}$, $\mathfrak{P}$ is almost always maximal in $\mathfrak{G}$. All the possible inclusions $\mathfrak{P} \subset \mathfrak{L} \subset \mathfrak{G}$, where the subalgebras $\mathfrak{P}$ and $\mathfrak{L}$ are simple and non-reducible, are listed in [**1**].

2. All the maximal non-reducible non-simple subalgebras of classical algebras are semisimple and to within conjugation are the following:

$$\begin{array}{lll}
\mathfrak{G} = \mathrm{sl}(mn), & \mathfrak{P} = \mathrm{sl}(m) \otimes \mathrm{sl}(n), & m \geq n \geq 2; \\
\mathfrak{G} = \mathrm{sp}(2mn), & \mathfrak{P} = \mathrm{sp}(2m) \otimes \mathrm{o}(n), & m \geq 1, \ n \geq 3, \\
 & & n = 4 \text{ or } m = 1, \ n = 4; \\
\mathfrak{G} = \mathrm{o}(mn), & \mathfrak{P} = \mathrm{o}(m) \otimes \mathrm{o}(n), & m \geq n \geq 3, \ m, n \neq 4; \\
\mathfrak{G} = \mathrm{o}(4mn), & \mathfrak{P} = \mathrm{sp}(2m) \otimes \mathrm{sp}(2n), & m \geq 2, \ n \geq 1.
\end{array}$$

(Note that $\mathrm{o}(4) = \mathrm{sp}(2) \otimes \mathrm{sp}(2)$, $\mathrm{sl}(2) = \mathrm{sp}(2)$.)

3. All the maximal ones among reducible reductive subalgebras of simple classical algebras to within conjugation are the following:

(1) $\mathfrak{G} = \mathrm{sl}(m+n)$, $\mathfrak{P} = (\mathfrak{GL}(m) + \mathfrak{GL}(n))_0$.
This is the subalgebra of matrices with trace zero;

(2) $\mathfrak{G} = \mathrm{sp}(2m+2n)$, $\mathfrak{P} = \mathrm{sp}(2n) \oplus \mathrm{sp}(2m)$, $n \geq m \geq 1$;

(3) $\mathfrak{G} = \mathrm{sp}(2n)$, $\mathfrak{P} = \mathfrak{GL}^*(n)$,

(4) $\mathfrak{G} = \mathrm{o}(2n+1+2m)$, $\mathfrak{P} = \mathrm{o}(2n+1) \oplus \mathrm{o}(2m)$, $n \geq 0$, $m \geq 1$, $2n+1+2m \geq 5$;

(5) $\mathfrak{G} = \mathrm{o}(2n+2m)$, $\mathfrak{P} = \mathrm{o}(2n) \oplus \mathrm{o}(2m)$, $n \geq m \geq 1$, $2n+2m \geq 6$;

(6) $\mathfrak{G} = \mathrm{o}(2n+2m+2)$, $\mathfrak{P} = \mathrm{o}(2n+1) \oplus \mathrm{o}(2m+1)$, $n \geq m \geq 1$, $2n+2m+2 \geq 6$;

(7) $\mathfrak{G} = \mathrm{o}(2n)$, $\mathfrak{P} = \mathfrak{GL}^*(n)$.

The following assumptions are made here: $\mathfrak{GL}(1) = \mathbb{C}$, $\mathrm{o}(1) = \{0\}$, $\mathfrak{GL}^*(n)$—the algebra of matrices written in the suitable basis in the form

$$\mathfrak{GL}^*(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} : A \in \mathfrak{GL}(n) \right\}.$$

4. All the subalgebras of item 3 are the maximal ones among all the subalgebras of $\mathfrak{G}$, except for the following ones, which are contained in the parabolic subalgebras [**5**], [**9**], [**2**]: (1); (3); (4) $m = 1$; (5) $m = 1$; (7).

Now we return to the classification problem of primitive reductive subalgebras.

LEMMA 1. *If $\mathfrak{P}$ is a primitive subalgebra of $\mathfrak{G}$, then either* $\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \mathfrak{P}$, *or* $\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \mathfrak{G}$.

Here

$$\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \{x \in \mathfrak{G},\ [x, \mathfrak{P}] \subset \mathfrak{P}\}$$

is the normalizer of $\mathfrak{P}$ *in* $\mathfrak{G}$.

Proof. Let $\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \mathfrak{L}$, $\mathfrak{L} \supset \mathfrak{P}$, $f \in \mathrm{Int}_{\mathfrak{P}} \mathfrak{G}$. If $x \in \mathfrak{L}$, then $[x, \mathfrak{P}] \subset \mathfrak{P}$, whence

$$[f(x), f(\mathfrak{P})] \subset f(\mathfrak{P}) \Leftrightarrow [f(x), \mathfrak{P}] \subset \mathfrak{P},$$

that is $f(x) \in \mathfrak{L}$. Hence $\mathrm{Int}_{\mathfrak{P}} \mathfrak{G}(\mathfrak{L}) = \mathfrak{L}$. Since $\mathfrak{P}$ is a primitive subalgebra, either $\mathfrak{L} = \mathfrak{P}$ or $\mathfrak{L} = \mathfrak{G}$.

COROLLARY 1. *If* $\mathfrak{P}$ *is a reductive primitive subalgebra of a simple Lie algebra* $\mathfrak{G}$, *then* $\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \mathfrak{P}$.

COROLLARY 2. *If* $\mathfrak{P}$ *is a semisimple primitive subalgebra of a simple Lie algebra* $\mathfrak{G}$, *then* $Z_{\mathfrak{G}}(\mathfrak{P}) = 0$, *where*

$$Z_{\mathfrak{G}}(\mathfrak{P}) = \{x \in \mathfrak{G}, [x, \mathfrak{P}] = 0\}$$

is the centralizer of $\mathfrak{P}$ *in* $\mathfrak{G}$.

LEMMA 2. *If* $\mathfrak{L} = k \oplus \mathfrak{z}$ *is a reductive Lie algebra with the centre* $\mathfrak{z}$, $f \in \mathrm{Aut}\, \mathfrak{L}$, *then* $f(k) = k$, $f(\mathfrak{z}) = \mathfrak{z}$.

Proof. The fact that $[\mathfrak{z}, \mathfrak{L}] = 0$ implies

$$[f(\mathfrak{z}), f(\mathfrak{L})] = 0 \Leftrightarrow [f(\mathfrak{z}), \mathfrak{L}] = 0,$$

that is, $f(\mathfrak{z})$ is the centre of $\mathfrak{L}$. Therefore $f(\mathfrak{z}) = \mathfrak{z}$, whence if

$$f(k) = f[k_1, k_2][f(k_1), f(k_2)] = [\tilde{k}_1 + z_1, \tilde{k}_2 + z_2] \in k.$$

Let $Z(\mathfrak{L})$ be a centre of algebra $\mathfrak{L}$.

LEMMA 3. *Let* $\mathfrak{P}$ *be a reductive primitive subalgebra in a simple complex Lie algebra* $\mathfrak{G}$. *Then the following cases are the only possible ones:*

(1) $\mathfrak{P}$ *is an* S*-subalgebra;*

(2) $\mathfrak{P}$ *is a maximal-rank subalgebra;*

(3) $\mathfrak{P}$ *is a semisimple* S*-subalgebra in a maximal-rank semisimple subalgebra.*

Proof. Suppose that $\mathfrak{P}$ is an R-subalgebra.

