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ON THE EXTENSION OF LIPSCHITZ FUNCTIONS FROM BOUNDARIES OF SUBVARIETIES TO STRONGLY PSEUDOCONVEX DOMAINS

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In this paper, we study the principal value integral on boundaries of subvarieties in strongly pseudoconvex domains and using it, we give a condition for the extendability of Lipschitz functions.

Introduction. Let D be a strongly pseudoconvex domain in C^n with C^∞ boundary. Henkin [6] and Ramírez [12] obtained independently the support function $g(\zeta, z)$ for D which depends holomorphically on z, and then, using this support function, they obtained the integral formula for holomorphic functions in $\overline{D}$. On the other hand, Stout [14], when $p = 1$, and then, Hatziafratis [5], when p is arbitrary, obtained the integral formula for a certain subvariety V of codimension p in D. By using the support function $g(\zeta, z)$ and the integral formula for V, we can obtain the kernel $\Omega(\zeta, z)$ for $(\zeta, z) \in \partial V \times \overline{D}$. In this paper, we shall define the principal value integral P.V. $\int_{\partial V} f(\zeta)\Omega(\zeta, z)$ for a Lipschitz function f on ∂V and $z \in \partial V$. The definition of the principal value integral is the same as that of Alt [2] when $V = D$ (cf. Dolbeault [4]). By using the principal value integral we can give the condition for a Lipschitz function on ∂V to be the boundary value of a function that is holomorphic in D and continuous on $\overline{D}$. Finally we end the introduction by giving an example which shows that the Lipschitz continuity is necessary in order to define the principal value integral.

EXAMPLE. Define $\varphi \in C^\infty(0, \infty)$ such that

$$\varphi(\theta) = \begin{cases} 1 & \text{if } 0 < \theta \le \frac{\pi}{4}, \\ 0 & \text{if } \theta \ge \frac{\pi}{2}. \end{cases}$$

Extend φ to an odd function on $R|\{0\}$. Let D be the unit disc in $\mathbb{C}$ and f be a function on ∂D such that

$$f(e^{i\theta}) = \begin{cases} \frac{\varphi(\theta)}{\log|\theta|} & \text{if } 0 < |\theta| \le \pi, \\ 0 & \text{if } \theta = 0. \end{cases}$$

Then f is continuous but not Lipschitz continuous on ∂D. Compute the principal value integral at $1 \in \partial D$. We find

$$\begin{aligned}
&\text{P.V.}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)d\zeta}{\zeta-1}\\
&= \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{\varepsilon\le|\theta|\le\pi}\frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta}-1}\,d\theta\\
&= \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{\varepsilon}^{\pi} f(e^{i\theta})\frac{e^{i\theta}+1}{e^{i\theta}-1}\,d\theta\\
&= \lim_{\varepsilon\to 0+}\frac{1}{2\pi}\int_{\varepsilon}^{\pi/4}\frac{1}{\log\theta}\frac{e^{i\theta}+1}{e^{i\theta}-1}\,d\theta + (\text{finite value}).
\end{aligned}$$

But we have

$$\frac{1}{\log\theta}\frac{e^{i\theta}+1}{e^{i\theta}-1} \sim \frac{-2i}{\theta\log\theta}$$

when $\theta \to 0+$. This shows that the principal value integral of f at 1 does not converge.

1. The integral formula on subvarieties. Let D be a bounded strongly pseudoconvex domain in C^n with C^∞ boundary. Let ρ be a defining function of D, i.e., $D = \{z: \rho(z) < 0\}$. We set

$$\begin{aligned}
F(\zeta, z) = &\sum_{i=1}^{n} 2\frac{\partial\rho}{\partial\zeta_i}(\zeta)(z_i-\zeta_i)\\
&+ \sum_{i,j=1}^{n}\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_j).
\end{aligned}$$

According to the construction of Henkin [**6**] (cf. Henkin-Leiterer [**8**], p. 108), there exists a pseudoconvex neighborhood $\widetilde{D}$ of $\overline{D}$, a neighborhood W of ∂D, and a C^∞ function $g: W \times \widetilde{D} \to C$ such that for each $\zeta \in W$, $g(\zeta, z)$ is holomorphic in $\widetilde{D}$. For $r > 0$, define $\Delta_r = \{(\zeta, z) \in W \times \widetilde{D}: |\zeta - z| < r\}$. Then there exist a constant $\sigma_1 > 0$ and a non-vanishing C^∞ function $Q(\zeta, z)$ on Δ_{σ_1} such that $g(\zeta, z) = F(\zeta, z)Q(\zeta, z)$ for $(\zeta, z) \in \Delta_{\sigma_1}$ and $g(\zeta, z) \neq 0$ for $(\zeta, z) \in W \times \widetilde{D}|\Delta_{\sigma_1}$. Moreover $g(\zeta, z)$ admits a division

$$g(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, z)(z_j - \zeta_j)$$

with $g_j: W \times \widetilde{D} \to C$ of class C^∞ and holomorphic in the second variable. Let $h_1, \dots, h_p$ $(p < n)$ be holomorphic functions in $\widetilde{D}$.

Define

$$\widetilde{V} = \{z \in \widetilde{D}: h_1(z) = \cdots = h_p(z) = 0\}, \qquad V = \widetilde{V} \cap D.$$

Assume that

$$(1) \qquad \partial h_1 \wedge \cdots \wedge \partial h_p \wedge \partial \rho \neq 0 \quad \text{on } \partial V.$$

By Hefer's theorem, there exist holomorphic functions $h_{ij}(\zeta, z)$ for $(\zeta, z) \in \widetilde{D} \times \widetilde{D}$ such that

$$h_i(\zeta) - h_i(z) = \sum_{j=1}^{n} h_{ij}(\zeta, z)(\zeta_j - z_j), \qquad i = 1, \dots, p.$$

