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ON ORTHOMORPHISMS BETWEEN VON NEUMANN PREDUALS AND A PROBLEM OF ARAKI

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A problem of Araki concerning the characterization of orthogonality preserving positive maps between preduals of von Neumann algebras is solved in a general setting.

Introduction. In an interesting recent article, Araki [1] initiated the study of orthogonal decomposition preserving positive linear maps (o.d. homomorphisms) between preduals of von Neumann algebras. (See below for definitions.)

Let M and N be von Neumann algebras and let $\phi: M_* \to N_*$ be a linear mapping. When either M or N is of Type I, with no direct summand of Type I_2, Araki proved that ϕ is a bijective o.d. homomorphism if, and only if, $\phi^* = z\pi$ where z is a positive invertible element of the centre of M and $\pi: N \to M$ is a Jordan isomorphism.

Araki posed the problem of establishing an analogous characterization when M and N were of Type II or Type III.

Araki used delicate Radon-Nikodym methods which seem very difficult to generalize to algebras which are not of Type I. However, by adopting a different approach, we are able to show, for arbitrary von Neumann algebras M and N, that if $\phi: M_* \to N_*$ is an o.d. homomorphism then $\phi^*\pi = z\,\mathrm{id}_M$ where z is a positive central element of M and π is a Jordan $*$ homomorphism, and we obtain a characterization in these terms. If ϕ is an o.d. isomorphism, we find that z is invertible and that π is a Jordan $*$ isomorphism. This proves that Araki's characterization of o.d. isomorphisms is valid for arbitrary von Neumann algebras M and N.

1. Preliminaries. Two positive linear functionals ρ, τ in the predual M_* of a W^*-algebra M are said to be orthogonal, written $\rho \perp \tau$, if the corresponding support projections $s(\rho)$, $s(\tau)$ are orthogonal elements in the algebra M. Every hermitian functional ρ in M_* admits a unique orthogonal decomposition $\rho = \rho_+ - \rho_-$, where $\rho_+, \rho_- \in M_*^+$ and $\rho_+ \perp \rho_-$. On the other hand every hermitian element x in M has a unique orthogonal decomposition $x = x_+ = x_-$, where $x_+, x_- \geq 0$ and $x_+ \cdot x_- = 0$.

In the language of orthogonally decomposable (o.d.) Banach spaces [**1**, **2**, **6**], given W^*-algebras M and N, a continuous linear map $M_* \to N_*$, or $M \to N$, is said to be an o.d. homomorphism if it preserves both order and the orthogonal decomposition, and to be an o.d. isomorphism if it is also bijective (and hence an order isomorphism, as is easy to see).

The dual $\phi^*: N \to M$ of an o.d. homomorphism $\phi: M_* \to N_*$ is not, in general, an o.d. homomorphism (see the example at the end of this section). But there is, nevertheless, a "duality" between the o.d. homomorphisms of preduals of W^*-algebras and the weak $*$ continuous o.d. homomorphisms of W^*-algebras. The latter, in fact, are positive central multiples of Jordan $*$ homomorphisms. Before considering maps between preduals, we need the following characterization of o.d. homomorphism between von Neumann algebras.

PROPOSITION 1.1. *Let $\psi: N \to M$ be a linear map between W^*-algebras. Then ψ is a weak $*$ continuous o.d. homomorphism if and only if $\psi = \pi(z\cdot)$ where $\pi: N \to M$ is a weak $*$ continuous Jordan $*$ homomorphism and $z \in Z(N)^+$.*

Moreover, ψ is an o.d. isomorphism if and only if π can be chosen to be a surjective Jordan $*$ isomorphism and z to be a positive central invertible element of N.

Let $\psi: N \to M$ be a weak $*$ continuous o.d. homomorphism between W^*-algebras. We may suppose without loss that $\|\psi\| = 1$. The o.d. property implies that $\psi \geq 0$ and that $\psi(a)\psi(b) = 0$ if $a, b \geq 0$ and $ab = 0$. In particular, given any projection p of N, $\psi(p)\psi(1-p) = 0$ so that

$$(*) \qquad \psi(p)\psi(1) = \psi(p)^2 = \psi(1) \cdot \psi(p).$$

Since, as a Banach space, N is generated by its projections, it follows that $\psi(1) \in Z(W)$, the centre of the W^*-subalgebra W of M generated by $\psi(N)$, and that the range projection $r(\psi(1)) = e$, where e is the identity element of W.

