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THE FLOW SPACE OF A DIRECTED G-GRAPH

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This paper deals with operator-algebraic aspects of the theory of infinite, locally finite directed graphs. A (complex-valued) function on the set of edges of a directed graph whose sum over the edges pointing out of each vertex equals the sum over the edges pointing in is called a flow. Of particular interest here is the projection of the Hilbert space of square-summable functions on the edges to the closed subspace consisting of the square-summable flows. The flow space projection can be identified in a meaningful and interesting way whenever a group G acts properly on the graph, with the latter finite modulo the action of G and connected. In general, a choice of vertex and edge orbit representatives gives a realization of the flow space projection in an algebra of matrices over the von Neumann algebra of G. Suppressing the dependence on the choice of orbit representatives yields a class in K_0 of this von Neumann algebra. This K_0-class is the sum of the classes arising from the stabilizers of a representative set of edges minus a corresponding sum for vertices. Furthermore, if G is non-amenable, all of the foregoing takes place within the reduced C^*-algebra of G rather than just in the group von Neumann algebra.

1. Preliminaries. We will largely follow the notation and terminology of the first chapter of [4] for directed graphs and group actions. A directed graph X consists of a set V of vertices, a set E of edges, and maps $i, t\colon E \to V$. The edge y joins the initial vertex $i(y)$ to the terminal vertex $t(y)$. We will assume that $(i, t)\colon E \to V \times V$ is injective with range missing the diagonal, i.e. that X has no loops or multiple edges. For a vertex v, we write $\operatorname{star}(v) = i^{-1}(v) \cup t^{-1}(v)$, the set of edges incident at v, and $N(v) = t(i^{-1}(v)) \cup i(t^{-1}(v))$, the set of vertices joined to v by an edge. The cardinality of $\operatorname{star}(v)$ is called the degree of v; we abbreviate $\deg(v) = |\operatorname{star}(v)|$. We will always require X to be locally finite of bounded degree, meaning that $\sup\{\deg(v)\colon v \in V\}$ (which we denote by $\deg(X)$) is finite. (This will ensure that the various Hilbert space operators considered below are all bounded.) A path p in X of length n is a sequence $v_1, y_1, \ldots, v_n, y_n, v_{n+1}$, where for $j = 1, \ldots, n$, the edge y_j joins the vertices v_j and v_{j+1}. We think of p as having a direction of traverse, from v_1 to v_{n+1}, so each edge y_j will point either forward or backward along p; we set $\langle p, y_j\rangle = 1$ or -1 depending on whether

$i(y_j)$ is v_j or v_{j+1}. The path p is said to be closed if $v_{n+1} = v_1$. We say that X is connected if every pair of vertices can be joined by a path in X.

We call a function $\eta: E \to \mathbb{C}$ a flow if

$$\sum\{\eta(y): y \in i^{-1}(v)\} = \sum\{\eta(x): x \in t^{-1}(v)\}$$

for every vertex v. (We use the complex numbers $\mathbb{C}$ purely out of functional-analytic habit. Everything we will do below works just as well over the reals, and indeed the definition we have just given only requires functions taking values in some specified abelian group.) Notice that every closed path p gives rise to a flow η_p defined by $\eta_p = \sum\langle p, y\rangle\delta_y$, where the sum extends over the edges y that occur in p and δ_y denotes the characteristic function of $\{y\}$. We call η_p a cyclical flow. When X is finite, the cyclical flows are easily seen to span the space of all flows, but as a by-product of our investigations below we will see that nothing like this is true for infinite X.

As usual, an action of a group G on a set C is a homomorphism from G into the group of permutations of C. This gives rise to an action map $G \times C \to C$ whose effect we denote by $(g, c) \mapsto gc$ (so our group actions are all on the left). For each c in C, let G_c denote the stabilizer subgroup $\{g \in G: gc = c\}$. The action is called proper if all of the stabilizers are finite. Notice the bijection between the right coset space G/G_c and the orbit Gc given by $hG_c \mapsto hc$. We write $G\backslash C$ for the set of orbits. An action of G on a directed graph X, as above, consists of actions of G on V and E, respectively, satisfying $i(gy) = gi(y)$ and $t(gy) = gt(y)$ for all g in G and y in E. Such an action is proper if the action on the vertices (and hence automatically on the edges, since $G_y \subseteq G_{i(y)} \cap G_{t(y)}$) is proper. Notice that $G\backslash V$ and $G\backslash E$ with orienting maps induced by i and t give rise to a directed graph which we denote by $G\backslash X$.