Define

$$\alpha^h(\zeta, z) = \begin{vmatrix} g_1 & h_{11} & \dots & h_{p1} & \overline{\partial}_\zeta g_1 & \dots & \overline{\partial}_\zeta g_1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ g_n & h_{1n} & \dots & h_{pn} & \overline{\partial}_\zeta g_n & \dots & \overline{\partial}_\zeta g_n \end{vmatrix},$$

$$|\nabla h(\zeta)|^2 = \sum_{1 \leq j_1 < \cdots < j_p \leq n} \left| \frac{\partial(h_1, \dots, h_p)}{\partial(\zeta_{j_1}, \dots, \zeta_{j_p})}(\zeta) \right|^2,$$

$$\beta^h(\zeta) = (-1)^{p(p+1)/2} |\nabla h(\zeta)|^{-2} \begin{vmatrix} \overline{\frac{\partial h_1}{\partial \zeta_1}} & \dots & \overline{\frac{\partial h_p}{\partial \zeta_1}} & d\zeta_1 & \dots & d\zeta_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{\frac{\partial h_1}{\partial \zeta_n}} & \dots & \overline{\frac{\partial h_p}{\partial \zeta_n}} & d\zeta_n & \dots & d\zeta_n \end{vmatrix}$$

and

$$K_V(\zeta, z) = c(n, p)\alpha^h(\zeta, z) \wedge \beta^h(\zeta),$$

where

$$c(n, p) = (-1)^{p(n+1)}(-1)^{n(n-1)/2} \frac{1}{(n-p)!(2\pi i)^{n-p}}.$$

Let $n - p = k$. We define the kernel $\Omega(\zeta, z)$ by

$$\Omega(\zeta, z) = \frac{K_V(\zeta, z)}{g(\zeta, z)^k}.$$

Let $A(D)$ (resp. $A(V)$) be the space of functions that are holomorphic in D (resp. V) and continuous on $\overline{D}$ (resp. $\overline{V}$). Then Hatziafratis [5] proved the following.

THEOREM 1. *For $f \in A(V)$ and $z \in V$, the integral formula*

$$(2) \qquad f(z) = \int_{\partial V} f(\zeta)\Omega(\zeta, z)$$

holds.

Now we begin by proving the following lemma.

Lemma 1. *If $h_1(z) = z_{k+1}, \dots, h_p(z) = z_n$, then we have*

$$K_V(\zeta, z) = (-1)^{k(k-1)/2}\frac{(k-1)!}{(2\pi i)^k} \cdot \sum_{j=1}^{k}(-1)^{j-1} g_j \bigwedge_{\substack{i=1\\ i\neq j}}^{k} \overline{\partial}_\zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k .$$

Proof. By the definition of α^h, β^h, we have

$$\alpha^h(\zeta, z) = \begin{vmatrix} g_1 & 0\cdots 0 & \overline{\partial}_\zeta g_1 \cdots \overline{\partial}_\zeta g_1 \\ \vdots & \vdots\ \vdots & \\ g_p & \begin{matrix} 0\cdots 0 \\ 1\cdots 0 \end{matrix} & \vdots \qquad \vdots \\ \vdots & \ddots & \\ g_n & 0\cdots 1 & \overline{\partial}_\zeta g_n \cdots \overline{\partial}_\zeta g_n \end{vmatrix} = (-1)^p \begin{vmatrix} g_1 & \overline{\partial}_\zeta g_1 \cdots \overline{\partial}_\zeta g_1 \\ \vdots & \vdots \qquad \vdots \\ g_k & \overline{\partial}_\zeta g_k \cdots \overline{\partial}_\zeta g_k \end{vmatrix}$$

$$= (-1)^{n-k}(k-1)! \sum_{j=1}^{k}(-1)^{j-1} g_j \bigwedge_{i\neq j}^{k} \overline{\partial}_\zeta g_i ,$$

$$\beta^h(\zeta) = (-1)^{\frac{p(p+1)}{2}} \begin{vmatrix} 0\cdots 0 & d\zeta_1 \cdots d\zeta_1 \\ \vdots\ \vdots & \\ \begin{matrix} 0\cdots 0 \\ 1\cdots 0 \end{matrix} & \vdots \qquad \vdots \\ \ddots & \\ 0\cdots 1 & d\zeta_n \cdots d\zeta_n \end{vmatrix} = (-1)^{\frac{p(p+1)}{2}} \begin{vmatrix} d\zeta_1 \cdots d\zeta_1 \\ \vdots \qquad \vdots \\ d\zeta_k \cdots d\zeta_k \end{vmatrix}$$

$$= (-1)^{\frac{(n-k)(n-k+1)}{2}} k! d\zeta_1 \wedge \cdots \wedge d\zeta_k .$$

Therefore we have

$$K_V(\zeta, z) = (-1)^{(n-k)(n+1)}(-1)^{n(n-1)/2}\frac{1}{k!(2\pi i)^k}\alpha^h(\zeta, z) \wedge \beta^h(\zeta, z)$$

$$= (-1)^{k(k-1)/2}\frac{(k-1)!}{(2\pi i)^k}\sum_{j=1}^{k}(-1)^{j-1} g_j \bigwedge_{i\neq j} \overline{\partial}_\zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k .$$

This completes the proof of Lemma 1.

For $\sigma > 0$, we set $S_{\zeta,\sigma} = \{z : |z-\zeta| < \sigma\}$. Then we have the following.

LEMMA 2. *Let* σ, $0 < \sigma < \sigma_1$, *be sufficiently small. Then for* $\zeta \in \partial V$, $z \in \widetilde{D} \cap S_{\zeta,\sigma}$, *we can choose a local coordinate system in such a way that*

$$\Omega(\zeta, z) = \frac{2^k}{(2\pi i)^k} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho)^{k-1}(\zeta) + e(\zeta, z)}{F(\zeta, z)^k},$$

where $e(\zeta, z)$ *is a* $(k, k-1)$ *form satisfying* $e(\zeta, z) = O(|\zeta - z|)$.

Proof. By a local holomorphic change of coordinates, we may assume that $h_1 = z_{k+1}, \dots, h_p = z_n$. We have (cf. Anderson and Berndtsson [3], Lemma 3),

$$\sum_{j=1}^k (-1)^{j-1} \frac{g_j \bigwedge_{i\neq j} \bar\partial_\zeta g_i \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_k}{g(\zeta, z)^k}$$
$$= c(k) \frac{(\sum_{j=1}^k g_j d\zeta_j) \wedge (\sum_{j=1}^\kappa \bar\partial_\zeta g_j \wedge d\zeta_j)^{k-1}}{g(\zeta, z)6k},$$

where $c(k) = (-1)^{k(k-1)/2} \frac{1}{(k-1)!}$.

If $|\zeta - z| < \sigma$, then we obtain

$$\sum_{i=1}^n g_i(\zeta, z)(z_i - \zeta_i)$$
$$= \sum_{i=1}^n \left\{ \left(2\frac{\partial\rho}{\partial\zeta_i}(\zeta) + \sum_{j=1}^n \frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(z_j - \zeta_j) \right) Q(\zeta, z) \right\} (z_i - \zeta_i).$$

Therefore we have

$$g_i(\zeta, \zeta) = 2\frac{\partial\rho}{\partial\zeta_i}(\zeta) Q(\zeta, \zeta), \qquad 1 \le i \le n.$$

By Lemma 1, we obtain

$$\Omega(\zeta, z) = \frac{1}{(2\pi i)^k} \frac{(\sum_{j=1}^k 2\frac{\partial\rho}{\partial\zeta_j}(\zeta)Q(\zeta, \zeta)d\zeta_j) \wedge (2\sum_{j=1}^k \bar\partial_\zeta(\frac{\partial\rho}{\partial\zeta_j}(\zeta)Q(\zeta, \zeta)) \wedge d\zeta_j)^{k-1}}{F(\zeta, z)^k Q(\zeta, z)^k}$$
$$+ \frac{e(\zeta, z)}{F(\zeta, z)^k Q(\zeta, z)^k}$$
$$= \frac{2^k}{(2\pi i)^k} \frac{\partial\rho(\zeta) \wedge (\bar\partial\partial\rho)^{k-1}(\zeta) + e(\zeta, z)}{F(\zeta, z)^k}.$$

This completes the proof of Lemma 2.