We note that if $\psi(1) = e$, then ψ preserves projections as well as orthogonality and hence is a Jordan $*$ homomorphism by elementary spectral theory. Consequently, in the case that $\psi(1)$ is merely invertible in W, we see that $\psi = \psi(1)\pi$, where π is a weak $*$ continuous Jordan $*$ homomorphism. (We note here that if ψ is an o.d. isomorphism then $\psi(1)$ must be invertible in M $(= W)$. This is because, then, the condition $(*)$ implies that the two-sided ideal $\psi(1)M$ is norm dense in M, and hence equals M as M is unital.)

In general, upon identifying $Z(W)$ with $C(X)$ and $\psi(1)$ with $f \in C(X)$ accordingly, where X is some compact hyperstonean space, we see that $0 \leq f \leq 1$ and that $\{x: f(x) > 0\}$ is dense in X (we had $r(\psi(1)) = e$). For each n let K_n be the closure in X of $\{x: f(x) > 1/n\}$. Then each K_n is a clopen subset of X. The characteristic functions $\mathcal{X}_{K_n}$, when translated back into $Z(W)$, give rise to an increasing sequence (e_n) of projections in $Z(W)$ converging strongly to e with the property that each $e_n\psi(1)$ is invertible in $e_n W$.

By the remark above applied to the o.d. homomorphism $e_n\psi: N \to e_n W$, this means that, for each n, there is a weak $*$ continuous Jordan $*$ homomorphism $\pi_n: N \to e_n W$ such that $e_n\psi = e_n\psi(1)\pi_n$. So, as $\sum(e_n - e_{n-1}) = e$ where $e_0 = 0$, we have $\psi = \psi(1)\pi$ where π is the weak $*$ continuous Jordan $*$ homomorphism from N onto W given by $\pi(x) = \sum(e_n - e_{n-1})\pi_n(x)$, $x \in N$. Since $\pi(N) = W$ we have $\pi(Z(N)) = Z(W)$, as in [5, Remark, p. 135]. Therefore choosing $z \in Z(N)$ with $\pi(z) = \psi(1)$ we have $\psi = \pi(z\cdot)$. The converse being obvious, this completes the proof.

The following example shows that a naive approach to duality of o.d. homomorphisms does not work.

We observe that given a W^*-algebra N without minimal central projections and any fixed $\rho \in N_*$ then, for all $n \geq 1$, the map $\phi: M_n(\mathbb{C})_* \to (M_n(\mathbb{C}) \otimes N)_*$ defined by $\phi(\tau) = \tau \otimes \rho$ is an o.d. homomorphism but its dual ϕ^* is not. In fact, there are no non-trivial weak $*$ continuous o.d. homomorphisms at all from $M_n(\mathbb{C}) \otimes N$ into $M_n(\mathbb{C})$.

2. o.d. homomorphisms of preduals. Given $\rho \in M_*$, where M is a W^*-algebra, and a central projection e of M let $\rho_e \in M_*$ be defined by $\rho_e(x) = \rho(ex)$, for all x in M. If $\rho \in M_*^+$, then $\rho_e \in M_*^+$, $s(\rho_e) = e \cdot s(\rho)$ and $\{\rho_e: \rho \in M_*\}$ is identified with $(eM)_*$.

PROPOSITION 2.1. *Let M and N be W^*-algebras and let $\phi: M_* \to N_*$ be an o.d. homomorphism. Then* $\ker\phi$ *is a norm closed invariant subspace of M_*. Hence* $\ker\phi = ((1-e)M)_*$, *for some central projection e of M, and ϕ is injective on the complement $(eM)_*$.*

Proof. We make use of fundamental results of Effros, as related in [4, III.4]. Recall, in particular, that each $\rho \in M_*$ has a polar decomposition, $\rho(\cdot) = |\rho|(\cdot u)$ where u is a partial isometry in M, $|\rho| \in M_*^+$ and $u^*u = s(|\rho|)$.

Put $K = (\ker\phi)_+$. Then K is a norm closed convex cone in M_*^+. The norm closed left invariant subspace of M_* generated by K is given by $V = \{\rho \in M_*: |\rho| \in K\}$, and $V_+ = K$. We will show first that $V \subset \ker\phi$.

Let, then, $\rho \in M_*$ such that $|\rho| \in K$. We may suppose that $\|\rho\| = 1$. Given x in N, where $x \geq 0$, we then have $|\rho|(\phi^*(x)) = \phi(|\rho|)(x) = 0$. Thus $\phi^*(x)$ is a positive element in the left kernel of $|\rho|$, as therefore is $\phi^*(x)^2$. Hence $|\rho|(\phi^*(x)^2) = 0$. But, by **[4**, III.4.6], $|\phi(\rho)(x)|^2 = |\rho(\phi^*(x))|^2 \leq |\rho|((\phi^*(x))^2)$. So $\phi(\rho)(x) = 0$ for all $x \in N_+$ implying that $\rho \in \ker\phi$, as required.