(a) $Z(\mathfrak{P}) = \mathfrak{z} \neq 0$, $\mathfrak{P} = \widetilde{\mathfrak{P}} \oplus \mathfrak{z}$, where $\widetilde{\mathfrak{P}}$ is the maximal semisimple ideal of $\mathfrak{P}$. If $f \in \operatorname{Int}_{\mathfrak{P}} \mathfrak{G}$, then $f \in \operatorname{Int}_{\mathfrak{z}} \mathfrak{G}$ since $f(\mathfrak{z}) = \mathfrak{z}$ from Lemma 2. Consider the subalgebra

$$\mathfrak{A} = Z_{\mathfrak{G}}(\mathfrak{z}) = \{x \in \mathfrak{G}, [x, \mathfrak{z}] = 0\}.$$

The fact that $[x, \mathfrak{z}] = 0$ implies $[f(x), f(\mathfrak{z})] = 0$ for $f \in \operatorname{Int}_{\mathfrak{z}} \mathfrak{G}$, that is, $f(x) \in \mathfrak{G}$, whence $f(\mathfrak{A}) \subseteq \mathfrak{A}$. Since $\widetilde{\mathfrak{P}} \subset \mathfrak{A}$, if $\mathfrak{P}$ is a primitive subalgebra, then $\mathfrak{P} = \mathfrak{A}$. It remains to note that $\mathfrak{A}$ is a maximal-rank subalgebra (as a torus centralizer) **[10]**.

(b) $Z(\mathfrak{P}) = 0$, that is, the subalgebra $\mathfrak{P}$ is semisimple and is not a maximal-rank subalgebra. Let $\mathfrak{L}$ be a regular subalgebra, which contains $\mathfrak{P}$. If $Z(\mathfrak{L}) = \tilde{\mathfrak{z}} \neq 0$, then $\mathfrak{P} \subset Z_{\mathfrak{G}}(\tilde{\mathfrak{z}})$ and one can show (like the item (a)) that $\operatorname{Int}_{\mathfrak{P}} \mathfrak{G}(\mathfrak{A}) = \mathfrak{A}$, that is, the subalgebra $\mathfrak{P}$ is not primitive.

If $Z(\mathfrak{L}) = 0$, but $\operatorname{rank} \mathfrak{L} < \operatorname{rank} \mathfrak{A}$, then $\mathfrak{P} \subset \mathfrak{L} \oplus \mathfrak{z}^r$ where $\mathfrak{L} \oplus \mathfrak{z}^r$ is a reductive subalgebra with the centre $\mathfrak{z}^r$, $r = \operatorname{rank} \mathfrak{G} - \operatorname{rank} \mathfrak{P}$. It has been shown above that such an inclusion cannot take place in the case of a primitive subalgebra $\mathfrak{P}$. So, in the present case any regular subalgebra, which contains $\mathfrak{P}$, is a maximal-rank semisimple subalgebra.

COROLLARY. *If $\mathfrak{P}$ is a primitive reductive subalgebra of a simple complex Lie algebra $\mathfrak{G}$ and $Z(\mathfrak{P}) \neq 0$, then* $\operatorname{rank} \mathfrak{P} = \operatorname{rank} \mathfrak{G}$.

LEMMA 4. (1) *If $\varphi(\mathfrak{P})$ is a reducible reductive subalgebra of an algebra* $\mathrm{sl}(n)$, *then* $Z_{\mathrm{sl}(n)}(\varphi(\mathfrak{P})) \neq 0$.

(2) *If $\varphi(\mathfrak{P})$ is a reducible reductive subalgebra of an algebra $\mathfrak{G} =$* $\mathrm{o}(n)$, $\mathrm{sp}(2n)$, *then $Z_{\mathfrak{G}}(\varphi(\mathfrak{P})) \neq 0$ if and only if there are two terms of the form $-\varphi_0 + \varphi_0^*$ in the decomposition of φ into irreducible components.*

Proof. (1) The maximal one among reducible reductive subalgebras $\mathrm{sl}(n)$ has a non-trivial centre, that is $\varphi(\mathfrak{P}) \subset \mathfrak{L} \oplus \mathfrak{z}$, $\mathfrak{z} = Z(\mathfrak{L} \oplus \mathfrak{z})$. Therefore,

$$Z_{\mathrm{sl}(n)}(\varphi(\mathfrak{P})) \supseteq Z_{\mathrm{sl}(n)}(\mathfrak{z}) \supseteq \mathfrak{z}.$$

(2) Let $\mathfrak{G} = \mathrm{sp}(2n)$, $\varphi = \varphi_0 + \varphi_0^* + f$, where the representation f is reducible and symplectic, generally speaking. In this case $\varphi(\mathfrak{P}) \subset$ $\mathfrak{G}\mathfrak{L}^*(k) \oplus \mathrm{sp}(2m) = \mathfrak{L}$ where

$$k = \dim \varphi_0, \ 2m = \dim f, \ k + m = n.$$

Hence $Z_{\mathrm{sp}(2n)}(\varphi(\mathfrak{P})) \neq 0$, since the subalgebra $\mathfrak{L}$ has a non-trivial centre.

Let $Z_{\mathrm{sp}(2n)}(\varphi(\mathfrak{P})) \neq 0$. Suppose that φ has no terms of the form $\varphi_0 + \varphi_0^*$. This means that $\varphi = \sum_{i=1}^{s} \varphi_i$, where the representations φ_i are symplectic and $\varphi_i \nsim \varphi_j$ with $i \neq j$. Clearly, in this case $Z_{\mathrm{sp}(2n)}(\varphi(\mathfrak{P})) = 0$. This contradiction proves our assertion.

If $\mathfrak{G} = \mathrm{o}(n)$, we can argue along the same lines substituting $\mathrm{o}(l)$ for $\mathrm{sp}(2m)$ and prove our assertion.

Corollary. *Let $\varphi(\mathfrak{P})$ be a primitive semisimple subalgebra of a simple complex classical Lie algebra $\mathfrak{G}$. Then the following cases are the only possible ones:*

(1) *$\varphi(\mathfrak{P})$ is non-reducible in $\mathfrak{G}$;*

(2) *$\mathfrak{G} = \mathrm{o}(n)$, $\varphi = \bigoplus_{i=1}^{s} \varphi_i$, $\varphi_i \nsim \varphi_j$, $i \neq j$, φ_i are orthogonal;*

(3) *$\mathfrak{G} = \mathrm{sp}(2n)$, $\varphi = \bigoplus_{i=1}^{t} \psi_i$, $\psi_i \nsim \psi_j$, $i \neq j$, ψ_i are symplectic.*

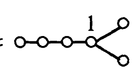
Lemma 5. (1) *Let $\mathfrak{G} = \mathrm{sl}(N)$, $\mathrm{o}(2N+1)$, $\mathrm{sp}(2N)$, $\varphi(k)$ is a reductive subalgebra of $\mathfrak{G}$, $\sigma \in \mathrm{Aut}\, k$. The automorphism σ can be extended to an inner automorphism $\mathfrak{G} \Leftrightarrow \varphi\sigma \sim \varphi$.*

(2) *Let $\mathfrak{G} = \mathrm{sl}(N)$. Then σ can be extended to an outer automorphism $\mathfrak{G} \Leftrightarrow \varphi\sigma \sim \varphi^*$.*

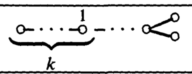
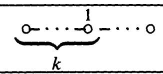
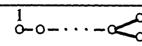
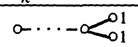
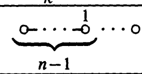
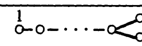
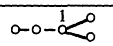
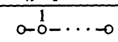
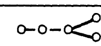
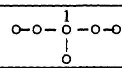
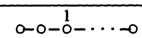
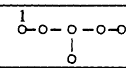
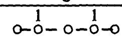
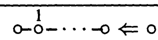
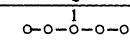
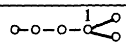
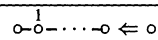
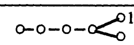
(3) *Let $\mathfrak{G} = \mathrm{o}(2N)$. Then σ can be extended to an automorphism $\mathfrak{G} \Leftrightarrow \varphi\sigma \sim \varphi$; σ can be extended to an outer automorphism only with $N \geq 5 \Leftrightarrow \varphi$ is decomposable into even-dimensional irreducible parts and contains a zero weight.*

The first and second parts of the lemma are given in **[12]**; *the third part is given in* **[5]**.