Now we prove the following lemma which will be used in the proof of Theorem 2 in order to calculate the principal value of the kernel function.

LEMMA 3. *For* $\zeta, z \in \partial V$, *it holds that*

$$F(\zeta, z) - \overline{F(z, \zeta)} = O(|\zeta - z|^3).$$

Proof. We may assume that $h_1 = z_{k+1}, \dots, h_p = z_n$. By the Taylor expansion, we have

$$\begin{aligned}
\frac{1}{2}F(\zeta, z) &= \sum_{i=1}^{k} \frac{\partial \rho}{\partial \zeta_i}(\zeta)(z_i - \zeta_i) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) \\
&= \sum_{i=1}^{k} \left(\frac{\partial \rho}{\partial \zeta_i}(z) + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_j - z_j) \right. \\
&\qquad \left. + \sum_{j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \overline{\zeta}_j}(z)(\overline{\zeta}_j - \overline{z}_j) \right)(z_i - \zeta_i) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j) + O(|\zeta - z|^3).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\frac{1}{2}\overline{F(\zeta, z)} &= \sum_{i=1}^{k} \frac{\partial \rho}{\partial \overline{\zeta}_i}(\zeta)(\overline{\zeta}_i - \overline{z}_i) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \overline{\zeta}_i \partial \overline{\zeta}_j}(\zeta)(\overline{\zeta}_i - \overline{z}_i)(\overline{\zeta}_j - \overline{z}_j).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\frac{1}{2}(F(\zeta, z) - \overline{F(z, \zeta)}) \\
&= -\sum_{i=1}^{k} \frac{\partial \rho}{\partial \zeta_i}(z)(\zeta_i - z_i) - \sum_{i=1}^{k} \frac{\partial \rho}{\partial \overline{\zeta}_i}(z)(\overline{\zeta}_i - \overline{z}_i) \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \zeta_j}(z)(\zeta_i - z_i)(\zeta_j - z_j) \\
&\quad - \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \overline{\zeta}_i \partial \overline{\zeta}_j}(z)(\overline{\zeta}_i - \overline{z}_i)(\overline{\zeta}_j - \overline{z}_j) \\
&\quad - \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \overline{\zeta}_j}(z)(\zeta_i - z_i)(\overline{\zeta}_j - \overline{z}_j) + O(|\zeta - z|^3) \\
&= -\rho(\zeta) + \rho(z) + O(|\zeta - z|^3) = O(|\zeta - z|^3).
\end{aligned}$$

This completes the proof of Lemma 3.

2. The principal value of the kernel function. Let $z \in \partial V$ and f be a continuous function on ∂V. If

$$\lim_{\varepsilon \to 0+} \int_{\partial V \cap \{\zeta : |g(\zeta, z)| > \varepsilon\}} f(\zeta)\Omega(\zeta, z)$$

exists, then we stand for the above limit by

$$\text{P.V.} \int_{\partial V} f(\zeta)\Omega(\zeta, z).$$

Now we are going to prove the following theorem which was obtained by Kerzman and Stein [9] when $k = n$.

THEOREM 2. *For $z \in \partial V$, it holds that*

$$\text{P.V.} \int_{\zeta \in \partial V} \Omega(\zeta, z) = \frac{1}{2}.$$

Proof. Let $q \in \partial V$ be fixed. We may assume that for $\delta > 0$ sufficiently small,

$$\widetilde{V} \cap S_{q,\delta} = \{z \in S_{q,\delta} : z_{k+1} = \cdots = z_n = 0\}.$$

For $z \in \widetilde{V} \cap S_{q,\delta}$, we have

$$\rho(z) = 2\operatorname{Re}\left(\sum_{i=1}^{k} \frac{\partial \rho}{\partial z_i}(q)(z_i - q_i) + \frac{1}{2}\sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(q)(z_i - q_i)(z_j - q_j)\right) + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(q)(z_i - q_i)(\overline{z}_j - \overline{q}_j) + O(|z-q|^3).$$

For simplicity, we may assume that $q = 0$. By (1), we can find a new local coordinate system $w_1, \dots, w_n$ by letting

$$w_1 = 2\sum_{i=1}^{k} \frac{\partial \rho}{\partial z_i}(0) z_i + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j,$$

and choosing $w_1, \dots, w_n$ suitably such that $w(0) = 0$. Then we have

$$\rho(w) = \operatorname{Re} w_1 + \sum_{i,j=1}^{k} \frac{\partial^2 \rho}{\partial w_i \partial \overline{w}_j}(0) w_i \overline{w}_j + O(|w|^3).$$

By a unitary transformation $w_i' = (w_1', \dots, w_n')$, we obtain

$$\rho(w') = \operatorname{Re}\left(\sum_{j=1}^{k} a_j w_j'\right) + \sum_{i=1}^{k} \frac{\partial^2 \rho}{\partial w_i' \partial \overline{w}_i'}(0)|w_i'|^2 + O(|w'|^3),$$

where $(a_1, \dots, a_k)$ is a non zero vector. Again we can find a new local coordinate system $\zeta = (\zeta_1, \dots, \zeta_n)$ such that

$$\rho(\zeta) = \operatorname{Re}\zeta_1 + \sum_{i=1}^{k} |\zeta_i|^2 + O(|\zeta|^3). \tag{3}$$

We set $\zeta_j = x_j + iy_j$. Then we have

$$\partial\rho(0) \wedge (\overline{\partial}\partial\rho)^{k-1}(0) = \left(\sum_{j=1}^{k} \frac{\partial \rho}{\partial \zeta_j}(0) d\zeta_j\right) \wedge \left(\sum_{i=1}^{k} \frac{\partial^2 \rho}{\partial \zeta_i \partial \overline{\zeta}_i}(0) d\overline{\zeta}_i \wedge d\zeta_i\right)^{k-1}.$$