It is worthwhile to have an example in hand to illustrate the matters discussed above.

1.1. EXAMPLE. Let G be the group with presentation $\langle a, b|a^2, b^3, (ab)^7\rangle$, in other words the triangle group $T(2, 3, 7)$; see [7] for a discussion of this group and some of its relatives. Write $c = ab$, and let A, B, and C denote the subgroups of G generated by a, b, and c respectively. Let X be the directed graph whose set of vertices is the disjoint union of G/A, G/B, and G/C, and whose edges are labelled by $G \times \{1, 2, 3\}$, with orienting maps defined by $i(g, 1) = gA = t(g, 3)$, $t(g, 1) = gB = i(g, 2)$, and $t(g, 2) = gC = i(g, 3)$.

The action of G is the obvious one on the left. There are three vertex orbits and three edge orbits, the action is free (trivial stabilizers) on the edges, while the vertex stabilizers are all conjugate to A, or B, or C. The vertex gC has degree 14, with "out" edges $i^{-1}(gC) = gC \times \{3\}$ and "in" edges $t^{-1}(gC) = gC \times \{2\}$, and likewise $\deg(gB) = 6$ and $\deg(gA) = 4$. For a picture of X, minus the orienting information, see Figure 23 in [7].

We now introduce some functional-analytic apparatus related to the general situation we are considering. For a set C, we write $\ell^2(C)$ for the usual Hilbert space of absolutely square-summable functions on C. The orienting maps give rise to the coboundary operator $S\colon \ell^2(V) \to \ell^2(E)$, defined by

$$(S\xi)(y) = \xi(i(y)) - \xi(t(y)).$$

This operator is bounded because i and t are at most $\deg(X)$-to-1. Its adjoint is given by

$$(S^*\eta)(v) = \sum\{\eta(y)\colon y \in i^{-1}(v)\} - \sum\{\eta(x)\colon x \in t^{-1}(v)\}.$$

Let $\mathscr{F} = \ker(S^*)$; this is the flow space of our title, consisting of all square-summable flows. We denote by $P_{\mathscr{F}}$ the orthogonal projection of $\ell^2(E)$ on $\mathscr{F}$. An easy calculation shows that the operator S^*S on $\ell^2(V)$ is given by

$$(S^*S\xi)(v) = \deg(v)\xi(v) - \sum\{\xi(u)\colon u \in N(v)\}.$$

In other words, S^*S is the difference Laplacian of X, an operator central to much of the considerable body of work surveyed in [9]. As will become apparent later on, S^*S is usually invertible (i.e. S is bounded below) in the context we are considering. When this happens, one has $P_{\mathscr{F}} = I - S(S^*S)^{-1}S^*$.

The group G acts by unitary operators on $\ell^2(G)$ via the right regular representation $g \mapsto R_g$, where $(R_g\varphi)(h) = \varphi(hg)$. We write $\mathbb{C}G$ for the linear span of R_G. The (right) reduced C^*-algebra of G, which we denote by $C^*(R_G)$, is the norm-closure of $\mathbb{C}G$ in the algebra of all bounded operators on $\ell^2(G)$. Closing up $\mathbb{C}G$ in the weak operator topology (or equivalently, passing to its double commutant) gives a considerably larger algebra $\mathrm{VN}(R_G)$, the (right) von Neumann algebra of G. (These algebras are usually encountered in their "left" forms; the use of the right regular representation here is made necessary by our decision to have G act on sets on the left.) Notice that if F is a finite subgroup of G, the average $|F|^{-1}\sum\{R_h\colon h \in F\}$ is a projection (selfadjoint idempotent) in $\mathbb{C}G$. The projections arising in this way from vertex and edge stabilizers will play an essential role in what follows.

2. Locating the flow space projection. Let X be a directed graph with apparatus as in the preceding section on which a group G acts properly, with $G\backslash X$ finite. (We observe that X is locally finite of bounded degree because of the finiteness of the stabilizers and of $G\backslash X$.) For c in V or E, let $Q_c = |G_c|^{-1}\sum\{R_h : h \in G_c\}$, a projection in $\mathbb{C}G$ whose range in $\ell^2(G)$ consists of all ℓ^2-functions invariant under right multiplication by elements of G_c. But this subspace is the same as $\ell^2(G/G_c)$, which is in turn the same as $\ell^2(Gc)$. We let $W_c : \ell^2(Gc) \to \ell^2(G)$ be the isometry that identifies $\ell^2(Gc)$ with the range of Q_c in this fashion. Now pick (finite) complete sets V_0 and E_0 of vertex (resp. edge) orbit representatives, and set $m = |V_0|$ and $n = |E_0|$. Write