On the other hand, since $\phi \geq 0$, given $\rho = \sigma + i\tau \in \ker\phi$ where $\sigma = \sigma^*$ and $\tau = \tau^*$, we have that $\sigma, \tau \in \ker\phi$. The o.d. condition now implies that $\sigma_\pm, \tau_\pm \in \ker\phi$. Hence $\rho \in \mathrm{lin}(K) \subset V$. Therefore $V = \ker\phi$.

But, as is easy to see, $\rho \in \ker\phi$ if and only if $\rho^* \in \ker\phi$. So $\ker\phi$ is also a right invariant subspace of M_*, completing the proof.

Notation. In the remainder M and N are (arbitrary) W^*-algebras and $\phi: M_* \to N_*$ is an o.d. homomorphism. We will also write

$$\phi(\rho) = \rho', \quad \text{for all } \rho \text{ in } M_*.$$

We define V_ϕ to be the weak $*$ closed linear span of $\{s(\rho'): \rho \in M_*^+\}$ and n_ϕ to be the W^*-subalgebra of N generated by V_ϕ. The identity element of N_ϕ is $1_\phi = \sup\{s(\rho'): \rho \in M_*^+\}$. The central projection e for which $\ker\phi = ((1-e)M)_*$ (Proposition 2.1) will be denoted by e_ϕ.

Recall that the projections $s(\rho)$ where $\rho \in M_*^+$ are precisely the σ-finite projections of M and that, by a standard argument using Zorn's Lemma, every projection in M is the sum of an orthogonal family of σ-finite projections.

LEMMA 2.2. (i) *If* $\rho, \tau \in M_*^+$ *with* $\rho \perp \tau$, *then* $\rho' \perp \tau'$.
(ii) $\phi^*(1) = \phi^*(1_\phi)$ *and* $r(\phi^*(1)) = e_\phi$.

Proof. (i) This is a direct consequence of the o.d. property (as $\rho - \tau$ is an orthogonal decomposition in this case).

(ii) $\phi^*(1 - 1_\phi) = 0$ because $\rho(\phi^*(1 - 1_\phi)) = \rho'(1 - 1_\phi) = 0$, for all $\rho \in M_*^+$.

By Proposition 2.1, $\phi^*(N) \subset e_\phi M$ and for any ρ in M_*^+ with $\rho' \neq 0$ we have $\rho(r(\phi^*(1))) \geq \rho(\phi^*(1)) = \rho'(1) \neq 0$, as required.

LEMMA 2.3. (i) $\phi^*(s(\rho')) = \phi^*(1)s(\rho)$, *for all* $\rho \in M_*^+$.
(ii) $\phi^*(1) \in Z(M)$.

Proof. (i) By the remark above, given $\rho \in M_*^+$ we can write $1 - s(\rho) = \sum s(\tau_i)$, where (τ_i) is an orthogonal family in M_*^+. We have $\rho \perp \tau_i$ for each i, so that $\rho' \perp \tau_i'$ and hence $\tau_i(\phi^*(s(\rho'))) = \tau_i'(s(\rho')) = 0$. As $\phi^*(s(\rho')) \geq 0$, this means that $\phi^*(s(\rho')) \cdot s(\tau_i) = 0$, for all i. Hence $\phi^*(s(\rho')) \cdot (1 - s(\rho)) = 0$. Also, $\rho(\phi^*(1 - s(\rho'))) = \rho'(1-s(\rho')) = 0$, so that $\phi^*(1-s(\rho'))\cdot s(\rho) = 0$. Therefore, $\phi^*(s(\rho')) = \phi^*(1)s(\rho)$.

(ii) By (i), $\phi^*(1)$ commutes with all support projections in M and hence with all projections in M. It follows that $\phi^*(1) \in Z(M)$.

The following is an immediate consequence of [3, Lemma 4.1].

LEMMA 2.4. *Let* $\psi: N \to M$ *be a positive linear map such that* $\|\psi\| \leq 1$ *and let* e *and* f *be projections in* N *and* M, *respectively, such that* $\psi(e) = f$. *Then* $\psi(ex + xe) = f\psi(x) + \psi(x)f$, *for all* x *in* N.

LEMMA 2.5. ϕ^* *is injective on* V_ϕ.