Since $d\rho = 0$ on ∂V, we have

$$\sum_{j=1}^{k} \frac{\partial \rho}{\partial x_j}(0) dx_j + \sum_{j=1}^{k} \frac{\partial \rho}{\partial y_j}(0) dy_j = 0.$$

Therefore we have

$$\begin{aligned}
&\partial\rho(0)\wedge(\overline{\partial}\partial\rho)^{k-1}(0)\\
&\quad=\frac{i}{2}\sum_{j=1}^{k}\left(\frac{\partial\rho}{\partial x_j}(0)dy_j-\frac{\partial\rho}{\partial y_j}(0)dx_j\right)\\
&\qquad\wedge(2i)^{k-1}\left(\sum_{j=2}^{k}\frac{\partial^2\rho}{\partial w_j\partial\overline{w}_j}(0)dx_j\wedge dy_j\right)^{k-1}\\
&\quad=2^{k-2}i^k(k-1)!dy_1\wedge dx_2\wedge dy_2\wedge\cdots\wedge dx_k\wedge dy_k\,.
\end{aligned}$$

Thus, by Lemma 2, we obtain

(4)

$$\Omega(\zeta\,,\,z)=\frac{2^{k-2}(k-1)!dy_1\wedge dx_2\wedge dy_2\wedge\cdots\wedge dx_k\wedge dy_k+O(|\zeta|+|\zeta-z|)}{\pi^kF(\zeta\,,\,z)^k}.$$

Let ν be the unit inner normal of ∂V at 0. Then we have

$$\nu=-\left(\frac{\partial\rho}{\partial x_1}(0)\,,\,\frac{\partial\rho}{\partial y_1}(0)\,,\,\ldots\,,\,\frac{\partial\rho}{\partial y_n}(0)\right)=(-1\,,\,0\,,\,\ldots\,,\,0)\,.$$

We set, for $\delta>0$ sufficiently small, $z=\nu\delta$. Then we have

$$\begin{aligned}
&F(\zeta\,,\,z)-F(\zeta\,,\,0)\\
&\quad=\sum_{i=1}^{k}(-\zeta_i+\nu_i\delta)\left(2\frac{\partial\rho}{\partial\zeta_i}(\zeta)+\sum_{j=1}^{k}\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(-\zeta_j+\nu_j\delta)\right)\\
&\qquad-\sum_{i=1}^{k}(-\zeta_i)\left(2\frac{\partial\rho}{\partial\zeta_i}(\zeta)+\sum_{j=1}^{k}\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)(-\zeta_j)\right)\\
&\quad=\sum_{i=1}^{k}2\nu_i\delta\frac{\partial\rho}{\partial\zeta_i}(\zeta)\\
&\qquad+\sum_{i\,,\,j=1}^{k}(\nu_i\nu_j\delta^2-\nu_i\zeta_j\delta-\nu_j\zeta_i\delta)\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(\zeta)\\
&\quad=-\delta+O(\delta|\zeta|+\delta^2)\,.
\end{aligned}$$

On the other hand we have

$$F(0\,,\,\zeta)=\sum_{i=1}^{k}\zeta_i\left(2\frac{\partial\rho}{\partial\zeta_i}(0)+\sum_{j=1}^{k}\frac{\partial^2\rho}{\partial\zeta_i\partial\zeta_j}(0)\zeta_j\right)=\zeta_1=x_1+iy_1\,,$$

and

$$0 = \rho(\zeta) = x_1 + \sum_{i=2}^{k} |\zeta_i|^2 + O(|\zeta|^3).$$

Thus we obtain

$$F(0, \zeta) = -\sum_{i=1}^{k} |\zeta_i|^2 + iy_1 + O(|\zeta|^3).$$

Taking account of Lemma 3, we have

$$\begin{aligned} F(\zeta, z) &= F(\zeta, 0) - \delta + O(|\zeta|\delta + \delta^2) \\ &= \overline{F(0, \zeta)} - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2) \\ &= -iy_1 - \sum_{i=2}^{k} |\zeta_i|^2 - \delta + O(|\zeta|^3 + |\zeta|\delta + \delta^2). \end{aligned}$$

We set

$$\beta(0, \varepsilon) = \lim_{\delta \to 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{K_V(\zeta, \nu\delta)}{F(\zeta, \nu\delta)^k}.$$

Then, by (4), we have

$$\beta(0, \varepsilon) = \lim_{\delta \to 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{2^{k-2}(k-1)!\sigma(d\zeta) + O(|\zeta| + \delta)}{\pi^k F(\zeta, \nu\delta)^k}$$

where $\sigma(d\zeta)$ is the surface element of $\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}$. We set $\zeta' = (\zeta_2, \dots, \zeta_k)$. It holds from (3) that $(\partial\rho/\partial x_1)(0) \neq 0$. Therefore by the implicit function theorem, $\rho = 0$ can be represented by $x_1 = \varphi(y_1, \zeta')$, where φ is a smooth function satisfying $\varphi(0, 0) = 0$. Thus we have $x_1 = O(|y_1| + |\zeta'|)$. Hence there exists a constant $c_1 > 0$ such that

$$\big|\, ||\zeta'|^2 + iy_1| - |F(\zeta, 0)| \,\big| \le c_1(|\zeta'| + |y_1|)^3.$$

Therefore, for $\varepsilon > 0$ sufficiently small, we have

$$\left\{\zeta \in \partial V : ||\zeta'|^2 + iy_1| < \frac{\delta}{2}\right\} \\ \subset \{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\} \subset \left\{\zeta \in \partial V : ||\zeta'|^2 + iy_1| < \frac{3}{2}\varepsilon\right\}.$$

Thus we have

$$\beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{\delta \to 0} \int_{\{\zeta \in \partial V : |F(\zeta, 0)| < \varepsilon\}} \frac{\sigma(d\zeta)}{F(\zeta, \nu\delta)^k},$$

provided the limit of the right-hand side exists. We set $A = \delta + |\zeta'|^2 + iy_1$. Then we have $F(\zeta, \nu\delta) = A + \mathscr{E}$ where $\mathscr{E} = O(|\zeta|^3 + \delta|\zeta| + \delta^2)$. Then for some constant $c_2 > 0$, it holds that

$$\left|\frac{\mathscr{E}}{A}\right| \le c_2(|\zeta'| + |y_1| + \delta).$$

Thus if we choose δ and ε sufficiently small, then we have

$$\left|\frac{\mathscr{E}}{A}\right| \le \frac{1}{2}.$$

On the other hand we have

$$\frac{1}{F(\zeta, \nu\delta)^k} = \frac{1}{(A + \mathscr{E})^k} = \frac{1}{A^k} + \mathscr{E}',$$

where $\mathscr{E}'$ satisfies, for some constant $c_3 > 0$,

$$|\mathscr{E}'| \le \frac{c_3(|\zeta'| + |y_1| + \delta)}{|A|^k}.$$

It holds that

$$\int_{\{|iy_1 + |\zeta'|^2 < \varepsilon\}} \frac{|\zeta'| + |y_1| + \delta}{|A|^k} \sigma(d\zeta) \to 0 \quad (\text{as } \delta \to 0, \varepsilon \to 0).$$