Proposition 1. *Let $\mathfrak{G}$ be a simple classical Lie algebra. If $\mathfrak{P}$ is a simple irreducible primitive subalgebra of $\mathfrak{G}$, then either $\mathfrak{P}$ is maximal in $\mathfrak{G}$ or $\mathfrak{P} = D_6$, $\mathfrak{G} = \mathrm{o}(495)$,*

$$\varphi(\mathfrak{P}) =$$ 

Proof. Simple irreducible non-maximal subalgebras of classical algebras are listed in [4]. Let us pick out of Table 5 of [4] all possible triples $\varphi(\mathfrak{P}) \subset f(\mathfrak{L}) \subset \mathfrak{G}$, where $\varphi(\mathfrak{P})$ is a simple irreducible subalgebra of $\mathfrak{G}$, for which there is such an outer automorphism θ that $\varphi\theta \sim \varphi$. Let ψ stand for a representation of $\mathfrak{P}$ in $\mathfrak{L}$ (ψ is defined to within transition to ψ_σ, where σ is a symmetry of the algebra $\mathfrak{P}$'s simple roots scheme).

N	$\mathfrak{P}$	$\mathfrak{L}$	$\mathfrak{G}$	$\varphi : \mathfrak{P} \to \mathfrak{G}$	$f : \mathfrak{L} \to \mathfrak{G}$	$\psi : \mathfrak{P} \to \mathfrak{L}$
1	D_n	A_{2n-1}	$\mathrm{sl}(C_{2n}^k)$			
2	D_n	A_{2n-1}	$\mathrm{o}(C_{2n}^{n-1})$			
3	D_5	A_{15}	$\mathrm{sl}(120)$			
4	E_6	A_{26}	$\mathrm{sl}(2925)$			
5	A_5	C_{10}	$\mathrm{o}(189)$			
6	D_6	C_{16}	$\mathrm{o}(495)$			

In cases 1, 2 and 5, $\psi\theta \sim \psi$ and θ can be extended to an inner automorphism $f(\mathfrak{L})$.

In cases 3 and 4 $\psi\theta \sim \psi^*$ and θ can be extended to an outer automorphism of $f(\mathfrak{L})$. In case 6 the outer automorphism D_6 can be extended to an inner automorphism of $\mathfrak{G}$, but it does not preserve $\mathfrak{L}$ from Lemma 5.

PROPOSITION 2. *Let $\mathfrak{P}$ be a primitive irreducible non-maximal non-simple subalgebra of a simple complex classical Lie algebra $\mathfrak{G}$. Then one of the following cases takes place*:

(1) $\mathfrak{G} = \mathrm{sl}(k^r)$, $\mathfrak{P} = \overbrace{\mathrm{sl}(k) \otimes \cdots \otimes \mathrm{sl}(k)}^{r}$, $r \geq 3$, $k \geq 2$.
(2) $\mathfrak{G} = \mathrm{sp}(4s)$, $\mathfrak{P} = \mathrm{sp}(s) \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2)$, $s > 2$.
(3) $\mathfrak{G} = \mathrm{o}(4s)$, $\mathfrak{P} = \mathrm{o}(s) \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2)$, $s > 2$, $s \neq 4$.
(4) $\mathfrak{G} = \mathrm{sp}(k^{2r+1})$, $\mathfrak{P} = \underbrace{\mathrm{sp}(k) \otimes \cdots \otimes \mathrm{sp}(k)}_{2r+1}$, $r \geq 1$, $k \geq 2$ *is even.*
(5) $\mathfrak{G} = \mathrm{o}(k^r)$, $\mathfrak{P} = \underbrace{\mathrm{o}(k) \otimes \cdots \otimes \mathrm{o}(k)}_{r}$, $r \geq 3$, $k \geq 3$.
(6) $\mathfrak{G} = \mathrm{o}(k^{2r})$, $\mathfrak{P} = \underbrace{\mathrm{sp}(k) \otimes \cdots \otimes \mathrm{sp}(k)}_{2r}$, $r \geq 2$, $k \geq 4$ *is even or* $r \geq 2$, $k = 2$.

Proof. Introduce denotations for some irreducible non-simple subalgebras:

$$\mathfrak{G} = \mathrm{sl}(n), \quad \mathfrak{L}_A(N_1, \ldots, N_s) = \mathrm{sl}(N_1) \otimes \cdots \otimes \mathrm{sl}(N_s),$$

where $N_1 \cdots N_s = n$.

$$\mathfrak{G}_t = \mathrm{o}(n),$$

$$\mathfrak{L}_0(M_1, \ldots, M_t, k_1, \ldots, k_s) = \mathrm{sp}(M_1) \otimes \cdots \otimes \mathrm{sp}(M_t) \otimes \mathrm{o}(k_1) \otimes \cdots \otimes \mathrm{o}(k_s)$$

where $\prod_{i=1}^{t} M_i \cdot \prod_{j=1}^{s} k_j = n$, t is even.

$$\mathfrak{G} = \mathrm{sp}(n),$$

$$\begin{aligned}&\mathfrak{L}_s(M_1, \dots, M_t, k_1, \dots, k_s)\\ &\quad = \mathrm{sp}(M_1) \otimes \cdots \otimes \mathrm{sp}(M_t) \otimes \mathrm{o}(k_1) \otimes \cdots \otimes \mathrm{o}(k_s)\end{aligned}$$

where $\prod_{i=1}^{t} M_i \cdot \prod_{j=1}^{s} k_j = n$, t is odd.

Non-simple irreducible subalgebras of classical Lie algebras have the form [4]:

$$\mathfrak{G} = \mathrm{sl}(n), \quad \psi(\mathfrak{P}) = \bigotimes_{i=1}^{s} \psi_i(\mathfrak{P}_i);$$

$$\mathfrak{G} = \mathrm{o}(n), \quad \mathrm{sp}(n), \quad \psi(\mathfrak{P}) = \bigotimes_{i=1}^{t} \varphi_i(\mathfrak{P}_i) \otimes \bigotimes_{j=1}^{s} f_j(\mathfrak{P}_j),$$

where ψ_i, φ_i, f_j are irreducible representations of simple algebras $\mathfrak{P}_i$, $\mathfrak{P}_j$; φ_i are symplectic; f_j are orthogonal; t is even with $\mathfrak{G} = \mathrm{o}(n)$ and t is odd with $\mathfrak{G} = \mathrm{sp}(n)$.