Proof. By Proposition 2.1 it can be supposed without loss that ϕ is injective, so that $e_\phi = 1$. We have $\phi^*(1) \in Z(M)$ and $r(\phi^*(1)) = 1$, by (ii) of Lemmas 2.2, 2.3. Choose (see the proof of Proposition 1.1) an increasing sequence of projections (e_n) in $Z(M)$, converging strongly to 1 and such that $e_n\phi^*(1)$ has an inverse, t_n, in $e_nZ(M)$ for all n.

Now let $x \in V_\phi$ such that $\phi^*(x) = 0$ and $x = x^*$. Fixing n, put $\psi = t_n e_n \phi^*$. Then $\psi: N \to e_nM$ is positive and $\psi(1) = e_n$. Hence $\|\psi\| = 1$. Let $\rho \in M_*^+$. Then $\psi(s(\rho')) = e_n s(\rho)$, by Lemma 2.3(i). So using Lemma 2.4 in the second equation below,

$$\begin{aligned} t_n e_n \phi^*(xs(\rho') + s(\rho')x) &= \psi(s(\rho')x + xs(\rho')) \\ &= e_n s(\rho) \cdot \psi(x) + \psi(x) e_n s(\rho) = 0, \end{aligned}$$

as $\psi(x) = 0$. Hence $e_n\phi^*(s(\rho')x + xs(\rho')) = 0$, for all n, which implies that $s(\rho')x + xs(\rho') \in \ker\phi^*$, for all ρ in M_*^+. Therefore, $xy + yx \in \ker\phi^*$, for all y in V_ϕ. In particular, $x^2 \in \ker\phi^*$.

We now have $x^2 \in N_\phi$, $x^2 \geq 0$ and $\rho'(x^2) = \rho(\phi^*(x^2)) = 0$, so that $s(\rho')x^2 = 0$, for all $\rho \in M_*^+$. By the definition of N_ϕ this implies that $x^2 = 0$. So $x = 0$ and it follows from this that ϕ^* is injective on V_ϕ.

We are now in a position to provide a detailed description of the properties of ϕ^*.

THEOREM 2.6. *There is a weak $*$ continuous and surjective Jordan $*$ homomorphism $\pi: M \to N_\phi$ such that $\phi^*(\pi(x)) = \phi^*(1)x$, for all x in M. Moreover, π maps $e_\phi M$ isomorphically onto N_ϕ and $\phi^*: N_\phi \to e_\phi M$ is an injective o.d. homomorphism with dense image. Also, $V_\phi = N_\phi$.*

Proof. We claim that for any x in M, there is a unique element x' in V_ϕ such that $\phi^*(x') = \phi^*(1)x$. Uniqueness follows directly from the injectivity of ϕ^* on V_ϕ (Lemma 2.5). Existence is explained as follows.

First, let e be any projection in M. Then $e = \sum s(\rho_i)$, for some orthogonal family (ρ_i) in M_*^+. By Lemma 2.2(i), (ρ_i') is an orthogonal family in N_*^+. Therefore $e' = \sum s(\rho_i')$ is a projection of N lying in V_ϕ and, by weak $*$ continuity together with Lemma 2.3(i), we have

$$\phi^*(e') = \sum \phi^*(s(\rho_i')) = \sum \phi^*(1)s(\rho_i) = \phi^*(1)e.$$

Now let $x \in M$. In order to establish the claim it is sufficient to suppose $0 \le x \le 1$. We can then write $x = \sum(e_n/2^n)$, for certain spectral projections e_n of x.

By the above, there exist projections e_n' in V_ϕ such that $\phi^*(e_n') = \phi^*(1)e_n$, for each n. Thus $x' = \sum(e_n'/2^n) \in V_\phi$ and

$$\phi^*(1)x = \sum \phi^*(1)\frac{e_n}{2^n} = \sum \frac{\phi^*(e_n')}{2^n} = \phi^*(x'),$$

thereby proving the claim.

So, in the notation of the previous paragraph, we see that we have a well-defined function $\pi: M \to V_\phi$, given by $\pi(x) = x'$, satisfying

(a) $\phi^*(\pi(x)) = \phi^*(1)x$, for all x in M.

(b) $\pi(s(\rho)) = s(\rho')$, for all ρ in M_*^+.

(c) If (e_i) is an orthogonal family of projections in M, then $(\pi(e_i))$ is an orthogonal family of projections in V_ϕ and $\pi(\sum e_i) = \sum \pi(e_i)$.

Furthermore,

(d) π is linear.