Thus we obtain

$$\beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{\delta\to 0} \int_{\{|iy_1 + |\zeta'|^2 < \varepsilon\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + \delta)^k}.$$

The dilation $(y-1, \zeta') \to (\delta y_1, \sqrt{\delta}\zeta')$ gives

$$\lim_{\varepsilon\to 0} \beta(0, \varepsilon) = \frac{2^{k-2}(k-1)!}{\pi^k} \lim_{R\to\infty} \int_{\{|iy_1 + |\zeta'|^2| < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}.$$

The calculation of this limit is contained in Korànyi and Vagi **[10]** as follows. Consider the integral

$$I(\varepsilon, R) = \int_{\{\varepsilon < |iy_1 + |\zeta'|^2| < R\}} \frac{\sigma(d\zeta)}{(iy_1 + |\zeta'|^2 + 1)^k}, \qquad 0 < \varepsilon < R.$$

Introduce polar coordinates in C^{k-1},

$$\rho = |\zeta'|, \quad \omega = \frac{\zeta'}{|\zeta'|}, \quad \sigma(d\zeta) = \rho^{2k-3} dy_1\, d\rho\, d\omega$$

where $d\omega$ is the surface element in the unit sphere S^{2k-3} in C^{k-1}. We denote the volume of the unit sphere S^{2k-3} by $|S^{2k-3}|$. Next we make the variable change

$$u = \rho^2, \qquad du = 2\rho\, d\rho$$

and then introduce polar coordinates in the u, y_1 halfplane,

$$u + iy_1 = se^{i\theta}, \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad \varepsilon < s < R,$$
$$du\, dy_1 = s\, ds\, d\theta.$$

We find

$$\begin{aligned} I(\varepsilon, R) &= |S^{2k-3}| \int_{\{\varepsilon^2 < y_1^2 + \rho^4 < R^2\}} \frac{\rho^{2k-3} d\rho\, dy_1}{(1 + \rho^2 + iy_1)^k} \\ &= \frac{|S^{2k-3}|}{2} \int_{\{\varepsilon^2 < y_1^2 + u^2 < R,\ 0 \leq u\}} \frac{u^{k-1}\, du\, dy_1}{(1 + u + iy_1)^k} \\ &= \frac{|S^{2k-3}|}{2} \int_{-\pi/2}^{\pi/2} \cos^{k-2}\theta\, d\theta \int_\varepsilon^R \frac{s^{k-1}\, ds}{(1 + se^{i\theta})^k}. \end{aligned}$$

The variable change $\theta \to \theta - \frac{\pi}{2}$, $s \to \frac{1}{\rho}$ finally gives

$$I(\varepsilon, R) = \frac{|S^{2k-3}|}{2} \int_0^\pi \sin^{k-2}\theta\, d\theta \int_{1/R}^{1/\varepsilon} \frac{d\rho}{\rho(\rho - ie^{i\theta})}.$$

Then Korànyi and Vagi ([**10**], Lemma 6.2, p. 613) gives

$$\lim_{\substack{\varepsilon \to 0+ \\ R \to \infty}} I(\varepsilon, R) = \frac{\pi^{k-1}}{(k-2)!} \cdot \frac{\pi}{2^{k-1}(k-1)!} = \frac{\pi^k}{2^{k-1}(k-1)!}.$$

Hence we obtain

$$\lim_{\varepsilon \to 0+} \beta(0, \varepsilon) = \frac{1}{2},$$

which completes the proof of Theorem 2.

3. The continuous extension to the boundary. The following proposition is proved essentially by Adachi [**1**] (cf. Henkin [**7**]). But for the comparison with the principal value integral, we give the sketch of the proof.

PROPOSITION 1. *Define, for* $z \in \overline{D} | \partial V$,

$$H(z) = \int_{\zeta \in \partial V} \Omega(\zeta, z).$$

Then the function

$$\widetilde{H}(z) = \begin{cases} H(z) & (z \in \overline{D} | \partial V), \\ 1 & (z \in \partial V) \end{cases}$$

belongs to $A(D)$.

Proof. Since $H(z)$ is holomorphic in $\overline{D}|\partial V$, it is sufficient to show that, for $z^0 \in \partial V$,

$$\lim_{z \to z^0,\, z \in \overline{D}|\partial V} H(z) = 1 .$$

We may assume that

$$\widetilde{V} \cap S_{z^0, \sigma_1} = \{z \in S_{z^0, \sigma_1} : z_{k+1} = \cdots = z_n = 0\} .$$

By (1), we may assume, without loss of generality, that $(\partial \rho / \partial \zeta_1)(z^0) \neq 0$. For $z \in S_{z^0, \sigma_1}$, we consider the system of equations for $\zeta^0 = (\zeta_1^0, \ldots, \zeta_n^0)$ of the following form:

$$\text{(5)} \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial \rho}{\partial \zeta_i}(\zeta^0)(\zeta_i^0 - z_i) = 0 , \\ \zeta_i^0 = z_i \ (i = 2, \ldots, k), \quad \zeta_{k+1}^0 = \cdots = \zeta_n^0 = 0 . \end{cases}$$

We set $\varepsilon = (|z_{k+1}|^2 + \cdots + |z_n|^2)^{1/2}$. Then by Adachi **[1]**, there exist positive constants σ_2 $(< \sigma_1)$, γ_1 and γ_2 such that for any $\sigma \leq \sigma_2$ and any $z \in S_{z^0, \sigma/2} \cap (\overline{D}|\partial V)$, there exists a unique solution $\zeta^0 = \zeta^0(z)$ of the system (5) which belongs to the set $S_{z^0, \sigma} \cap V$ and satisfies the following.

$$\text{(6)} \qquad \varepsilon \leq |z - \zeta^0| \leq \gamma_1 \varepsilon ,$$

$$\text{(7)} \qquad \left| \sum_{i=1}^{n} \frac{\partial g}{\partial z_i}(\zeta, z)(\zeta_i^0 - z_i) \right| \leq \gamma_2 \varepsilon (|\zeta - z| + \varepsilon) .$$