Let

$$\dim \psi_i = N_i, \quad \dim \varphi_i = M_i, \quad \dim f_j = k_j.$$

Then

$$\begin{aligned}&\psi(\mathfrak{P}) \subset \mathfrak{L}_A(N_1, \dots, N_s), && \text{if } \mathfrak{G} = \mathrm{sl}(N);\\ &\psi(\mathfrak{P}) \subset \mathfrak{L}_0(M_1, \dots, M_t, k_1, \dots, k_s), && \text{if } \mathfrak{G} = \mathrm{o}(n);\\ &\psi(\mathfrak{P}) \subset \mathfrak{L}_s(M_1, \dots, M_t, k_1, \dots, k_s), && \text{if } \mathfrak{G} = \mathrm{sp}(n).\end{aligned}$$

Let

$$\psi(\mathfrak{P}) \subset \mathfrak{G} = \mathrm{sl}(n), \quad f \in \mathrm{Int}_{\psi(\mathfrak{P})} \mathfrak{G}.$$

The automorphism f is a product of automorphisms of the form

$$X_i \to s_i X_i s_i^{-1}, \quad X_i \in \mathfrak{P}_i, \quad s_i \in \mathrm{Sl}(N_i),$$

and of automorphisms which permute the conjugate simple ideals $\psi_i(\mathfrak{P}_i)$ (see Lemma 5), that is, f preserves $\mathfrak{L}_A$. Analogously, $\mathrm{Int}_{\psi(\mathfrak{P})} \mathfrak{G}(\mathfrak{L}_0) = \mathfrak{L}_0$ if $\mathfrak{G} = \mathrm{o}(n)$ and $\mathrm{Int}_{\psi(\mathfrak{P})} \mathfrak{G}(\mathfrak{L}_s) = \mathfrak{L}_s$ if $\mathfrak{G} = \mathrm{sp}(n)$. In other words, those primitive subalgebras which are irreducible and simple are among subalgebras of $\mathfrak{L}_A$, $\mathfrak{L}_0$, $\mathfrak{L}_s$.

(a) $\mathfrak{G} = \mathrm{sl}(n)$.

Let

$$\mathfrak{L}_A = \mathfrak{L}_A(N_1, \dots, N_s) \quad \text{where } N_1 \geq N_2 \geq \cdots \geq N_s \geq 2, \quad N_1 \neq N_s.$$

Then

$$\mathfrak{L}_A \subset \widetilde{\mathfrak{L}}_A = \mathfrak{L}_A(\widetilde{N}_1, \widetilde{N}_2), \quad \widetilde{N}_1 = \prod_{N_i = N_1} N_i, \quad \widetilde{N}_2 = \prod_{N_i \neq N_1} N_i.$$

From Lemma 5,

$$\mathrm{Int}_{\mathfrak{L}_A} \mathrm{sl}(n)(\widetilde{\mathfrak{L}}_A) = \widetilde{\mathfrak{L}}_A$$

that is in the assumptions made, $\mathfrak{L}_A$ being a primitive subalgebra implies $s = 2$. But in this case the subalgebra is maximal.

Let $\mathfrak{P} = \mathfrak{L}_A(\underbrace{k, \dots, k}_{r})$, where $r \geq 3$, $k \geq 2$.

Any proper subalgebra $\mathrm{sl}(n)$ which contains $\mathfrak{P}$ strictly, has the form $\mathfrak{L}_A(N_1, \dots, N_s)$ where

$$N_i = k^{l_i}, \quad \sum_{i=1}^{s} l_i = r, \quad s < r.$$

Consider an automorphism $\sigma \in \mathrm{Int}_{\mathfrak{P}} \mathrm{sl}(n)$ which permutes the conjugate ideals of $\mathfrak{P}$ and is a cycle of the length r. It is evident that

$$\sigma \mathfrak{L}_A(N_1, \dots, N_s) \neq \mathfrak{L}_A(N_1, \dots, N_s)$$

that is the subalgebra $\mathfrak{P}$ is primitive.

(b) $\mathfrak{G} = \mathrm{o}(n)$.

Let $\mathfrak{L}_0^M = \mathfrak{L}_0(M_1, \dots, M_t)$. Then t is even and all M_i are even. If

$$M_1 \geq M_2 \geq \cdots \geq M_t \geq 2, \quad t > 2, \quad M_1 \neq M_t,$$

then

$$\mathfrak{L}_0^M \subset \widetilde{\mathfrak{L}}_0^M = \mathfrak{L}_0(\widetilde{M}_1, \widetilde{M}_2), \quad \widetilde{M}_1 = \prod_{M_i = M_1} M_i, \quad \widetilde{M}_2 = \prod_{M_i \neq M_1} M_i.$$

Since

$$\mathrm{Int}_{\mathfrak{L}_0^M} \mathrm{o}(n)(\widetilde{\mathfrak{L}}_0^M) = \widetilde{\mathfrak{L}}_0^M,$$

and the subalgebra $\mathfrak{L}_0^M$ is maximal with $t = 0$, we see that $\mathfrak{L}_0^M$ may be a primitive subalgebra only if $M_1 = \cdots = M_t$.

Consider the subalgebra

$$\mathfrak{P} = \mathfrak{L}_0(\underbrace{M, \dots, M}_{2r}), \quad r \geq 2.$$

Any proper subalgebra $\mathrm{o}(n)$ which contains $\mathfrak{P}$ strictly has the form

$$\widetilde{\mathfrak{L}}_0 = \mathfrak{L}_0(M_1, \dots, M_p, k_1, \dots, k_s)$$

where p is even,

$$M_i = M^{M_i}, \quad k_j = M^{k_j}, \quad \sum_{i=1}^{p} M_i + \sum_{j=1}^{s} k_j = 2r, \quad p+s < 2r.$$

Take an automorphism $\sigma \in \mathrm{Int}_\rho\, \mathrm{o}(n)$ which represents conjugate simple ideals of $\mathfrak{P}$ and is a cycle of the length $2r$. It is evident that $\sigma\widetilde{\mathfrak{L}}_0 \neq \widetilde{\mathfrak{L}}_0$ that is the subalgebra $\mathfrak{P}$ is primitive.

Let $\mathfrak{L}_0^k = \mathfrak{L}_0(k_1, \dots, k_t)$. Reasoning on the same lines proves that the subalgebra $\mathfrak{L}_0^k$ is primitive but not maximal if and only if $k_1 = \cdots = k_t$, $t > 2$.

It remains to consider the subalgebras of the form

$$\mathfrak{L}_0 = \mathfrak{L}_0(M_1, \dots, M_t, k_1, \dots, k_s) \quad \text{with } ts \geq 2.$$

The following inclusion holds:

$$\mathfrak{L}_0 \subset \widetilde{\mathfrak{L}}_0 = \mathfrak{L}_0(\tilde{k}_1, \tilde{k}_2), \quad \tilde{k}_1 = \prod_{i=1}^{t} M_i, \quad \tilde{k}_2 = \prod_{j=1}^{s} k_j.$$

Here

$$\mathrm{Int}_{\mathfrak{L}_0}\, \mathrm{o}(n)(\widetilde{\mathfrak{L}}_0) = \widetilde{\mathfrak{L}}_0 \quad (\tilde{k}_1 \neq 4);$$

that is, non-maximal subalgebras of the form considered here cannot be primitive. If $\tilde{k}_1 = 4$, we have the primitive subalgebra

$$\mathrm{o}(s) \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2), \quad s > 2, \ s \neq 4 \text{ in } \mathfrak{G} = \mathrm{o}(4s).$$

(c) $\mathfrak{G} = \mathrm{sp}(2n)$.