$$Q_{V_0} = \bigoplus\{Q_v : v \in V_0\},$$

an idempotent in the matrix algebra $\mathbb{C}G \otimes M_m$, which acts on the Hilbert space $\ell^2(G)^m$. Decomposing $\ell^2(V)$ as $\bigoplus\{\ell^2(Gv) : v \in V_0\}$ and direct-summing the corresponding W_v's yields an isometry $W_{V_0} : \ell^2(V) \to \ell^2(G)^m$ onto the range of Q_{V_0}. In like manner we obtain the projection Q_{E_0} in $\mathbb{C}G \otimes M_n$ and the isometry $W_{E_0} : \ell^2(E) \to \ell^2(G)^n$ onto the range of Q_{E_0}. We may thus regard S as an operator from $\ell^2(G)^m$ to $\ell^2(G)^n$, with a corresponding matrix representation whose entries are indexed row-and-column by $E_0 \times V_0$. We now observe that each such entry lies in $\mathbb{C}G$.

2.1. Lemma. *The operator* $W_{E_0} S W_{V_0}^* : \ell^2(G)^m \to \ell^2(G)^n$ *belongs to* $\mathbb{C}G \otimes M_{n,m}$.

Proof. Fix y in E_0 and v in V_0, and let $T : \ell^2(G) \to \ell^2(G)$ be the corresponding matrix entry. For g in G, we have

$$W_v^*\delta_g = |G_v|^{-1/2}\delta_{gv},$$

$$SW_v^*\delta_g = |G_v|^{-1/2}\left(\sum\{\delta_{gx} : x \in i^{-1}(v)\} - \sum\{\delta_{gz} : z \in t^{-1}(v)\}\right),$$

and

$$W_y S W_v^*\delta_g = (|G_v||G_y|)^{-1/2}\left(\sum\{\delta_{gh} : hy \in i^{-1}(v)\} - \sum\{\delta_{gk} : ky \in t^{-1}(v)\}\right).$$

Since $R_h^*\delta_g = \delta_{gh}$, it follows that

$$T^* = (|G_v||G_y|)^{-1/2}\Big(\sum\{R_h : hy \in i^{-1}(v)\} \\ - \sum\{R_k : ky \in t^{-1}(v)\}\Big). \qquad \square$$

We thus have $W_{E_0}SS^*W_{E_0}^* \in \mathbb{C}G \otimes M_n$. Projection on the closure of the range of this operator is just $Q_{E_0} - W_{E_0}P_{\mathscr{F}}W_{E_0}^*$. Passing to this projection does not take us outside the enveloping von Neumann algebra, so $W_{E_0}P_{\mathscr{F}}W_{E_0}^* \in \mathrm{VN}(R_G) \otimes M_n$. Of course, this way of locating $P_{\mathscr{F}}$ in $\mathrm{VN}(R_G) \otimes M_n$ depends upon the choice of E_0, but only up to conjugation by a unitary operator in $\mathbb{C}G \otimes M_n$, since the stabilizers G_y and G_x are conjugate in G for x and y in the same edge-orbit. We can thus remove the dependence on E_0 by considering the class $[P_{\mathscr{F}}]$ of the flow space projection in $K_0(\mathrm{VN}(R_G))$. (Recall that for a unital ring A, the abelian semigroup $K_0^+(A)$ consists of equivalence classes of idempotent matrices over A, two such being identified if they can be made similar by direct-summing with zero-matrices of appropriate sizes. Constructing an abelian group from $K_0^+(A)$ by taking formal differences and building in cancellation yields $K_0(A)$. See [3] for a comprehensive treatment.)

2.2. THEOREM. *Let X be an infinite, connected directed graph on which a group G acts properly, with $G\backslash X$ finite. Let $n = |G\backslash E|$. When $\ell^2(E)$ is identified with a subspace of $\ell^2(G)^n$ by choosing a set of edge orbit representatives, the flow space projection $P_{\mathscr{F}}$ lies in $\mathrm{VN}(R_G) \otimes M_n$. Its class in $K_0(\mathrm{VN}(R_G))$ is given by*

$$[P_{\mathscr{F}}] = \sum [Q_y] - \sum [Q_v],$$

where the sum on y (resp. v) extends over a complete set of edge (resp. vertex) orbit representatives.