From the integral formula (2) we have

$$H(\zeta^0) = 1 .$$

Hence it is sufficient to show that

$$\lim_{z \to z^0,\, z \in \overline{D}|\partial V} |H(z) - H(\zeta^0)| = 0 .$$

Let $z \in S_{z^0, \sigma/2} \cap (\overline{D}|\partial V)$. Let V' be an open subset in $\widetilde{V}$ with smooth boundary such that $\overline{V} \subset V' \subset \overline{V}' \subset \widetilde{V}$. By using Stokes' formula, we have

$$H(z) = \int_{\partial V'} \Omega(\zeta, z) - \int_{V' - V} \overline{\partial}_\zeta \Omega(\zeta, z) .$$

Define

$$\Psi(z) = \int_{(V' - V) \cap S_{z^0, \sigma}} \overline{\partial}_\zeta \Omega(\zeta, z) .$$

It is sufficient to show that

$$\lim_{z \to z^0,\, z \in \overline{D} | \partial V} |\Psi(z) - \Psi(\zeta^0)| = 0.$$

We can write $\Psi(z)$ in the following form

$$\Psi(z) = \int_{(V'-V)\cap S_{z^0,\sigma}} \frac{A(\zeta, z)}{g(\zeta, z)^k} + \int_{(V'-V)\cap S_{z^0,\sigma}} \sum_{j=1}^{n} \frac{(\zeta_j - z_j)B_j(\zeta, z)}{g(\zeta, z)^{k+1}},$$

where $A(\zeta, z)$, $B_j(\zeta, z)$ are (k, k) forms that are smooth in (ζ, z) and holomorphic in z. By using (6), (7), there are positive constants γ_3 and γ_4 such that

$$\left|\frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\Big|_{\lambda=1}\right| \le \gamma_3 \int_{(V'-V)\cap S_{z^0,\sigma}} \frac{\varepsilon\, dV}{|g(\zeta, z)|^{k+1}} + \gamma_4 \int_{(V'-V)\cap S_{z^0,\sigma}} \frac{|\zeta - z|\varepsilon(|\zeta - z| + \varepsilon) dV}{|g(\zeta, z)|^{k+2}}.$$

By the estimates obtained by Henkin [7], we have for some constant $\gamma_5 > 0$

$$\text{(8)} \qquad \left|\frac{d\Psi(\zeta^0 + \lambda(z - \zeta^0))}{d\lambda}\Big|_{\lambda=1}\right| \le \gamma_5(\varepsilon|\log \varepsilon| + \varepsilon).$$

Let

$$z(\theta) = \zeta^0 + \theta(z - \zeta^0) \quad \text{for } \theta \in [0, 1].$$

Then the uniqueness of the solution of the system (5) implies $\zeta^0(z(\theta)) = \zeta^0(z)$. Therefore from (8), we have, for some constant $\gamma_6 > 0$,

$$\left|\frac{d\Psi(\zeta^0 + \lambda\theta(z - \zeta^0))}{d\lambda}\Big|_{\lambda=1}\right| = \left|\frac{d\Psi(\zeta^0(z(\theta)) + \lambda(z(\theta) - \zeta^0(z(\theta))))}{d\lambda}\Big|_{\lambda=1}\right| \le \gamma_6(\theta\varepsilon|\log(\theta\varepsilon)| + \theta\varepsilon).$$

Thus we obtain for some constant $\gamma_7 > 0$,

$$\begin{aligned}|\Psi(z) - \Psi(\zeta^0)| &= \left|\int_0^1 \frac{d}{d\theta}\Psi(\zeta^0 + \theta(z - \zeta^0))\, d\theta\right| \\ &= \left|\int_0^1 \frac{1}{\theta}\left(\frac{d\Psi(\zeta^0 + \lambda\theta(z - \zeta^0))}{d\lambda}\right)_{\lambda=1} d\theta\right| \\ &\leq \gamma_7 \int_0^1 (\varepsilon|\log\varepsilon| + \varepsilon|\log\theta| + \varepsilon)d\theta \to 0 \qquad (\varepsilon \to 0),\end{aligned}$$

which completes the proof of Proposition 1.

4. The extension of Lipschitz functions from the boundary. In order to prove Lemma 5, we need the following lemma which is the modified version of Lemma 3.1 of Henkin [6].

LEMMA 4. *Let* $t = (t_1, \dots, t_{2k}) \in R^{2k}$, $\varepsilon > 0$, $0 < \delta < 1$, $t' = (t_2, \dots, t_{2k})$. *Then we have*

$$I_1 = \int_{\{\delta^2 \leq |t'|^2 + \varepsilon^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \varepsilon^2)^2 + t_2^2]^{k/2}} \leq \gamma \log\frac{\gamma}{\delta},$$

where γ *is the constant which is independent of* ε *and* δ.

Proof. (a) In case $\varepsilon^2 \leq \frac{1}{2}\delta^2$. Since $\delta^2 \leq |t'|^2 + \varepsilon^2$, we have $|t'|^2 \geq \frac{1}{2}\delta^2$. Therefore we have

$$I_1 \leq \int_{\{\delta^2/2 \leq |t'|^2 \leq 1\}} \frac{dt_2 \cdots dt_{2k}}{(|t'|^2 + t_2)^{k/2}}.$$

If $k = 1$, then we have

$$I_1 \leq \int_{\delta/2}^1 \frac{dt_2}{t_2} \leq \gamma \log\frac{\gamma}{\delta}.$$

If $k \geq 2$, by using polar coordinates, we have for some $\gamma_1 > 0$,

$$\begin{aligned} I_1 &\le \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{r^{2k-2} \sin^{2k-3} \varphi \, d\theta}{(r^4 + r^2 \cos^2 \varphi)^{k/2}} \\ &= \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{r^{k-2} \sin^{2k-3} \varphi \, d\varphi}{(r^2 + \cos^2 \varphi)^{k/2}} \\ &\le \gamma_1 \int_{\delta/2}^1 dr \int_0^\pi \frac{\sin^{2k-3} \varphi}{r^2 + \cos^2 \varphi} \, d\varphi \\ &\le \gamma_1 \int_{\delta/2}^1 dr \int_{-1}^1 \frac{ds}{r^2 + s^2} = \gamma_1 \int_{\gamma/2}^1 \frac{2}{r} \tan^{-1}\left(\frac{1}{r}\right) dr \\ &\le \pi\gamma_1 \int_{\delta/2}^1 \frac{dr}{r} \le \gamma \log \frac{\gamma}{\delta}. \end{aligned}$$

(b) In case $\varepsilon^2 \ge \frac{1}{2}\delta^2$. Then we have

$$I_1 \le \int_{\{|t'| \le 1\}} \frac{dt_2 \cdots dt_{2k}}{[(|t'|^2 + \varepsilon^2)^2 + t_2^2]^{k/2}}.$$