Let

$$\mathfrak{L}_s = \mathfrak{L}_s(M_1, \dots, M_t, k_1, \dots, k_s)$$

where t is odd and $s \neq 0$. The following inclusion holds:

$$\mathfrak{L}_s \subset \widetilde{\mathfrak{L}}_s = \mathfrak{L}_s(M, k), \quad M = \prod_{i=1}^{t} M_i, \quad k = \prod_{j=1}^{s} k_j$$

where $\widetilde{\mathfrak{L}}_s$ is the maximal subalgebra and $\mathrm{Int}_{\mathfrak{L}_s}\, \mathrm{sp}(2n)(\widetilde{\mathfrak{L}}_s) = \widetilde{\mathfrak{L}}_s$ with $k \neq 4$.

If $k = 4$, we have a primitive subalgebra

$$\mathrm{sp}(s) \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2), \quad s > 2 \text{ in } \mathfrak{G} = \mathrm{sp}(4s).$$

Now consider the case $s = 0$. Let

$$\mathfrak{L}_s = \mathfrak{L}_s(M_1, \dots, M_{2t+1}) \cdot M_1 \geq \cdots \geq M_{2t+1}, \quad t \geq 1, \ M_1 \neq M_{2t+1}.$$

Consider the subalgebra $\widetilde{\mathfrak{L}}_s = \mathfrak{L}_s(\widetilde{M}, \tilde{k})$, where

$$\begin{cases} \widetilde{M} = \prod\limits_{M_i = M_1} M_i, & \text{the number of factors being odd,} \\ \tilde{k} = \prod\limits_{M_i \neq M_1} M_i, & \end{cases}$$

$$\begin{cases} \tilde{k} = \prod\limits_{M_i = M_1} M_i, & \text{the number of factors being even,} \\ \widetilde{M} = \prod\limits_{M_i \neq M_1} M_i. & \end{cases}$$

The subalgebra $\mathfrak{L}_s$ is maximal in $\mathrm{sp}(n)$ and $\mathrm{Int}_{\mathfrak{L}_s} \mathrm{sp}(n)(\mathfrak{L}_s) = \mathfrak{L}_s$. Consequently, for the non-maximal subalgebra $\mathfrak{L}_s$ to be primitive it is necessary that $s = 0$ and $M_1 = \cdots = M_{2t+1}$, $t \geq 1$.

Consider the subalgebra

$$\mathfrak{P} = \mathfrak{L}_s(\overbrace{M, \dots, M}^{2t+1}).$$

Any proper subalgebra $\mathrm{sp}(n)$ which strictly contains $\mathfrak{P}$, has the form

$$\mathfrak{L}_s = \mathfrak{L}_s(M_1, \dots, M_p, k_1, \dots, k_q)$$

where

$$M_i = M^{M_i}, \ k_j = M^{k_j}, \ p + q < 2t + 1.$$

Take an automorphism $\sigma \in \mathrm{Int}_{\mathfrak{P}} \mathrm{sp}(n)$ which permutes the conjugate ideals of $\mathfrak{P}$ and is a cycle of the length $2t + 1$. It is evident that $\sigma(\mathfrak{L}_s) \neq \mathfrak{L}_s$; that is, the subalgebra $\mathfrak{P}$ is primitive.

PROPOSITION 3. *Let $\mathfrak{P}$ be a reducible semisimple primitive non-maximal subalgebra in a simple complex classical Lie algebra $\mathfrak{G}$. Then one of the following cases holds*:

(1) $\mathfrak{G} = \mathrm{o}(2k)$, $\mathfrak{P} = \underbrace{\mathrm{o}(k) \oplus \cdots \oplus \mathrm{o}(k)}_{r}$, $r \geq 3$, $k \geq 3$;

(2) $\mathfrak{G} = \mathrm{sp}(2rk)$, $\mathfrak{P} = \underbrace{\mathrm{sp}(2k) \oplus \cdots \oplus \mathrm{sp}(2k)}_{r}$, $r \geq 3$, $k \geq 1$.

Proof. Let $\mathfrak{G} = \mathrm{o}(n)$ and $\varphi(\mathfrak{P})$ be its primitive subalgebra, $\varphi(\mathfrak{P})$ being reducible and semisimple. According to the corollary from Lemma 1, the decomposition of φ into irreducible terms has the form $\varphi = \bigoplus_{i=1}^{s} \varphi_i$ where $\varphi_i \not\sim \varphi_j$ with $i \neq j$ and φ_i is orthogonal. Let $\dim \varphi_i = k_i$. Consider subalgebras

$$\mathscr{M}_0(k_1, \dots, k_s) = \bigoplus_{i=1}^{s} \mathrm{o}(k_i) \subset \mathrm{o}(n)$$

where $\sum_{i=1}^{s} k_i = n$. It is clear that

$$\varphi(\mathfrak{P}) \subset \mathscr{M}_0(k_1, \ldots, k_s) = \mathscr{M}_0.$$

Moreover, from Lemma 5

$$\operatorname{Int}_{\varphi(\mathfrak{P})} \mathrm{o}(n)(\mathscr{M}_0) = \mathscr{M}_0;$$

that is, if the subalgebra $\mathfrak{P}$ is primitive, $\mathfrak{P} = \mathscr{M}_0(k_1, \ldots, k_s)$. Note that the subalgebra $\mathscr{M}_0$ is maximal in $\mathrm{o}(n)$ if $s = 2$. Assume that $k_1 \geq \cdots \geq k_s$ and $k_1 \neq k_s$. Then

$$\mathscr{M}_0 = \mathscr{M}_0(k_1, \ldots, k_s) \subset \mathscr{M}_0(\tilde{k}_1, \tilde{k}_2) = \widetilde{M}_0,$$

where $k_1 = \sum_{k_i = k_1} k_i$, $k_2 = \sum_{k_i \neq k_1} k_i$ and $\operatorname{Int}_{\mathscr{M}_0} \mathrm{o}(n)(\widetilde{\mathscr{M}_0}) = \widetilde{\mathscr{M}_0}$ from Lemma 5. Thus, if the subalgebra $\mathfrak{P}$ is primitive but non-maximal, $\mathfrak{P} = \mathscr{M}_0(\underbrace{k, \ldots, k}_{r})$, $r \geq 3$. The condition of semisimplicity of $\mathfrak{P}$ is $k \geq 3$.

Consider the subalgebra

$$\mathfrak{P} = \mathscr{M}_0(\underbrace{k, \ldots, k}_{r}), \quad r \geq 3, \ k \geq 3.$$

Any proper subalgebra $\mathrm{o}(n)$, which strictly contains $\mathfrak{P}$, has the form $\mathscr{M}_0(l_1 k_1, \ldots, l_p k_p) = \widetilde{\mathscr{M}_0}$ where $\sum_{i=1}^{p} l_i = r$, $p < r$. If $\tau \in \operatorname{Int}_{\mathfrak{P}} \mathrm{o}(n)$ and $\tau|_{\mathfrak{P}}$ is a permutation $\mathrm{o}(k)$ of simple ideals, which is a cycle of the length r, then τ does not preserve $\widetilde{\mathscr{M}_0}$; that is, the subalgebra $\mathfrak{P}$ is primitive.

Case $\mathfrak{G} = \mathrm{sp}(n)$ can be treated in the same way.