Proof. Choose cross-sections E_0 and V_0, and consider the operator $\mathscr{S}$ on $\ell^2(G)^m \oplus \ell^2(G)^n$ (where $m = |V_0|$) whose matrix is

$$\begin{pmatrix} 0 & 0 \\ W_{E_0}SW_{V_0}^* & 0 \end{pmatrix}.$$

We have $\mathscr{S} \in \mathbb{C}G \otimes M_{m+n}$ by Lemma 2.1. Our assumption that X is infinite and connected forces $\ker(S) = (0)$, so S^* has dense range in $\ell^2(V)$. This means that $\mathscr{S}^*$ has range projection $Q_{V_0} \oplus 0$. On the other hand, the range projection of $\mathscr{S}$ is $0 \oplus (Q_{E_0} - W_{E_0}P_{\mathscr{F}}W_{E_0}^*)$.

These two range projections are Murray-von Neumann equivalent in $\mathrm{VN}(R_G)$ (V.1.5 of **[12]**); we conclude that they represent the same element of K_0. □

Except in rather special cases (such as the first example in §4 below), the actual calculation of $P_{\mathscr{F}}$, i.e. the determination of the inner products $(P_{\mathscr{F}}\delta_y, \delta_x)$ for all edges x and y, appears to be a very strenuous undertaking. It is worth noting, however, that the theorem provides at least a bit of specific information about these numbers when we apply the state on K_0 that arises from the natural trace on $\mathrm{VN}(R_G)$. Recall that this trace is defined by $\tau(T) = (T\delta_1, \delta_1)$. It satisfies $\tau(T_1T_2) = \tau(T_2T_1)$ for all T_1, T_2 in $\mathrm{VN}(R_G)$, and extends to a trace τ_n on $\mathrm{VN}(R_G)\otimes M_n$ by summing the diagonal entries of the matrix argument and then applying τ. We obtain a homomorphism $\tau_*\colon K_0(\mathrm{VN}(R_G)) \to \mathbb{C}$ (actually $\mathbb{R}$) satisfying $\tau_*([P]) = \tau_n(P)$ for any idempotent P in $\mathrm{VN}(R_G)\otimes M_n$.

2.3. COROLLARY. *In the situation of the theorem, we have*

$$\sum |G_y|^{-1}(1-(P_{\mathscr{F}}\delta_y, \delta_y)) = \sum |G_v|^{-1},$$

where the sum on y (resp. v) extends over a complete set of edge- (resp. vertex-) orbit representatives.

Proof. Apply τ_* to the equality asserted by the theorem, after noticing that

$$\tau_n(P_{\mathscr{F}}) = \sum (W_yP_{\mathscr{F}}W_y^*\delta_1, \delta_1) = \sum |G_y|^{-1}(P_{\mathscr{F}}\delta_y, \delta_y),$$

and that $\tau(Q_c) = |G_c|^{-1}$ for each edge or vertex c. □

3. When G is non-amenable. We turn now to the case in which the difference Laplacian S^*S is not only one-to-one, but actually invertible. This has to do with the notion of isoperimetric number introduced and studied for infinite graphs by B. Mohar (**[8]**, **[2]**; see also **[5]**). It is defined as follows. For a non-empty subset U of the vertex set of an infinite locally finite graph, let ∂U denote the set of all edges with one end in U and the other end not in U. The isoperimetric number of the graph in question is the infimum of $|\partial U|/|U|$ as U ranges over all non-empty finite sets of vertices. Mohar shows in **[8]** that a graph has invertible difference Laplacian if and only if its isoperimetric number is positive. (The forward implication, used below in the proof of 3.1, is straightforward. Namely, let ξ be the characteristic function of the finite set U of vertices. Then $\|\xi\|^2 = |U|$, while $\|S\xi\|^2 = |\partial U|$, so if S^*S is strictly positive, the isoperimetric number must be positive.)

The definition of the isoperimetric number is reminiscent of Følner's condition characterizing amenable groups. Recall that these are the groups G for which there exists a right- (or left-) translation invariant nonzero positive linear functional on $\ell^\infty(G)$. A good general reference on this subject is **[10]**. Our point of view here is that amenability is a rather exceptional property for a group to enjoy. For instance, the group in our Example 1.1 is non-amenable, as is $T(2, 3, k)$ for any $k > 6$. The following theorem is a modest extension of a result obtained in **[6]** for Cayley graphs, i.e. connected G-graphs with free action and one vertex orbit.