If $k = 1$, then we have for some $\gamma_2 > 0$,

$$I_1 \le \gamma_2 \int_0^1 \frac{dt_2}{t_2 + \delta^2} \le \gamma \log \frac{\gamma}{\delta}.$$

If $k \ge 2$, then we obtain for some $\gamma_3 > 0$,

$$\begin{aligned} I_1 &\le \gamma_3 \int_0^1 dr \int_0^\pi \frac{r^{2k-2} \sin^{2k-3} \varphi \, d\theta}{[(r^2 + \delta^2)^2 + r^2 \cos^2 \varphi]^{k/2}} \\ &\le \gamma_3 \int_0^1 dr \int_0^\pi \frac{\sin^{2k-3} \varphi \, d\varphi}{(r + \delta^2/r)^2 + \cos^2 \varphi} \\ &\le \gamma_3 \int_0^1 dr \int_{-1}^1 \frac{ds}{(r + \delta^2/r)^2 + s^2} \\ &\le \pi\gamma_3 \int_0^1 \frac{dr}{\delta^2/r + r} \le \frac{\pi\gamma_3}{2} \int_{\delta^2}^{1+\delta^2} \frac{d\lambda}{\lambda} \le \gamma \log \frac{\gamma}{\delta}. \end{aligned}$$

This completes the proof of Lemma 4.

Define, for $\delta > 0$,

$$(\partial V)_\delta = \{w : |w - \zeta| < \delta \text{ for some } \zeta \in \partial V\}.$$

Then we have

LEMMA 5. *There exists* $\delta_0 > 0$ *such that for any* $z \in \overline{D}$ *and any* δ $(0 < \delta < \delta_0)$, *it holds that*

$$I_2 = \int_{(S_{z,\delta_0}|S_{z,\delta})\cap\partial V} \frac{\sigma(d\zeta)}{|F(\zeta, z)|^k} \le \gamma \log\frac{\gamma}{\delta}.$$

where γ *is independent of* z *and* δ.

Proof. There exist positive constants δ_0 and γ_1 such that

$$-\operatorname{Re} F(\zeta, z) \ge \rho(\zeta) - \rho(z) + \gamma_1|\zeta - z|^2 \quad \text{for } z \in (\partial V)_{\delta_0},\ \zeta \in S_{z,\delta_0}.$$

We may assume that $z \in (\partial V)_{\delta_0} \cap \overline{D}$, $d\rho \ne 0$ on $(\partial V)_{\delta_0}$, and that

$$\widetilde{V} \cap S_{z,\delta_0} = \{w \in S_{z,\delta_0}: w_{k+1} = \cdots = w_n = 0\}.$$

We can find a new local coordinate system $t = (t_1, \dots, t_{2n})$ by letting

$$t_1 + it_2 = \rho(\zeta) - \rho(z) + i \operatorname{Im} F(\zeta, z),$$
$$t_{2j-1} + it_{2j} = \zeta_j - z_j \quad \text{for } j = k+1, \dots, n,$$

and choosing $t_3, \dots, t_{2k}$ suitably such that $t(z) = 0$. Then there exist positive constants γ_2, γ_3 and γ_4 such that

$$\gamma_2|\zeta - z|^2 \le |t|^2 \le \gamma_3|\zeta - z|^2,$$
$$|F(\zeta, z)| \ge \gamma_4[(t_1 + |t|^2)^2 + t_2^2]^{1/2}.$$

Define $\varepsilon^2 = |\rho(z)| + |z_{k+1}|^2 + \cdots + |z_n|^2$. Taking account of the relation

$$(S_{z,\delta_0}|S_{z,\delta}) \cap \partial V \subset \{t: t_1 = -\rho(z),\ \gamma_2\delta^2 \le |t|^2 \le \gamma_3^2\delta_0^2,$$
$$t_{2j-1} + it_{2j} = -z_j\ (j = k+1, \dots, n)\},$$

we have, together with Lemma 4, for some $\gamma_5 > 0$,

$$I_2 \le \gamma_5 \int_{\{\gamma_2\delta^2 \le |t|^2 \le \gamma_3^2\delta_0^2\}} \frac{dt_2 \cdots dt_{2k}}{[(t_2^2 + \cdots + t_{2k}^2 + \varepsilon^2)^2 + t_2^2]^{k/2}} \le \gamma \log\frac{\gamma}{\delta},$$

which completes the proof of Lemma 5.

We set

$$K_V(\zeta, z) = N_V(\zeta, z)\sigma(d\zeta).$$

Then we have the following.

LEMMA 6. *Let* $0 < \alpha \leq 1$. *Then there exists a positive constant* δ_0 *such that for any* $z \in \overline{D}$ *and any* δ $(0 < \delta < \delta_0)$,

$$I_3 = \int_{S_{z,\delta} \cap \partial V} \frac{|\zeta - z|^\alpha |N_V(\zeta, z)|}{|g(\zeta, z)|^k} \sigma(d\zeta) \leq \gamma \delta^\alpha \log \frac{\gamma}{\delta},$$

where γ *is independent of* δ *and* z.

Proof. There exists a positive constant γ_1 such that

$$I_3 \leq \gamma_1 \int_{S_{z,\delta} \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(d\zeta)}{|F(\zeta, z)|^k}.$$

Thus, by Lemma 5, we have

$$\begin{aligned} I_3 &\leq \gamma_1 \sum_{i=0}^{\infty} \int_{(S_{z,\delta 2^{-i}} | S_{z,\delta 2^{-(i+1)}}) \cap \partial V} \frac{|\zeta - z|^\alpha \sigma(d\zeta)}{|F(\zeta, z)|^k} \\ &= \sum_{i=0}^{\infty} \left(\frac{i+1}{2^{i\alpha}} \log 2 \right) \gamma_1 \delta^\alpha + \gamma_1 \delta^\alpha \log \left(\frac{\gamma_2}{\delta} \right) \left(\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \right) \\ &\leq \gamma \delta^\alpha \log \frac{\gamma}{\delta}, \end{aligned}$$

which completes the proof of Lemma 6.

Let F be a closed subset of C^n. According to the definition of Stein [13], we define the Lipschitz space for $0 < \alpha \leq 1$ such that

$$\operatorname{Lip}(\alpha, F) = \{f : |f(x)| \leq M,\ |f(x) - f(y)| \leq M|x - y|^\alpha,\ x, y \in F\}.$$

From the extension theorem (Stein [13], Theorem 3, p. 174), $f \in \operatorname{Lip}(\alpha, F)$ can be regarded as an element of $\operatorname{Lip}(\alpha, C^n)$. We shall prove the following theorem which was proved by Martinelli in the case when the kernel is Bochner-Martinelli kernel (cf. Martinelli [11], Dolbeault [4]).