COROLLARY. *Let $\mathfrak{P}$ be a reducible primitive non-maximal reductive subalgebra of a simple complex classical Lie algebra $\mathfrak{G}$ and* $\operatorname{rank}\mathfrak{P} < \operatorname{rank}\mathfrak{G}$. *Then one of the following cases holds:*

(a) $\mathfrak{G} = \mathrm{o}((2r+1)(2n+1))$, $\mathfrak{P} = \overbrace{\mathrm{o}(2n+1) \oplus \cdots \oplus \mathrm{o}(2n+1)}^{2r+1}$, $r \geq 1$, $n \geq 1$, $\operatorname{rank}\mathfrak{G} = 2rn + r + n$, $\operatorname{rank}\mathfrak{P} = 2rn + n$;

(b) $\mathfrak{G} = \mathrm{o}(2r(2n+1))$, $\mathfrak{P} = \overbrace{\mathrm{o}(2n+1) \oplus \cdots \oplus \mathrm{o}(2n+1)}^{2r}$, $r \geq 2$, $n \geq 1$, $\operatorname{rank}\mathfrak{G} = 2rn + r$, $\operatorname{rank}\mathfrak{P} = 2rn$.

Proof. Note that maximal-range semisimple subalgebras in $\mathfrak{G} = \mathrm{o}(n)$, $\mathrm{sp}(2n)$ have the form **[5]**:

$$\mathfrak{G} = \mathrm{o}(2n), \quad \bigoplus_{i=1}^{s} \mathrm{o}(2k_i), \quad \sum_{i=1}^{s} 2k_i = 2n;$$

$$\mathfrak{G} = \mathrm{o}(2n+1), \quad \bigoplus_{i=1}^{s} \mathrm{o}(2k_i) \oplus \mathrm{o}(2k+1),$$

$$\sum_{i=1}^{s} 2k_i + 2k + 1 = 2n + 1;$$

$$\mathfrak{G} = \mathrm{sp}(2n), \quad \bigoplus_{i=1}^{s} \mathrm{sp}(2M_i), \quad \sum_{i=1}^{s} 2M_i = 2n,$$

and compare them with the subalgebras listed in Proposition 3.

Let us say that the reductive subalgebra $\mathfrak{P}$ is primitive among the reductive subalgebras of the algebra $\mathfrak{G}$ if $\mathrm{Int}_{\mathfrak{P}} \mathfrak{G}(\mathfrak{L}) \neq \mathfrak{L}$ for any reductive subalgebra $\mathfrak{L}$ for which $\mathfrak{P} \subset \mathfrak{L} \subset \mathfrak{G}$, $\mathfrak{L} \neq \mathfrak{P}$, $\mathfrak{L} \neq \mathfrak{G}$.

PROPOSITION 4. *Let $\mathfrak{P}$ be a reductive subalgebra of a simple complex classical algebra $\mathfrak{G}$, the following provided*:

$\mathrm{rank}\,\mathfrak{P} = \mathrm{rank}\,\mathfrak{G}$;

$\mathfrak{P}$ is primitive among the reductive subalgebras of $\mathfrak{G}$;

$Z(\mathfrak{P}) \neq 0$;

$\mathfrak{P}$ is not maximal among the reductive subalgebras.

Then

$$\mathfrak{G} = \mathrm{sl}(n), \quad \mathfrak{P} = \left(\bigoplus^{r} \mathfrak{G}\mathfrak{L}(k) \right)_0, \quad n = rk, \ r \geq 3, \ k \geq 2.$$

Proof. (1) $\mathfrak{G} = \mathrm{sl}(n)$.

The maximal-range reductive subalgebra in $\mathrm{sl}(n)$ has the form

$$R_A(n_1, \ldots, n_s) = \left(\bigoplus_{i=1}^{s} \mathrm{sl}(n_i) \right)_0, \quad n_1 \geq n_2 \geq \cdots \geq n_s \geq 1,$$

$$\sum_{i=1}^{s} n_i = n.$$

The condition of the inequality $R_A \neq Z(R_A)$ (which means that the subalgebra R_A is non-Abelian) is $n_1 > 1$. The condition of R_A being the maximal one among the reductive subalgebras is $s = 2$.

Suppose that $s \geq 3$, $n_1 > 1$ and $n_1 \neq n_s$. Then

$$R_A = R_A(n_1, \ldots, n_s) \subset R_A(\tilde{n}_1, \tilde{n}_2) = \widetilde{R}_A$$

where $\tilde{n}_1 = \sum_{n_i = n_1} n_i$, $\tilde{n}_2 = \sum_{n_i \neq n_1} n_i$. If $f \in \mathrm{Int}_{R_4} \mathrm{sl}(n)$, Lemma 5 implies that $f|_{R_A}$ is a product of an inner automorphism of R_A and a permutation of conjugate ideals of $\mathrm{sl}(n_i)$. Therefore $f(\widetilde{R}_A) \subset \widetilde{R}_A$; that is, the subalgebra R_A cannot be primitive in this case.

Let

$$\mathfrak{P} = \left(\bigoplus^{r} \mathfrak{G}\mathfrak{L}(k)\right)_0, \quad n = rk, \ r \geq 3, \ k \geq 2;$$

that is, $\mathfrak{P} = R_A(\underbrace{k, \ldots, k}_{r})$. Any proper reductive subalgebra $\mathrm{sl}(n)$, which contains $\mathfrak{P}$ strictly, has the form $\widetilde{R}_A = R_A(n_1, \ldots, n_s)$ where $n_i = l_i k$, $\sum_{i=1}^{s} l_i = r$, $s < r$. Let τ be such an automorphism $\mathrm{Int}_{\mathfrak{P}}\,\mathrm{sl}(n)$ that $\tau|_{\mathfrak{P}}$ is a permutation of the summands of $\mathfrak{G}\mathfrak{L}(n)$ which is a cycle of the length r. Evidently, $\tau\widetilde{R}_A \neq \widetilde{R}_A$; that is, the subalgebra $\mathfrak{P}$ is primitive among the reductive subalgebras.

(2) $\mathfrak{G} = \mathrm{sp}(2n)$.

The maximal-range reductive subalgebra k in $\mathrm{sp}(2n)$, which satisfies the condition $Z(k) \neq 0$, has the form

$$R_S = R_s(M_1, \ldots, M_t, n_1, \ldots, n_k) = \bigoplus_{i=1}^{t} \mathrm{sp}(M_i) \oplus \bigoplus_{j=1}^{k} \mathfrak{G}\mathfrak{L}_k^*(n_i)$$

where $n_1 \geq \cdots \geq n_k \geq 1$, M_i are even, $\sum_{i=1}^{t} M_i + 2\sum_{j=1}^{k} n_j = 2n$, $k \neq 0$. Consider the inclusion $R_s \subset \widetilde{R}_s = R_s(M, \tilde{n})$ where $M = \sum_{i=1}^{t} M_i$, $\tilde{n} = \sum_{j=1}^{k} n_j$. Lemma 5 implies that $\mathrm{Int}_{R_s}\,\mathrm{sp}(2n)(\widetilde{R}_s) = \widetilde{R}_s$; that is, if the subalgebra $R_s\,\mathrm{sp}(2n)$ is primitive among the reductive subalgebras, it has the form $\widetilde{R}_s = \mathrm{sp}(M) \oplus \mathfrak{G}\mathfrak{L}(\tilde{n})$ where $M + 2\tilde{n} = 2n$, $\tilde{n} \geq 1$, $M \geq 0$. If $M = 0$, the subalgebra $\mathfrak{G}\mathfrak{L}^*(n)$ is maximal among the reductive subalgebras of $\mathrm{sp}(2n)$ [5]; in the other case $M \neq 0$,

$$\widetilde{R}_s \subset \widetilde{\widetilde{R}}_s = \mathrm{sp}(M) \oplus \mathrm{sp}(2\tilde{n})$$

and Lemma 5 implies

$$\mathrm{Int}_{\widetilde{R}_s}\,\mathrm{sp}(2n)(\widetilde{\widetilde{R}}_s) = \widetilde{\widetilde{R}}_s$$

which means that the subalgebra $\widetilde{R}_s$ is not primitive. Since the subalgebra $\widetilde{\widetilde{R}}_s$ is maximal among all the subalgebras of $\mathrm{sp}(2n)$, the algebra $\mathrm{sp}(2n)$ has no subalgebras subject to conditions listed in Proposition 4.