3.1. THEOREM. *Let X be a connected directed graph on which a group G acts properly, with $G\backslash X$ finite. Then S^*S is invertible if and only if G is non-amenable.*

Proof. ($\Rightarrow$) We suppose that S is not bounded below and show that G must then be amenable. For v in V, define $A_v : \ell^2(V) \to \ell^2(G)$ by $(A_v\xi)(g) = \xi(gv)$, and similarly for y in E, define $B_y : \ell^2(E) \to \ell^2(G)$ by $(B_y\eta)(g) = \eta(gy)$. (Our hypotheses imply that the maps $g \mapsto gv$ and $g \mapsto gy$ are most M-to-1 for some integer M, so the operators A_v and B_y are bounded.) Take h in G and v in V. Since X is connected, there is a path p from v to hv, say with edges (in order) $y_1, y_2, \dots, y_n$. We have for any ξ in $\ell^2(G)$ that

$$\begin{aligned}((I - R_h)A_v\xi)(g) &= \xi(gv) - \xi(ghv) \\ &= \sum \langle p, y_j\rangle (S\xi)(gy_j) \\ &= \sum \langle p, y_j\rangle (B_{y_j}S\xi)(g).\end{aligned}$$

There is thus a bounded operator $C_{h,v} : \ell^2(V) \to \ell^2(G)$ such that $(I - R_h)A_v = C_{h,v}S$. Let V_0 be a complete set of vertex orbit representatives, finite by assumption, and set $A = \sum\{A_v : v \in V_0\}$, $C_h = \sum\{C_{h,v} : v \in V_0\}$. Then $(I - R_h)A = C_hS$. Furthermore, A is bounded below (by $(\sum\{|G_v| : v \in V_0\})^{1/2}$) because the vertex orbits are disjoint. Thus, if $\{\xi_n\}$ is a sequence in $\ell^2(V)$ which is bounded away from 0 in norm and for which $\|S\xi_n\| \to 0$, the sequence $\{A\xi_n\}$ in $\ell^2(G)$ is bounded away from zero in norm and satisfies $\|(I - R_h)A\xi_n\| \to 0$ for every h in G. Let $\varphi_n = \|A\xi_n\|^{-1}A\xi_n$, and let ω be any w^*-limit of the vector states on $\mathscr{L}(\ell^2(G))$ arising from the φ_n's. We have $\omega(R_n) = 1$ for all h, whence it follows easily that restriction of ω to $\ell^\infty(G)$ (represented on $\ell^2(G)$ by pointwise multiplication) is a right-invariant mean.

($\Leftarrow$) We assume that G is amenable and show that S is not bounded below. Let V_0 be a complete set of distinct vertex-orbit representatives, and consider

$$F = \{g \in G\colon gu \in N(v) \text{ for some } u, v \text{ in } V_0\},$$

which is finite because V_0 is finite and the action of G is proper. Given $\varepsilon > 0$, use Følner's condition (4.10 of [10]) to find a non-empty finite subset K of G such that

$$(*) \qquad |Kg \triangle K| < \frac{\varepsilon}{r|F|\deg(X)}|K|$$

for all g in F, where $r = \max\{|G_v|\colon v \in V_0\}$. Let $U = KV_0$, a finite subset of V. For an edge y in ∂U, let $\mathrm{Out}(y)$ be the end of y that lies in $V\backslash U$. The other end of y, the one in U, has the form kv for some k in K and v in V_0. Since $\mathrm{Out}(y) \in kN(v)$, we may write $\mathrm{Out}(y) = kgu$, where $g \in F$, $u \in V_0$, and (since $kgu \in V\backslash U$) $kg \notin K$. Thus, $\mathrm{Out}(\partial U) \subseteq (KF\backslash K)V_0$. Since the map Out is at most $\deg(X)$-to-1, we have

$$|\partial U| \leq \deg(X)|(KF\backslash K)V_0| \leq \deg(X)|KF\backslash K||V_0|.$$

Using $(*)$ and the observation that the action map $K \times V_0 \to KV_0$ is at most r-to-1, we then obtain

$$|\partial U| \leq \frac{\varepsilon}{r}|K||V_0| \leq \varepsilon|KV_0| = \varepsilon|U|.$$

It follows that X has isoperimetric number zero, i.e. that S is not bounded below. □

In the non-amenable case, we can improve upon Theorem 2.2 as follows.