THEOREM 3. *Let* $f \in \operatorname{Lip}(\alpha, \partial V)$. *Then it holds that for any* $z \in \partial V$,

$$\lim_{t \to z,\, t \in \overline{D} | \partial V} \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, t) = \int_{\zeta \in \partial V} (f(\zeta) - f(z)) \Omega(\zeta, z).$$

Proof. Since $f \in \operatorname{Lip}(\alpha, \partial V)$, the integral of the right-hand side converges. In view of Lemma 6, for $\varepsilon > 0$, there exists $\delta > 0$ such that for any $w \in \overline{D}$, we have

$$\int_{S_{w,3\delta} \cap \partial V} \frac{|f(\zeta) - f(w)|\,|N_V(\zeta, w)|}{|g(\zeta, z)|^k} \sigma(d\zeta) < \varepsilon .$$

We set

$$T(\zeta, w) = \frac{(f(\zeta) - f(w))N_V(\zeta, w)}{g(\zeta, w)^k},$$

$$A_\delta = \{(\zeta, w) \in \partial V \times \overline{D}: |w - \zeta| \geq \delta\}.$$

Since $T(\zeta, w)$ is continuous on A_δ, there exists ρ $(0 < \rho < \delta)$ such that

$$|T(\zeta, z) - T(\zeta, w)| < \varepsilon, \quad |f(z) - f(w)| < \varepsilon$$
$$\text{for } |w - z| < \rho, \quad |\zeta - z| \geq 2\rho, \qquad \zeta \in \partial V .$$

By Proposition 1, there exists a constant γ_1 such that

$$|H(t)| \leq \gamma_1 \quad \text{for } t \in \overline{D} | \partial V .$$

Thus we have for $|t - z| < \rho$, $t \in \overline{D}|\partial V$,

$$\left| \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, t) - \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \right|$$
$$= \left| \int_{\partial V} T(\zeta, t)\sigma(d\zeta) + (f(t) - f(z))H(t) - \int_{\partial V} T(\zeta, z)\sigma(d\zeta) \right|$$
$$\leq \gamma_1 |f(t) - f(z)| + \int_{\partial V | S_{z,2\delta}} |T(\zeta, t) - T(\zeta, z)|\sigma(d\zeta)$$
$$+ \int_{S_{z,2\delta} \cap \partial V} |T(\zeta, z)|\sigma(d\zeta) + \int_{S_{t,3\delta} \cap \partial V} |T(\zeta, t)|\sigma(d\zeta)$$
$$\leq \gamma_1 \varepsilon + \varepsilon \int_{\partial V | S_{z,2\delta}} \sigma(d\zeta) + \varepsilon + \varepsilon \leq \gamma_2 \varepsilon .$$

This completes the proof of Theorem 3.

Now we shall prove the following theorem which shows that any Lipschitz function on ∂V is the continuous boundary value of a function of $A(D)$ by adding $\int_{\partial V} (f(\zeta) - f(\cdot))K_V(\zeta, \cdot)$.

THEOREM 4. *Let* $f \in \mathrm{Lip}(\alpha, \partial V)$, $0 < \alpha \leq 1$. *Define*

$$\tilde{f}(z) = f(z) + \int_{\zeta \in \partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \quad \textit{for } z \in \partial V ,$$

and

$$F(z) = \int_{\zeta \in \partial V} f(\zeta)\Omega(\zeta, z) \quad \textit{for } z \in \overline{D} | \partial V .$$

Then the holomorphic function $F(z)$ *can be extended continuously to* $\overline{D}$ *and it has the boundary value* $F|_{\partial V} = \tilde{f}$.

Proof. For $z \in \partial V$, we have from Theorem 3,

$$\begin{aligned} &\lim_{t \to z, t \in \overline{D} | \partial V} (F(t) - \tilde{f}(z)) \\ &\quad = \lim_{t \to z, t \in \overline{D} | \partial V} \left(\int_{\partial V} (f(\zeta) - f(z))\Omega(\zeta, t) \right. \\ &\qquad \left. - \int_{\partial V} (f(\zeta) - f(z))\Omega(\zeta, z) \right) = 0 . \end{aligned}$$

Since $F(z)$ is holomorphic in $\overline{D} | \partial V$, $F \in A(D)$. This completes the proof of Theorem 4.

The boundary value $\tilde{f}$ is also represented by

$$\tilde{f}(z) = \frac{1}{2} f(z) + \text{P.V.} \int_{\partial V} f(\zeta)\Omega(\zeta, z)$$

in view of the following.

Lemma 7. *Let* $f \in \text{Lip}(\alpha, \partial V)$ *and* $z \in \partial V$. *Then we have*

$$\text{P.V.} \int_{\partial V} f(\zeta)\Omega(\zeta, z) = \int_{\partial V} (f(\zeta) - f(z))\Omega(\zeta, z) + \frac{1}{2} f(z) .$$

Proof. We set, for $\varepsilon > 0$, $M(\varepsilon) = \partial V \cap \{\zeta : |g(\zeta, z)| > \delta\}$. In view of Lemma 6 and Theorem 2, we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{M(\varepsilon)} f(\zeta)\Omega(\zeta, z) &= \lim_{\varepsilon \to 0} \int_{M(\varepsilon)} (f(\zeta) - f(z))\Omega(\zeta, z) \\ &\quad + f(z) \lim_{\varepsilon \to 0} \int_{M(\varepsilon)} \Omega(\zeta, z) \\ &= \int_{\partial V} (f(\zeta) - f(z))\Omega(\zeta, z) + \frac{1}{2} f(z) . \end{aligned}$$

This completes the proof of Lemma 7.

Now we are going to prove the following which is a main theorem in this paper.

THEOREM 5. *Let* $f \in \mathrm{Lip}(\alpha, \partial V)$, $0 < \alpha \leq 1$. *If* f *satisfies for any* $z \in \partial V$,

$$\mathrm{P.V.}\int_{\partial V} f(\zeta)\Omega(\zeta, z) = \frac{1}{2}f(z),$$

then there exists a function $F \in A(D)$ *such that* $F|_{\partial V} = f$.

Proof. We set

$$F(z) = \int_{\partial V} f(\zeta)\Omega(\zeta, z) \quad \text{for } z \in \overline{D}|\partial V.$$

Then F can be extended continuously to $\overline{D}$ and satisfies, for $z \in \partial V$,

$$\begin{aligned} F(z) &= f(z) + \int_{\partial V}(f(\zeta) - f(z))\Omega(\zeta, z) \\ &= \frac{1}{2}f(z) + \mathrm{P.V.}\int_{\partial V} f(\zeta)\Omega(\zeta, z) = f(z), \end{aligned}$$

which completes the proof of Theorem 5.

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