(3) $\mathfrak{G} = \mathrm{o}(2n)$. Proof is the same as in item (2) substituting $\mathrm{o}(2k)$ for all the subalgebras $\mathrm{sp}(2k)$.

(4) $\mathfrak{G} = \mathrm{o}(2n+1)$. The maximal-range reductive subalgebra in $\mathrm{o}(2n+1)$ has the form **[5]**

$$\begin{aligned} R_0(k_0, \dots, k_t, n_1, \dots, n_l) \\ = \mathrm{o}(2k_0+1) \oplus \bigoplus_{i=1} \mathrm{o}(2k_i) \oplus \bigoplus_{j=1} \mathfrak{G}\mathfrak{L}^*(n_i) = R_0, \end{aligned}$$

where

$$2k_0 + 1 + \sum_{i=1}^{t} 2k_i + 2\sum_{j=1}^{l} n_j = 2n+1.$$

Consider the inclusion

$$R_0 \subset \widetilde{R}_0 = \mathrm{o}(2k+1) \oplus \mathrm{o}(2L)$$

where

$$2k+1 = 2k_0 + 1 + \sum_{i=1}^{t} 2k_i, \quad L = \sum_{j=1}^{l} n_j.$$

Lemma 5 implies that if $f \in \mathrm{Int}_{R_0} \mathrm{o}(2n+1)$ then $f(\widetilde{R}_0) = \widetilde{R}_0$. Hence, if the subalgebra R_0 is primitive, $R_0 = \widetilde{R}_0$. But the subalgebra $\widetilde{R}_0$ is the maximal subalgebra of $\mathrm{o}(2n+1)$. Thus in $\mathrm{o}(2n+1)$ there are no subalgebras subject to the conditions listed in Proposition 4.

Note. Let

$$\mathfrak{G} = \mathrm{sl}(n), \quad \mathfrak{P} = \left(\bigoplus^{r} \mathfrak{G}\mathfrak{L}(k) \right)_0, \quad n = rk, \ r \geq 3, \ k \geq 3.$$

Any proper subalgebra $\mathfrak{L}$ which contains $\mathfrak{P}$ strictly, to within conjugation, has the form

$$\mathfrak{L} = R_A(l_1 k, \dots, l_s k) + \mathfrak{N},$$

where $\sum_{i=1}^{s} l_i = r$, $s < r$, and $\mathfrak{N}$ is a nilpotent ideal:

$$\mathfrak{N} = \begin{pmatrix} \mathrm{o}_{l,k} & A_{l_1 k, l_2 k \cdots} & \cdots & A_{l_1 k, l_s k} \\ 0 & \mathrm{o}_{l_2 k} & \cdots & A_{l_2 k, l_s k} \\ & \cdots & & \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$f \in \mathrm{Int}\,\mathrm{sl}(n), \quad f(x) = TxT^{-1}, \quad x \in \mathrm{Sl}(n),$$

$$T = \begin{pmatrix} \mathrm{o}_k & 1_k & \cdots & \\ & \mathrm{o}_k & \ddots & \\ & & \ddots & 1_k \\ 1_k & \cdots & & \mathrm{o}_k \end{pmatrix}.$$

One can see that $f(\mathfrak{P}) = \mathfrak{P}$ but $f(\mathfrak{L}) \neq \mathfrak{L}$; that is, the subalgebra $\mathfrak{P}$ is primitive among all the subalgebras of the algebra $\mathrm{sl}(n)$. In the root terms it has been proved in [7].

PROPOSITION 5. *Let $\mathfrak{P}$ be a reductive Abelian subalgebra which is primitive in a simple complex algebra $\mathfrak{Z}$. Then $\mathfrak{G}$ is an algebra all the roots of which have equal length and $\mathfrak{P}$ is a Cartan subalgebra of $\mathfrak{G}$.*

Let $\mathfrak{P}$ be an Abelian reductive subalgebra of the algebra $\mathfrak{G}$. Since the equality $\mathrm{Norm}_{\mathfrak{G}} \mathfrak{P} = \mathfrak{P}$ is a necessary condition for $\mathfrak{P}$ to be primitive, the subalgebra $\mathfrak{P}$ must coincide with the Cartan subalgebra $\mathfrak{h}$ of the algebra $\mathfrak{G}$. It is well-known that $\mathrm{Int}_{\mathfrak{h}} \mathfrak{G} = W \cdot \mathrm{Exp}\,\mathrm{ad}\,\mathfrak{h}$ where W is the Weyl Group of algebra $\mathfrak{G}$, which acts transitively on the roots of one length. If an algebra's root system consists of roots which have different lengths, we consider the regular subalgebra $\mathfrak{L} = \mathfrak{h} + \sum_{j \in R_0} \mathbb{C} l_j$, where R_0 is the subsystem of long roots. For any $w \in W$ the action at a root vector l_j is defined as follows: $w l_j = \pm l_{wj}$. Thus $\mathrm{Int}_{\mathfrak{h}} \mathfrak{G}$ preserves the subalgebra $\mathfrak{L}$ invariant. Let the root system of the algebra $\mathfrak{G}$ consist of roots of one length. In this case any subalgebra $\mathfrak{L}$, which contains $\mathfrak{h}$ strictly, contains some root vector. Since in this case W acts transitively on $\mathfrak{G}$ algebra's root system, $\mathrm{Int}_{\mathfrak{h}} \mathfrak{G}(\mathfrak{L})$ contains any root vector of the algebra $\mathfrak{G}$, that is, $\mathrm{Int}_{\mathfrak{z}} \mathfrak{G}(\mathfrak{L}) = \mathfrak{L}$, and hence the subalgebra $\mathfrak{h}$ is primitive in this case.

THEOREM. (1) *Let $\mathfrak{G}$ be a simple complex classical Lie algebra and $\mathfrak{P}$ be its primitive reductive subalgebra, $\mathfrak{P}$ not being maximal among the $\mathfrak{G}$ algebra's reductive subalgebras. All the possible* (*to within conjugation*) *pairs* $(\mathfrak{G}, \mathfrak{P})$ *are listed in Table* 1 (*next page*).

(2) *Let $\mathfrak{G}$ be a simple complex classical Lie algebra and $\mathfrak{P}$ be its primitive reductive subalgebra; let $\mathfrak{P}$ be maximal among the reductive subalgebras of the algebra $\mathfrak{G}$ but not be maximal among all the subalgebras of $\mathfrak{G}$. All the possible* (*to within conjugation*) *pairs* $(\mathfrak{G}, \mathfrak{P})$ *are listed in Table* 2 (*next page*).