3.2. THEOREM. *Under the hypotheses of Theorem* 2.2 *and the further assumption that* G *is non-amenable, we have* $P_{\mathcal{G}} \in C^*(R_G) \otimes M_n$ *and*

$$[P_{\mathcal{G}}] = \sum[Q_y] - \sum[Q_v] \quad \text{in } K_0(C^*(R_G)).$$

Proof. By 3.1, the difference Laplacian S^*S is invertible, and we have already observed that in this situation $P_{\mathcal{G}} = I - S(S^*S)^{-1}S^*$ (as operators on $\ell^2(E)$). Since $(S^*S)^{-1}$ is the norm-limit of polynomials in S^*S, it follows that $P_{\mathcal{G}}$ is the norm-limit of polynomials in SS^*.

We have $W_{E_0}SS^*W_{E_0}^* \in \mathbb{C}G \otimes M_n$ for any choice of edge-orbit representatives (by 2.1), so $W_{E_0}P_{\mathscr{G}}W_{E_0}^* \in C^*(R_G) \otimes M_n$. Using the notation of the proof of 2.2, write $P = Q_{V_0} \oplus 0$ and $Q = 0 \oplus (Q_{E_0} - W_{E_0}P_{\mathscr{G}}W_{E_0}^*)$, two projections in the C^*-algebra $\mathscr{A} = C^*(R_G) \otimes M_{m+n}$. Because S^*S is invertible, the operators $\mathscr{S}^*\mathscr{S}$ and $\mathscr{S}\mathscr{S}^*$ are invertible in the corners $P\mathscr{A}P$ and $Q\mathscr{A}Q$, respectively. Let H be the inverse of $\mathscr{S}^*\mathscr{S}$ relative to $P\mathscr{A}P$, and set $T = \mathscr{S}H^{1/2}$. Then $T^*T = P$, and $TT^*\mathscr{S}\mathscr{S}^* = \mathscr{S}P\mathscr{S}^* = \mathscr{S}\mathscr{S}^*$, so $TT^* = Q$. This is enough to make $[Q] = [P]$ in $K_0(C^*(R_G))$. □

4. Examples and remarks. The case in which X is a tree (i.e. connected, with no nonreversing closed paths) is quite special in our context. For one thing, most groups (like the one in Example 1.1, by a result in I.6.5 of **[11]**) are incapable of acting properly on any tree. The groups that can so act are, roughly speaking, the ones that arise from a collection of finite groups by a sequence of amalgamated products and HNN-constructions; see **[11]** or **[4]**. In terms of analysis, explicit computations are sometimes possible in the tree case that seem to be out of reach otherwise.

4.1. EXAMPLE. Consider the group G with presentation $\langle a, b | a^m, b^n \rangle$, the free product of cyclic groups of order m and n, where $m > 1$ and $n > 2$. It is well known that G is non-amenable. Let A and B be the subgroups of orders m and n generated by a and b respectively. Let X be the directed graph whose vertices are labeled by the disjoint union of G/A and G/B, and whose edges are labeled by G, with $i(g) = gA$ and $\iota(g) = gB$. It is not hard to see that X is the directed tree whose edges point from vertices of degree m to vertices of degree n. Between any two distinct vertices of X, there is a unique nonreversing path, called the geodesic, whose length measures the distance between the two vertices. Similarly, we can speak of geodesics between distinct pairs of edges. For any g in $G \backslash \{1\}$, let v $(= A$ or $B)$ and w $(= gA$ or $gB)$ be the ends of the edges 1 and g that are farthest apart, and let $p(g) = v, 1, \ldots, g, w$ be the geodesic from v to w. Define nonnegative integer-valued functions c, d on G by setting $c(1) = d(1) = 0$, and $c(g)$ (resp. $d(g)$) $=$ number of vertices in G/A (resp. G/B) encountered on $p(g)$ between 1 and g. (For example, with $g = ab^2aba$, we have

$$\begin{gathered} p(g) = B, 1, A, a, aB, ab^2, ab^2A, ab^2a, ab^2aB, \\ ab^2ab, ab^2abA, ab^2aba, ab^2abaB, \end{gathered}$$

and $c(g) = 3$, $d(g) = 2$; in general, c and d count the number of powers of a and powers of b in the reduced form of a group element.) Define $\varphi\colon G \to \mathbb{R}$ by

$$\varphi(g) = \left(1 - \frac{1}{m} - \frac{1}{n}\right)(1-m)^{-c(g)}(1-n)^{-d(g)}.$$