In order to see (d), let $x, y \in M$. By (a),

$$\phi^*(\pi(x+y)) = \phi^*(1)(x+y) = \phi^*(\pi(x) + \pi(y)).$$

So, $\pi(x+y) = \pi(x) + \pi(y)$, by Lemma 2.5.

By (c) and (d) $\pi: M \to N$ is a linear map that preserves projections and is completely additive on projections. Hence π is a weak $*$ continuous Jordan homomorphism. Hence $\pi(M)$ is a W^*-subalgebra of M. But now $\pi(M) = V_\phi$, by (b). Hence $V_\phi = N_\phi$. By Lemma 2.2, $\phi^*(1) \in e_\phi M$ and $r(\phi^*(1)) = e_\phi$. Thus $\phi^*(\pi(1)) = \phi^*(1)e_\phi = \phi^*(\pi(e_\phi))$, by (a), so that $\pi(1) = \pi(e_\phi)$, by Lemma 2.5, and so $\pi(e_\phi M) = N_\phi$. If $x \in e_\phi M$ are $\pi(x) = 0$, then $\phi^*(1) \cdot x = 0$ so that $x = e_\phi x = r(\phi^*(1))x = 0$. This proves that π maps $e_\phi M$ isomorphically onto N_ϕ. Finally, if ψ is the inverse of $\pi: e_\phi M \to N_\phi$, then $\phi^* = \phi^*(1)\psi$ on N_ϕ, completing the proof.

For a W^*-algebra M, $z \in Z(M)^+$ and $\rho \in M_*$, the functional $\rho_z \in M_*$ is defined by $\rho_z(x) = \rho(zx)$, for all x in M. We extract the following characterisations.

COROLLARY 2.8. *Let M and N be W^*-algebras. Then a continuous linear map $\phi: M_* \to N_*$ is an o.d. homomorphism if and only if there is a positive central element z of M and a weak $*$ continuous Jordan $*$ homomorphism $\pi: M \to N$ such that*

$$\begin{aligned} \phi^*\pi(x) &= zx, &&\text{for all } x \text{ in } M \quad\text{and} \\ \|\phi(\rho)\| &= \|\rho_z\|, &&\text{for all } \rho \in M_*^+. \end{aligned}$$

Proof. It remains only to prove the 'if' part. Suppose then that the stated conditions hold as written. Let $\rho \in M_*^+$. Then

$$\|\phi(\rho)\| = \rho(z) = \rho(\phi^*(\pi(1))) = \phi(\rho)(\pi(1)).$$

So $\phi(\rho) \in N_*^+$ and further

$$\phi(\rho)(\pi(s(\rho))) = \rho(zs(\rho)) = \rho(z) = \phi(\rho)(1),$$

so that $s(\phi(\rho)) \leq \pi(s(\rho))$, from which the orthogonality condition follows.

COROLLARY 2.9. *Let M and N be W^*-algebras and let $\phi: M_* \to N_*$ be a linear map. Then the following are equivalent:*

(i) *$\phi: M_* \to N_*$ is an o.d. isomorphism;*
(ii) *$\phi^*: N \to M$ is an o.d. isomorphism;*
(iii) *$\phi^* = z\pi$ for some positive invertible central element z in M and a surjective Jordan $*$ isomorphism $\pi: N \to M$.*

Proof. (i) $\Rightarrow$ (ii) If (i) holds then $\phi^*: N \to M$ is a linear bijection by duality which, by Theorem 2.6, restricts to an o.d. homomorphism on V_ϕ. But $V_\phi = N$, by assumption. The implication (ii) $\Rightarrow$ (iii) follows from Proposition 1.1, and (iii) $\Rightarrow$ (ii) is immediate.

Corollary 2.9. solves the open problem posed in **[1]**.

References

[1] H. Araki, *An application of Dye's Theorem on projection lattices to orthogonally decomposable isomorphisms*, Pacific J. Math., **137** (1989), 1–13.

[2] T. B. Dang and S. Yamamuro, *On homomorphisms of an orthogonally decomposable Hilbert space*, J. Funct. Anal., **68** (1986), 366–373.

[3] E. Størmer, *Decomposition of positive projections on C^*-algebras*, Math. Ann., **247** (1980), 21–41.

[4] M. Takesaki, *Theory of Operator Algebras* I, Springer Verlag, 1979.

[5] J. Vesterstrøm, *On the homomorphic image of the centre of a C^*-algebra*, Math. Scand., **29** (1971), 134–136.

[6] S. Yamamuro, *On orthogonally decomposable ordered Banach spaces*, Bull. Austral. Math. Soc., **30** (1984), 357–380.

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