(3) *Let $\mathfrak{G}^\tau$ be a simple compact classical Lie algebra and $\mathfrak{P}^\tau$ be its primitive reductive subalgebra, $\mathfrak{P}^\tau$ not being the maximal subalgebra of the algebra $\mathfrak{G}^\tau$. All the possible* (*to within conjugation*) *pairs* $(\mathfrak{G}^\tau, \mathfrak{P}^\tau)$ *can be obtained from the pairs listed in Table* 1 *by transition to the compact forms.*

Proof. (1) Assertion of Theorem's first item follows from Propositions 1, 2, 3, 4 and Note for Proposition 4.

TABLE 1. Primitive reductive subalgebras, which are not maximal among the reductive subalgebras.

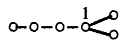
Algebra	Primitive subalgebras	$\mathrm{Int}_{\mathfrak{P}}\, \mathfrak{G} / \mathrm{Int}\, \mathfrak{P}$	
1. sl(n)	$\left(\overset{r}{\bigoplus} \mathfrak{GL}(k)\right)_0$ regular	Sym r	$kr = n$, $r \geq 3$, $k \geq 2$
2.	$\overset{r}{\bigotimes} \mathrm{sl}(k)$.	Sym r	$n = k^r$, $k \geq 2$, $r \geq 3$
3.	Cartan subalgebra	$W_{A_{n-1}}$	$n \geq 3$
4. sp($2n$)	$\overset{r}{\bigoplus} \mathrm{sp}(2k)$ regular	Sym r	$kr = n$, $k \geq 1$, $r \geq 3$
5.	$\overset{2r+1}{\bigotimes} \mathrm{sp}(2k)$	Sym($2r + 1$)	$2n = (2k)^{2r+1}$; if $k = 1$, $r \geq 2$
6.	$\mathrm{sp}(2s) \otimes \mathrm{sp}(2) \otimes \mathrm{sp}(2)$	Z_2	$n = 4s$, $s > 1$
7. o(n)		Z_2	$n = 4s$, $s > 1$
8.	$\overset{r}{\bigotimes} \mathrm{o}(k)$	Sym r, k odd Sym $r(z_2)^2$, k even	$n = k^r$, $k \geq 3$, $k \neq 4$, $r \geq 3$
9.	$\mathrm{o}(s) \otimes \mathrm{o}(4)$	z_2, s odd $z \times z_2$, s even	$s \geq 3$, $s \neq 4$, $n = 4s$
10.	$\overset{2r}{\bigotimes} \mathrm{sp}(2k)$	Sym $2r$	$n = (2k)^{2r}$ $k \geq 1$, $r \geq 2$
11.	$\overset{r}{\bigoplus} \mathrm{o}(k)$ (regular if n and k are even)	Sym r, k odd Sym $r(z_2)^k$, k even	$n = rk$, $r \geq 3$, $k \geq 3$
12.	Cartan subalgebra	W_{D_n}	n even

TABLE 2. Primitive subalgebras, which are maximal among the reductive subalgebras, but not among all the subalgebras.

Algebra	Primitive subalgebras	$\mathrm{Int}_{\mathfrak{P}}\, \mathfrak{G} / \mathrm{Int}_{\mathfrak{P}}$
sl($2n$)	$(\mathfrak{GL}(n) \oplus \mathfrak{GL}(n))_0$	$Z_2 \times Z_2$
o($2n$)	$\mathrm{o}(2n-2) \oplus \mathfrak{GL}^*(1)$	$Z_2 \times Z_2$
	$\mathfrak{GL}^*(n)$	Z_2 (with n even there are two classes of subalgebras which are not conjugate in Int O($2n$))

(2) Above we have presented the list of those subalgebras which are maximal among the reductive subalgebras of $\mathfrak{G}$, but not among all the subalgebras of $\mathfrak{G}$ (they certainly are maximal-range subalgebras with a non-trivial centre). All the primitive subalgebras of this class were presented in [7].

(3) Let $\mathfrak{P}^\tau$ be a primitive non-maximal subalgebra of $\mathfrak{G}^\tau$. Assume that there is such a reductive subalgebra $\mathfrak{L}$ that

$$\mathfrak{P} \subset \mathfrak{L} \subset \mathfrak{G}, \quad \mathfrak{L} \neq \mathfrak{P}, \quad \mathfrak{L} \neq \mathfrak{G}, \quad \mathrm{Int}_{\mathfrak{P}} \mathfrak{G}(\mathfrak{L}) = \mathfrak{L}.$$

Then $\mathfrak{P}^\tau \subset \mathfrak{L}^\tau \subset \mathfrak{G}^\tau$ and $\mathfrak{P}^\tau \neq \mathfrak{L}^\tau$, $\mathfrak{L}^\tau \neq \mathfrak{G}^\tau$. Take $f \in \mathrm{Int}_{\mathfrak{P}^\tau} \mathfrak{G}^\tau$. Since $\mathrm{Int}_{\mathfrak{P}^\tau} \mathfrak{G}^\tau < \mathrm{Int}_{\mathfrak{P}} \mathfrak{G}$, we have $f(\mathfrak{L}) = \mathfrak{L}$, whence $f(\mathfrak{L}^\tau) = \mathfrak{L}^\tau = \mathfrak{G}^\tau \cap \mathfrak{L}$. Thus, we have proved that if $\mathfrak{P}^\tau$ is a primitive non-maximal subalgebra, $\mathfrak{P}$ is primitive among all the reductive subalgebras of $\mathfrak{G}$.

Now let $\mathfrak{P}$ be a reductive and primitive subalgebra among reductive subalgebras of the algebra $\mathfrak{G}$, which is not maximal. Assume that there exists such a subalgebra $\mathfrak{L}^\tau$ that

$$\mathfrak{P}^\tau \subset \mathfrak{L}^\tau \subset \mathfrak{G}^\tau, \quad \mathfrak{P}^\tau \neq \mathfrak{L}^\tau, \quad \mathfrak{L}^\tau \neq \mathfrak{G}^\tau, \quad \mathrm{Int}_{\mathfrak{P}^\tau} \mathfrak{G}^\tau(\mathfrak{L}^\tau) \neq \mathfrak{L}^\tau.$$

For any reductive subalgebra $\mathfrak{P}$, which is primitive among the reductive subalgebras, we have presented such a finite-order automorphism σ that $\sigma \in \mathrm{Int}_{\mathfrak{P}} \mathfrak{G}$, $\sigma(\mathfrak{L}) \neq \mathfrak{L}$ for any reductive subalgebra $\mathfrak{L}$ such that $\mathfrak{P} \subset \mathfrak{L} \subset \mathfrak{G}$, $\mathfrak{P} \neq \mathfrak{L}$, $\mathfrak{G} \neq \mathfrak{L}$. Since σ has a finite order, one may assume (to within conjugation) that $\sigma \in \mathrm{Int}\, \mathfrak{G}^\tau$. Since $\sigma(\mathfrak{P}) = \mathfrak{P}$, $\mathfrak{P}^\tau = \mathfrak{P} \cap \mathfrak{G}^\tau$, we have $\sigma \in \mathrm{Int}_{\mathfrak{P}^\tau} \mathfrak{G}^\tau$.

From the supposition made above, $\sigma(\mathfrak{L}^\tau) = \mathfrak{L}^\tau$. But since $\mathfrak{L}^\tau + i\mathfrak{L}^\tau = \mathfrak{L}$, we have $\sigma(\mathfrak{L}) = \mathfrak{L}$. The resulting contradiction proves that our subalgebra $\mathfrak{P}$ is primitive.

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BYELORUSSIAN POLYTECHNIC INSTITUTE
MINSK, USSR