In other words, $\varphi(g)$ is obtained from $\varphi(1)$ by moving geodesically in X from edge 1 to edge g, multiplying by $(1-m)^{-1}$ (resp. $(1-n)^{-1}$) each time a vertex in G/A (resp. G/B) is crossed. *We claim that* $\varphi = P_{\mathscr{F}}\delta_1$. [Give G the obvious left action on X. It follows from 2.3 above that $(P_{\mathscr{F}}\delta_1)(1) = 1-m^{-1}-n^{-1} = \varphi(1)$. For $g \neq 1$, suppose vertex gA is closer to edge 1 than its vertex gB. Then there is an automorphism of X fixing edge 1 (and all other edges whose geodesic to 1 does not pass through g) and interchanging any given pair of edges taken from among $\{gb, gb^2, \dots, gb^{n-1}\} = \operatorname{star}(gB)\backslash\{g\}$. Such an automorphism must leave $P_{\mathscr{F}}\delta_1$ unchanged. This shows that $P_{\mathscr{F}}\delta_1$ is constant on $\operatorname{star}(gB)\backslash\{g\}$, with $P_{\mathscr{F}}\delta_1(gb^k) = (1-n)^{-1}P_{\mathscr{F}}\delta_1(g)$ for $k = 1, \dots, n-1$ because $P_{\mathscr{F}}\delta_1$ is a flow. Similarly $P_{\mathscr{F}}\delta_1(ga^k) = (1-m)^{-1}P_{\mathscr{F}}\delta_1(g)$ for $k = 1, \dots, m-1$ if gB is closer to 1 than is gA. This growth recipe completely determines $P_{\mathscr{F}}\delta_1$ once its value at 1 is known.] It now follows readily that $(P_{\mathscr{F}}\delta_h, \delta_g) = \varphi(h^{-1}g)$ for all g, h in G, and that $P_{\mathscr{F}}$ is the operator of right-convolution by φ on $\ell^2(G)$.

A direct calculation shows that $\varphi \in \ell^2(G)\backslash\ell^1(G)$. It is by no means obvious that right-convolution by φ even takes $\ell^2(G)$ into itself, but in fact (by 3.2 above) this operator lies in $C^*(R_G)$. The existence of a projection in $C^*(R_G)$ of trace $1 - m^{-1} - n^{-1}$ is not obvious either; this is related to the matters addressed in **[1]**. Indeed, 3.2 and the construction above show *mutatis mutandis* that $C^*(R_H)$ contains such a projection whenever H is a non-amenable quotient of $\mathbb{Z}_m * \mathbb{Z}_n$. (The graph one obtains will of course no longer be a tree, but it will be connected because the images of a and b in H generate H.)

When is the flow space non-trivial? For trees with only finitely many vertices of degree greater than 2, it is clear that $\mathscr{F} = (0)$. On the other hand, any tree with nonzero isoperimetric number has non-trivial flows. [If X is an infinite connected graph, the range of S cannot contain δ_y for any edge y. This rules out the possibility that S could be bounded below with $\ker(S^*) = (0)$.] As noted in §1, each nonreversing closed path in a graph gives rise to a non-trivial flow; thus, it very rarely happens that $\mathscr{F} = (0)$.

One might in any event expect that in a graph with an abundance of nonreversing closed paths, the corresponding cyclical flows would span a dense subspace of $\mathcal{F}$. We will now show that this doesn't happen in Example 1.1, and in so doing indicate a general procedure for measuring the discrepancy between $\mathcal{F}$ and [{cyclical flows}]. We denote by $\mathcal{F}_0$ the closed linear span of the cyclical flows, and by $P_{\mathcal{F}_0}$ the orthogonal projection on it.

4.2. PROPOSITION. *In the situation of Example* 1.1, *we have* $P_{\mathcal{F}_0} \in C^*(R_G)\otimes M_3$, *with* $[P_{\mathcal{F}}-P_{\mathcal{F}_0}] = [I]-[Q_A]-[Q_B]-[Q_C]$ *in* $K_0(C^*(R_G))$. *In particular,* $P_{\mathcal{F}} \neq P_{\mathcal{F}_0}$.

Proof. Taking advantage of the realization of $G = T(2, 3, 7)$ as a group of linear fractional transformations of the open disc, and of the resulting picture of X [7], we observe that every cyclical flow is a linear combination of those arising from the closed triangular paths

$$\begin{aligned} p_g &= gA, (g, 1), gB, (g, 2), gC, (g, 3), gA \quad \text{and} \\ \pi_g &= gA, (ga, 3), gaC, (gb, 2), gB, (g, 1), gA. \end{aligned}$$

(Note that $aC = bC$.) Write $K = p_G \cup \pi_G$; there is an obvious left action of G on K with two free G orbits. Recalling that η_p denotes the cyclical flow corresponding to the path p, we define $T: \ell^2(E) \to \ell^2(K)$ by $(T\eta)(p) = (\eta, \eta_p)$ for η in $\ell^2(E)$ and p in K. To see what T^* does, we note that each edge y has a path $\rho(y)$ in K on its right and another $\lambda(y)$ on its left. (Here, we are imagining a counterclockwise direction of traverse for the paths in K.) These are given by $\rho(g, 1) = \pi_g$, $\rho(g, 2) = \pi_{gb^{-1}}$, $\rho(g, 3) = \pi_{ga}$, and $\lambda(g, 1) = \lambda(g, 2) = \lambda(g, 3) = p_g$. One now checks that $T^*: \ell^2(K) \to \ell^2(E)$ is given by $(T^*\theta)(y) = \theta(\lambda(y)) - \theta(\rho(y))$ for θ in $\ell^2(K)$ and y in E. We have $T^*\delta_p = \eta_p$ for each p in K, so the closure of the range of T^* is precisely $\mathcal{F}_0$. Now write $\ell^2(E) = \ell^2((G, 1)) \oplus \ell^2((G, 2)) \oplus \ell^2((G, 3)) = \ell^2(G)^3$, and $\ell^2(K) = \ell^2(p_G) \oplus \ell^2(\pi_G) = \ell^2(G)^2$. The formulas for ρ and λ show that the matrix of $T: \ell^2(G)^3 \to \ell^2(G)^2$ is

$$\begin{pmatrix} I & I & I \\ -I & -R_b^* & -R_a \end{pmatrix},$$

so the matrix for TT^* is $\left(\begin{smallmatrix} 3I & -W \\ -W^* & 3I \end{smallmatrix}\right)$, where $W = I + R_b + R_a$. We have $\|W\| \leq 3$, and in fact $\|W\| < 3$, since otherwise we would obtain by quite routine arguments a state on $C^*(R_G)$ with value 1 at

R_a and R_b, which is impossible because here a and b generate a non-amenable group. It follows that TT^* is invertible in $C^*(R_G) \otimes M_2$. As in the proof of Theorem 3.2, this implies that $P_{\mathscr{F}_0}$, the projection on the range of T^* (which is closed because T^* is bounded below), belongs to $C^*(R_G) \otimes M_3$ and moreover $[P_{\mathscr{F}_0}] = 2[I]$ in $K_0(R_G)$. We have $[P_{\mathscr{F}}] = 3[I]-[Q_A]-[Q_B]-[Q_C]$ by Theorem 3.2, so $[P_{\mathscr{F}} - P_{\mathscr{F}_0}] = [I] - [Q_A] - [Q_B] - [Q_C]$. This shows that

$$\tau_3(P_{\mathscr{F}} - P_{\mathscr{F}_0}) = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7},$$

and in particular $P_{\mathscr{F}} \neq P_{\mathscr{F}_0}$. □

References

[1] J. Anderson, B. Blackadar and U. Haagerup, *Minimal projections in the reduced group C^*-algebra of $\mathbb{Z}_n * \mathbb{Z}_m$*, J. Operator Theory, to appear.

[2] N. L. Biggs, B. Mohar and J. Shawe-Taylor, *The spectral radius of infinite graphs*, Bull. London Math. Soc., **20** (1988), 116–120.

[3] B. Blackadar, *K-Theory for Operator Algebras*, MSRI Series No. 5, Springer, New York, 1986.

[4] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics vol. 17, Cambridge University Press, New York, 1989.

[5] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc., **284** (1984), 787–794.

[6] P. Gerl, *Amenable groups and amenable graphs*, in *Harmonic Analysis* (ed. P. Eymard and J.-P. Pier), Springer Lecture Notes in Mathematics No. **1359** (1987), 181–190.

[7] W. Magnus, *Noneuclidean Tesselations and their Groups*, Academic Press, New York, 1974.

[8] B. Mohar, *Isoperimetric inequalities, growth, and the spectrum of graphs*, Linear Algebra Appl., **103** (1988), 119–131.

[9] B. Mohar and W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc., **21** (1989), 209–234.

[10] A. L. T. Paterson, *Amenability*, Math. Surveys Monographs, vol. 29, Amer. Math. Soc., Providence, RI, 1988.

[11] J.-P. Serre, *Trees*, Springer, New York, 1980.

[12] M. Takesaki, *Theory of Operator Algebras* I, Springer, New York, 1979.

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